Basic Theorems in Dynamic Elasticity

1. Stress-Strain relationships
2. Equation of motion
3. Uniqueness and reciprocity theorems
4. Elastodynamic Green’s function
5. Representation theorems

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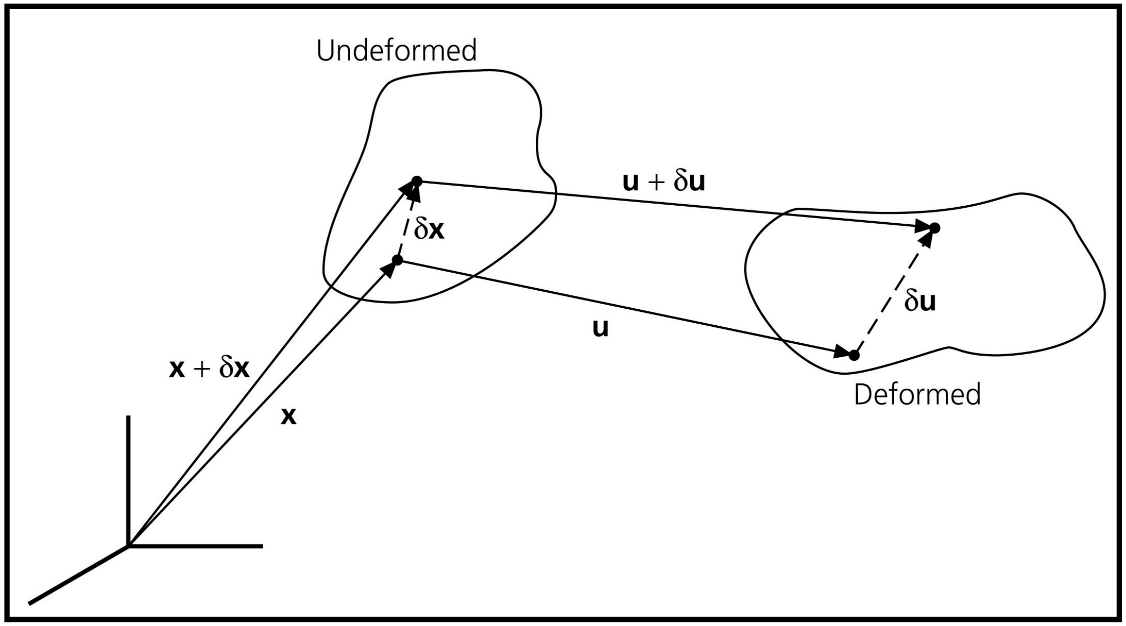
Strain in Lagrangian Coordinates

Distorsion in the continuum



Displacement of a given point

$$u_i(\mathbf{x} + \delta\mathbf{x}) \approx u_i(\mathbf{x}) + \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j = u_i(\mathbf{x}) + \delta u_i$$

Relative displacement between two points

$$\delta u_i = \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j$$

Strain in Lagrangian Coordinates

$$\delta u_i = \frac{\partial u_i(\mathbf{x})}{\partial x_j}\,\delta x_j$$

Relative displacement between two points

Although we are interested in deformation that distorts the body, there can also be a rigid body translation or a rigid body rotation, neither of which produces deformation. To distinguish these effects, we add and subtract $\partial u_j/\partial x_i$ and then separate it into two parts

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j = (e_{ij} + \omega_{ij})\delta x_j$$

Strain + Rotation

$$e_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Cauchy's Infinitesimal Strain Tensor

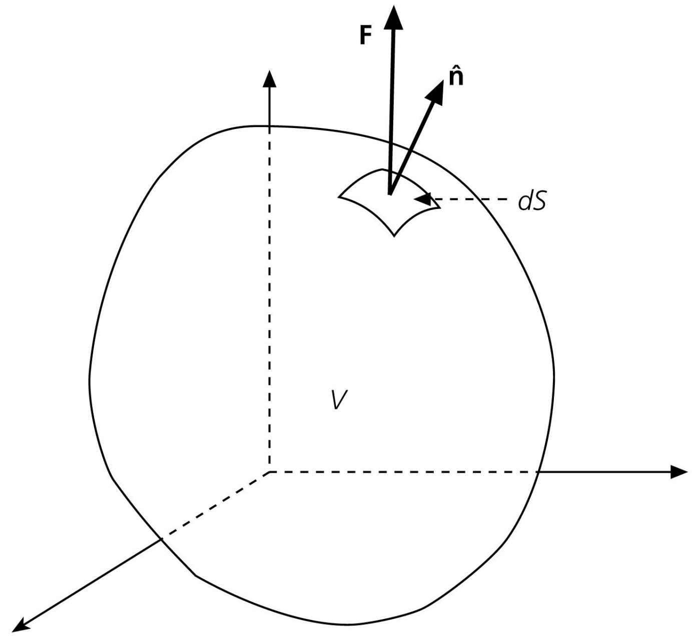
$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

Body and Contact Forces

Two kinds of forces:

Body forces (e.g., gravity)

Surface forces (e.g., pressure underwater, stresses)



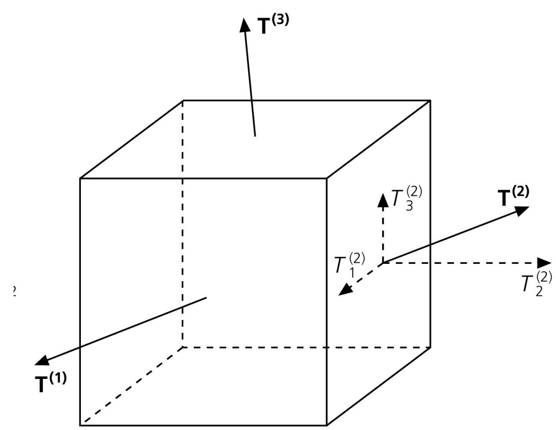
Vectorial Representation of Stresses (Tractions):

$$\mathbf{T}(\hat{\mathbf{n}}) = \lim_{dS \to 0} \frac{\mathbf{F}}{dS}$$

Stress Tensor (Three traction vectors):

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}$$

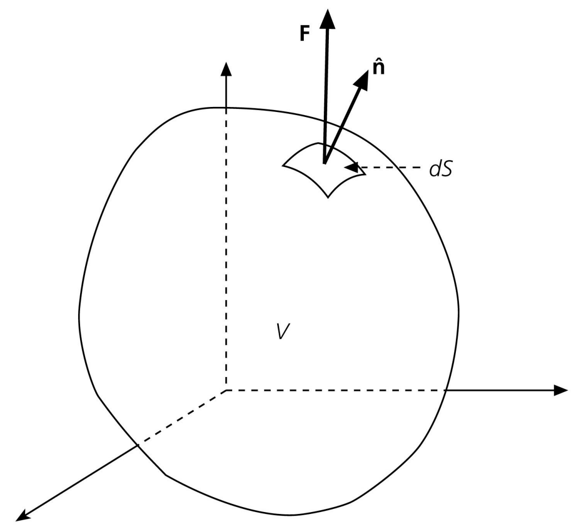
The tensor's rows are the three traction vectors.



Stress Tensor and Traction Vectors

Tensorial representation of the state of stresses (Stress Tensor):

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}$$



General relationship between the Stress Tensor and the Traction Vector:

$$T_i = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 = \sum_{j=1}^{3} \sigma_{ji} n_j = \sigma_{ji} n_j$$

$$\mathbf{T} = \sigma \hat{\mathbf{n}}$$

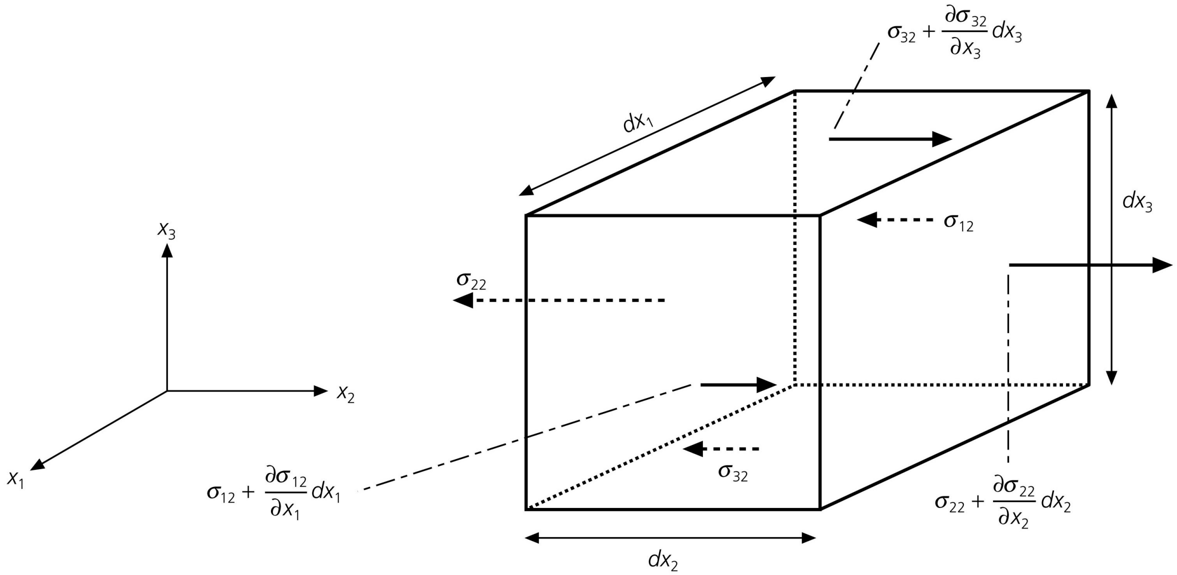
Cauchy-Descartes relationship: Traction vector as a function of both the stress tensor and the surface normal (unit) vector

Equation of Motion

Newton's Second Law:

$$\Sigma \mathbf{F} = m\mathbf{a}$$

Summation of all body and surface forces acting in the unit volume equals the linear momentum rate.



Summation of surface forces normal to the x_1x_3 plane:

$$\left[\sigma_{22}(\mathbf{x} + dx_2\hat{\mathbf{e}}_2) - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \left[\sigma_{22}(\mathbf{x}) + \frac{\partial \sigma_{22}(\mathbf{x})}{\partial x_2} dx_2 - \sigma_{22}(\mathbf{x})\right] dx_1 dx_3 = \frac{\partial \sigma_{22}(\mathbf{x})}{\partial x_2} dx_1 dx_2 dx_3$$

And for all forces parallel to x_2 but applied over the x_2x_3 and x_1x_2 planes, and including body forces:

$$\left[\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right] dx_1 dx_2 dx_3 + f_2\, dx_1 dx_2 dx_3 = \rho \frac{\partial^2 u_2}{\partial t^2}\, dx_1 dx_2 dx_3$$

Equation of Motion

Summation of all body and surface forces acting in the same direction must equate the rate of linear momentum (i.e. the unit volume acceleration).

$$\left[\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3}\right] dx_1 dx_2 dx_3 + f_2 \, dx_1 dx_2 dx_3 = \rho \frac{\partial^2 u_2}{\partial t^2} \, dx_1 dx_2 dx_3$$

Surface forces Body forces

Thus, the first component of the Equation of Motion may be written as:

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = \sum_{j=1}^{3} \frac{\partial \sigma_{j2}}{\partial x_j} + f_2 = \rho \frac{\partial^2 u_2}{\partial t^2}$$

Including the three components:

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + f_i(\mathbf{x}, t) = \rho \frac{\partial^2 u_i(\mathbf{x}, t)}{\partial t^2}$$

$$\sigma_{ij,j}(\mathbf{x}, t) + f_i(\mathbf{x}, t) = \rho \ddot{u}_i(\mathbf{x}, t)$$

Equation of Motion

Equation of Motion

(An alternative deduction)

$$\frac{\partial}{\partial t}\iiint_V \rho \frac{\partial \mathbf{u}}{\partial t}\, dV = \iiint_V \mathbf{f}\, dV + \iint_S \mathbf{T}(\mathbf{n})\, dS.$$

Momentum Rate — Body forces — Surface forces

Summation of all body and surface forces acting in the unit volume equals the linear momentum rate

Since $\mathbf{T} = \sigma\hat{\mathbf{n}}$ and using the Gauss divergence theorem:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

$$\iint_S T_i\, dS = \iint_S \tau_{ji} n_j\, dS = \iiint_V \tau_{ji,j}\, dV$$

Integrating over the whole volume

$$\iiint_V (\rho \ddot{u}_i - f_i - \tau_{ji,j})\, dV = 0.$$

Equation of Motion

$$\rho \ddot{u}_i = f_i + \tau_{ji,j}$$

Constitutive Equations

(stress – strain relationship)

Constitutive equations give the relation between stress and strain.

The simplest type of materials are *linearly elastic,* such that there is a linear relation between the stress and strain tensors.

Others could describe viscous (Newtonian and non-Newtonian), viscoelastic, elastic-plastic, etc.

Linearly elastic constitutive equations gives rise to seismic waves.

Linearly elastic material: The constitutive equation is *Hooke's law*:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

The constants c_{ijkl}, the *elastic moduli,* describe the properties of the material.

Since both the stress and strain tensors are symmetric and assuming the medium is isotropic, the 81 elastic moduli reduce to only 2 constant, the Lamé Coefficients, so that:

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

Constitutive Equations

(stress – strain relationship)

One useful pair are the *Lame´ constants* λ and μ:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

Dilation

$$\theta = e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

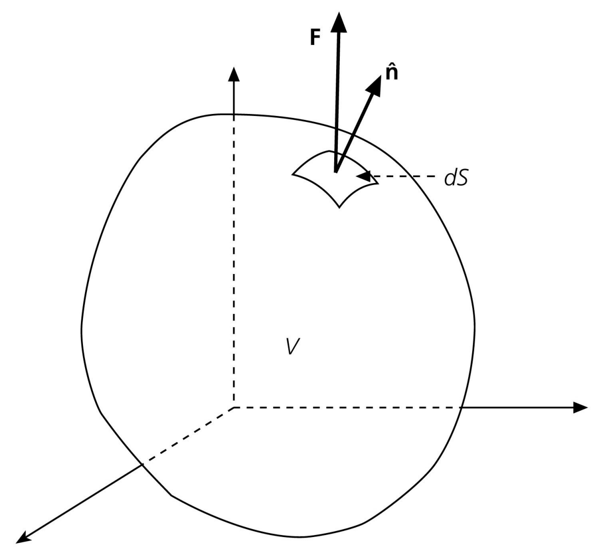
Hooke's Law for isotropic materials

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

Uniqueness Theorem

The displacement field u(x,t) throughout the volume V with surface S is uniquely determined after time t_0 by the initial displacement and velocity values at t_0 throughout V (initial conditions) and by values at all times $t > t_0$ of:

1. The body forces f and heat supplied throughout V
2. The tractions T over any part S_1 of S, and
3. The displacement over the remainder S_2 of S, with $S=S_1+S_2$.

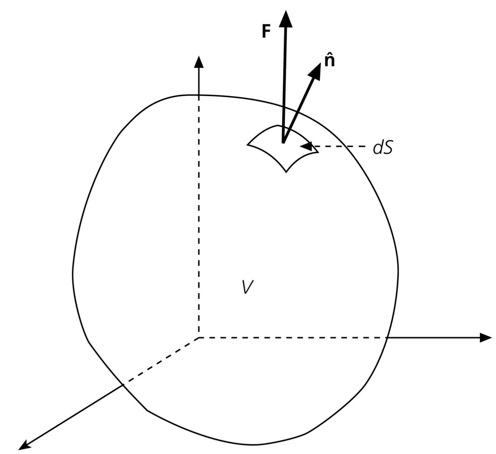


Reciprocity Theorem
(Betti's Theorem)

If displacement field u due to a body force f, boundary conditions on S and initial conditions throughout V induces traction T(u,n), and displacement field v due to a body force g, boundary conditions on S and initial conditions throughout V induces traction T(v,n), thus the following scalar equality holds:

$$\iiint_V (\mathbf{f} - \rho\ddot{\mathbf{u}}) \cdot \mathbf{v}\, dV + \iint_S \mathbf{T}(\mathbf{u},\mathbf{n}) \cdot \mathbf{v}\, dS$$
$$= \iiint_V (\mathbf{g} - \rho\ddot{\mathbf{v}}) \cdot \mathbf{u}\, dV + \iint_S \mathbf{T}(\mathbf{v},\mathbf{n}) \cdot \mathbf{u}\, dS$$

Homework:
Prove Betti's Theorem



Enrico Betti (1823-1892) was an Italian "mathematician who wrote a pioneering memoir on topology, the study of surfaces and higher-dimensional spaces, and wrote one of the first rigorous expositions of the theory of equations developed by the noted French mathematician Evariste Galois" (Encyclopaedia Britannica)

Reciprocity Theorem

(Betti's Theorem)

Betti's theorem remains true even if quantities associated with both displacement fields u and v are evaluated at two different times t_1 and t_2, respectively.

Let us suppose that u and v are everywhere zero throughout V before a given time t_0 (i.e. quiescent past). If we take $t_1 = t$ and $t_2 = \tau - t$, and time integrate the reciprocity equality from zero to τ, then Betti's formula becomes:

$$\int_{-\infty}^{\infty} dt \iiint_V \{\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, \tau - t) - \mathbf{v}(\mathbf{x}, \tau - t) \cdot \mathbf{f}(\mathbf{x}, t)\} \, dV$$
$$= \int_{-\infty}^{\infty} dt \iint_S \{\mathbf{v}(\mathbf{x}, \tau - t) \cdot \mathbf{T}(\mathbf{u}(\mathbf{x}, t), \mathbf{n}) - \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{T}(\mathbf{v}(\mathbf{x}, \tau - t), \mathbf{n})\} \, dS.$$

This major result relates, in a straightforward manner, both displacement fields throughout V, the body forces that originate them and the associated traction vectors they induce over S.

Green's Function for Elastodynamics

Consider the special case of a body force $f_i(x,t)$ applied impulsively, in space and time, to a given particle at $x = \xi$ and time $t = \tau$. If such a force is applied in the x_n-axis direction, and it is proportional to both the spatial and temporal one-dimensional Dirac functions, then:

$$f_i(\mathbf{x}, t) = A\, \delta(\mathbf{x} - \xi)\, \delta(t - \tau)\, \delta_{in}$$

where A is a constant giving the strength of the body force and δ_{in} is Kronecker delta function.

The displacement field from such a simple source (for A equal to unit) is the elastodynamic Green's function.

We thus denote the *i*th displacement component due to f_i at general coordinates (x,t) by

$$G_{in}(\mathbf{x}, t; \xi, \tau)$$

which clearly represents a second order tensor and depends on both source and receiver locations.

Green's Function for Elastodynamics

The Green tensor satisfy the Equation of Motion:

$$\rho \ddot{u}_i = f_i + \tau_{ji,j}$$

where $\tau_{ij} = c_{ijpq} e_{pq}$ and $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ and $u_i = G_{in}\left(x,t\,;\,\xi,\tau\right)$

so that the following equation holds throughout V

$$\rho \frac{\partial^2}{\partial t^2} G_{in} = \delta_{in}\, \delta(\mathbf{x} - \xi)\, \delta(t - \tau) + \frac{\partial}{\partial x_j}\left(c_{ijkl} \frac{\partial}{\partial x_l} G_{kn}\right)$$

To uniquely determine G_{ij} we should specify the boundary conditions over S, that may differ depending on the problem we study (e.g. free surface or rigid boundary conditions).

Green's Function for Elastodynamics

If G_{ij} satisfies homogeneous boundary conditions on S (i.e. a stress-free condition, which is a first order approximation of the Earth's surface), then an important reciprocal relationship for source and receiver coordinates is detached from Betti's theorem by assuming f and g to be two impulse forces applied in the m- and n-directions:

Space-time reciprocity of Green's function

$$G_{nm}(\xi_2, \tau_2; \xi_1, \tau_1) = G_{mn}(\xi_1, -\tau_1; \xi_2, -\tau_2).$$

Representation Theorems

A representation theorem is a formula for the displacement field (at a general point in space and time) in terms of quantities that originated the motion. As stated by the uniqueness theorem, these quantities may be body forces within V and applied tractions or displacements over S.

Integrated Betti's Theorem

$$\int_{-\infty}^{\infty} dt \iiint_V \{\mathbf{u}(\mathbf{x},t)\cdot\mathbf{g}(\mathbf{x},\tau-t) - \mathbf{v}(\mathbf{x},\tau-t)\cdot\mathbf{f}(\mathbf{x},t)\}\, dV$$
$$= \int_{-\infty}^{\infty} dt \iint_S \{\mathbf{v}(\mathbf{x},\tau-t)\cdot\mathbf{T}(\mathbf{u}(\mathbf{x},t),\mathbf{n}) - \mathbf{u}(\mathbf{x},t)\cdot\mathbf{T}(\mathbf{v}(\mathbf{x},\tau-t),\mathbf{n})\}\, dS.$$

If we substitute into Betti's theorem $g_i = \delta_{in}\,\delta(x-\xi)\,\delta(t)$ for which the corresponding displacement is $v_i = G_{in}(x,t\,;\,\xi,0)$ we obtain a formula for $u_n(x,t)$

First Representation Theorem

$$u_n(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\xi,\tau)G_{in}(\xi,t-\tau;\mathbf{x},0)\, dV(\xi)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\xi,t-\tau;\mathbf{x},0)T_i(\mathbf{u}(\xi,\tau),\mathbf{n})$$
$$- u_i(\xi,\tau)c_{ijkl}n_j G_{kn,l}(\xi,t-\tau;\mathbf{x},0)\}\, dS(\xi).$$

Useful Hint

$$\int_{-\infty}^{\infty} f(x)\,\delta\{dx\} = f(0)$$

Representation Theorems

Suppose that the Green's function is determined with S as a rigid boundary. This implies that $G_{in}(\xi, t-\tau\ ;\ x,0) = 0$ for ξ in S and considering the spatial reciprocity $G_{in}(\xi, t-\tau\ ;\ x,0) = G_{ni}(x, t-\tau\ ;\ \xi,0)$, thus:

Second Representation Theorem

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt \iiint_V f_i(\xi, \tau) G_{ni}^{\text{rigid}}(\mathbf{x}, t-\tau; \xi, 0)\, dV$$
$$- \int_{-\infty}^{\infty} dt \iint_S u_i(\xi, \tau) c_{ijkl} n_j \frac{\partial}{\partial \xi_l} G_{nk}^{\text{rigid}}(\mathbf{x}, t-\tau; \xi, 0)\}\, dS$$

Alternatively we can use the Green's function determined for a free surface, so the traction $c_{ijkl} n_j (\partial/\partial \xi_l) G_{kn}^{\text{free}}(\xi, t-\tau; \mathbf{x}, 0)$ is zero for ξ in S and thus:

Third Representation Theorem

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} dt \iiint_V f_i(\xi, \tau) G_{in}^{\text{free}}(\mathbf{x}, t-\tau; \xi, 0)\, dV$$
$$+ \int_{-\infty}^{\infty} dt \iint_S G_{ni}^{\text{free}}(\mathbf{x}, t-\tau; \xi, 0) T_i(\mathbf{u}(\xi, \tau), \mathbf{n})\, dS$$