Representation of a Faulting Source

1. Uniqueness theorem
2. Proof of Betti's theorem
3. Internal surface representation theorems
4. Body-force Equivalents

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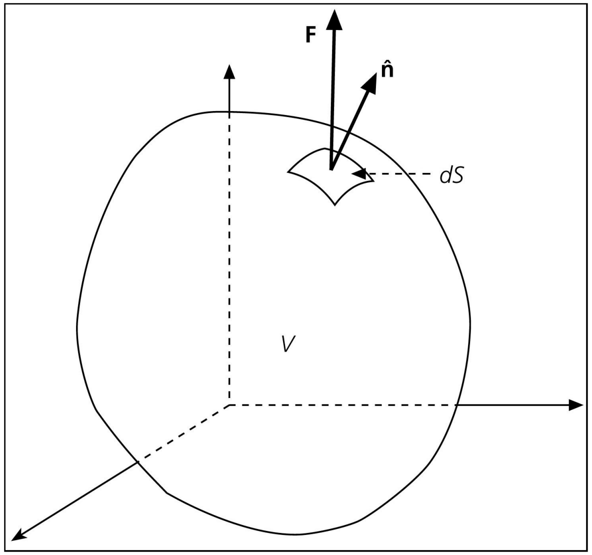
Uniqueness Theorem

The displacement field u(x,t) throughout the volume V with surface S is uniquely determined after time t_0 by the initial displacement and velocity values at t_0 throughout V (initial conditions) and by values at all times $t > t_0$ of:

1. The body forces f and heat supplied throughout V

1. The tractions T over any part S_1 of S, and

2. The displacement over the remainder S_2 of S, with $S=S_1+S_2$.



Reciprocity Theorem

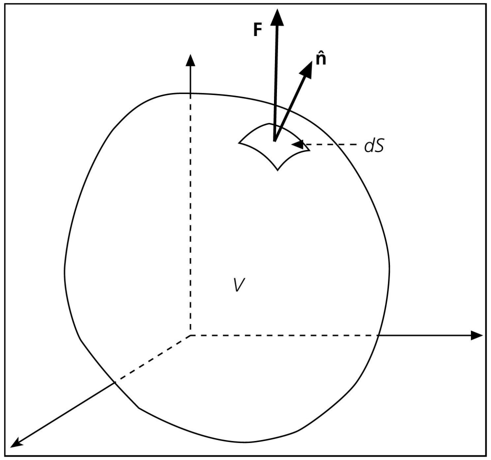
(Betti's Theorem)

If the displacement field u due to a body force f, boundary conditions on S and initial conditions at t_0 induces traction T(u,n), and the displacement field v due to a body force g, boundary conditions on S and initial conditions at t_0 induces traction T(v,n), thus the following scalar equality holds:

$$\iiint_V (\mathbf{f} - \rho\ddot{\mathbf{u}}) \cdot \mathbf{v}\, dV + \iint_S \mathbf{T}(\mathbf{u},\mathbf{n}) \cdot \mathbf{v}\, dS = \iiint_V (\mathbf{g} - \rho\ddot{\mathbf{v}}) \cdot \mathbf{u}\, dV + \iint_S \mathbf{T}(\mathbf{v},\mathbf{n}) \cdot \mathbf{u}\, dS$$

$$\mathbf{T} = \sigma\hat{\mathbf{n}}$$

Cauchy-Descartes relationship



Betti's Theorem
(Proof)

The reciprocity relationship

$$\int_V (\mathbf{f} - \rho\,\ddot{\mathbf{u}}) \cdot \mathbf{v}\,\mathrm{d}V + \int_S \mathbf{v} \cdot \boldsymbol{\tau} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_V (\mathbf{g} - \rho\,\ddot{\mathbf{v}}) \cdot \mathbf{u}\,\mathrm{d}V + \int_S \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\,\mathrm{d}S,$$

may be written in the index form:

$$\int_V (f_i - \rho\,\ddot{u}_i)\, v_i\,\mathrm{d}V + \int_S v_i\,\tau_{ij}\,\hat{n}_j\,\mathrm{d}S = \int_V (g_i - \rho\,\ddot{v}_i)\, u_i\,\mathrm{d}V + \int_S u_i\,\sigma_{ij}\,\hat{n}_j\,\mathrm{d}S.$$

Since

$$\int_S \mathbf{v} \cdot \boldsymbol{\tau} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_V \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})\,\mathrm{d}V.$$

(Divergence theorem)

Betti's theorem takes the form:

$$\int_V [(\mathbf{f} - \rho\,\ddot{\mathbf{u}}) \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})]\,\mathrm{d}V = \int_V [(\mathbf{g} - \rho\,\ddot{\mathbf{v}}) \cdot \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \boldsymbol{\sigma})]\,\mathrm{d}V.$$

Betti’s Theorem
(Proof)

Betti’s theorem (repeated):

$$\int_V [(\mathbf{f} - \rho\, \ddot{\mathbf{u}}) \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})]\; \mathrm{d}V = \int_V [(\mathbf{g} - \rho\, \ddot{\mathbf{v}}) \cdot \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \boldsymbol{\sigma})]\; \mathrm{d}V.$$

From the equation of motion $\rho \frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{f} - \nabla \cdot \boldsymbol{\tau} = \mathbf{0}.$

we replace $\mathbf{f} - \rho\, \ddot{\mathbf{u}}$ with $-\nabla \cdot \boldsymbol{\tau}$ in the left-hand side, and likewise $\mathbf{g} - \rho\, \ddot{\mathbf{v}}$ with $-\nabla \cdot \boldsymbol{\sigma}$ in the right-hand side, leading to

$$\int_V [(\nabla \cdot \boldsymbol{\tau}) \cdot \mathbf{v} - \nabla \cdot (\mathbf{v} \cdot \boldsymbol{\tau})]\; \mathrm{d}V = \int_V [(\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} - \nabla \cdot (\mathbf{u} \cdot \boldsymbol{\sigma})]\; \mathrm{d}V.$$

Betti's Theorem
(Proof)

Which in index form becomes

$$\int_V \left[\frac{\partial \tau_{ij}}{\partial x_i} v_j - \frac{\partial}{\partial x_j}(v_i \tau_{ij})\right] \mathrm{d}V = \int_V \left[\frac{\partial \sigma_{ij}}{\partial x_i} u_j - \frac{\partial}{\partial x_j}(u_i \sigma_{ij})\right] \mathrm{d}V.$$

By definition, the stress tensor is symmetric so that

$$\frac{\partial}{\partial x_j}(v_i \tau_{ij}) = \frac{\partial v_i}{\partial x_j}\tau_{ij} + v_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial v_i}{\partial x_j}\tau_{ij} + v_j \frac{\partial \tau_{ij}}{\partial x_i},$$

By substituting the corresponding term into both equation terms, we finally get

$$\frac{\partial v_i}{\partial x_j}\tau_{ij} = \frac{\partial u_i}{\partial x_j}\sigma_{ij}.$$

Since $\tau_{ij} = c_{ijkl}\frac{\partial u_l}{\partial x_k}$ and $\sigma_{ij} = c_{ijkl}\frac{\partial v_l}{\partial x_k}$

and thanks to the symmetry of the elastic moduli tensor, the equality is verified and Betti's theorem is proved.

Representation Theorem Recall

Let us take Betti's theorem in its integrated form, where displacements u and v are everywhere zero throughout V before a given time t_0 (repeated):

$$\int_{-\infty}^{\infty} dt \iiint_V \{\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, \tau - t) - \mathbf{v}(\mathbf{x}, \tau - t) \cdot \mathbf{f}(\mathbf{x}, t)\}\, dV$$
$$= \int_{-\infty}^{\infty} dt \iint_S \{\mathbf{v}(\mathbf{x}, \tau - t) \cdot \mathbf{T}(\mathbf{u}(\mathbf{x}, t), \mathbf{n}) - \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{T}(\mathbf{v}(\mathbf{x}, \tau - t), \mathbf{n})\}\, dS.$$

Substituting into this theorem the body force $g_i = \delta_{in}\, \delta(x-\xi)\, \delta(t)$ for which the corresponding Green function is $v_i = G_{in}(x,t\,;\,\xi,0)$ we obtain a formula for $u_n(x,t)$, the first representation theorem (repeated):

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_i(\xi, \tau) G_{in}(\xi, t - \tau; \mathbf{x}, 0)\, dV(\xi)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_S \{G_{in}(\xi, t - \tau; \mathbf{x}, 0) T_i(\mathbf{u}(\xi, \tau), \mathbf{n})$$
$$- u_i(\xi, \tau) c_{ijkl} n_j G_{kn,l}(\xi, t - \tau; \mathbf{x}, 0)\}\, dS(\xi).$$

Representation Theorem for an Internal Surface

Internal seismic sources, like earthquakes and underground explosions, may be of two kinds: 1) Faulting sources (e.g., slip across a fault), or 2) Volumetric sources (e.g., sudden expansion).

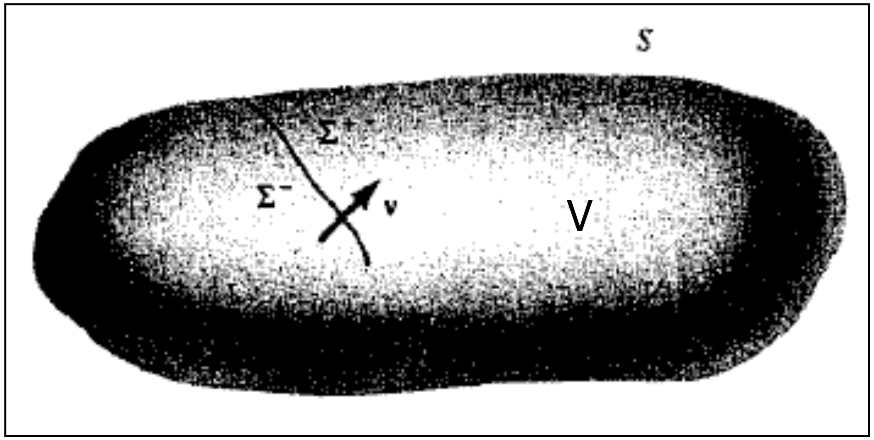
These sources may be mathematically described by considering either body forces applied to certain elements of the volume; or discontinuities in displacement or strain across a particular surface.

The second approach may be incorporated into the first throughout body force equivalents to discontinuities in displacement and strain.

The surface of V consists of an external surface S and two adjacent internal surfaces Σ^+ and Σ^- representing the fault.

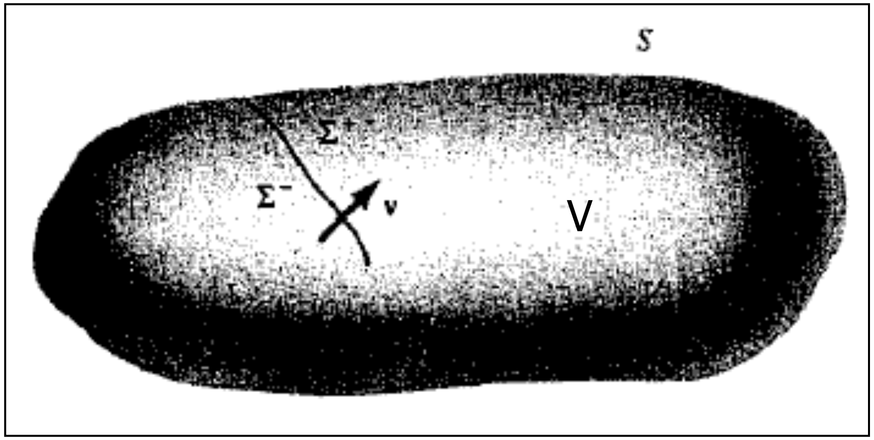
If slip occurs across Σ, then the displacement is discontinuous there and the equation of motion is only satisfied in the interior of surface $S + \Sigma^+ + \Sigma^-$

Volume V with internal surface Σ



Representation Theorem for an Internal Surface

Volume V with internal surface Σ



Slip across Σ implies that displacements on the Σ^+ side of Σ differ from those on the Σ^- side of Σ.

Such displacement discontinuity (i.e. slip) is denoted by $[u(\xi,\tau)]$ for ξ on Σ, and refers to the difference $u(\xi,\tau)|_{\Sigma+} - u(\xi,\tau)|_{\Sigma-}$

Then, in virtue of the Green's function spatial reciprocity, our first representation theorem far from S (i.e., homogeneous conditions for u and G over S) may be written in index form as:

$$u_n(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta,\tau) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta)$$

$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \left\{ \left[u_i(\xi,\tau) c_{ijpq} \nu_j \partial G_{np}(\mathbf{x}, t-\tau; \xi, 0)/\partial \xi_q \right] \right.$$

$$\left. - \left[G_{np}(\mathbf{x}, t-\tau; \xi, 0) T_p(\mathbf{u}(\xi,\tau), \nu) \right] \right\} d\Sigma.$$

Representation Theorem for an Internal Surface

Without assuming any boundary condition on Σ, our first representation theorem including an internal surface reads (repeated):

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\boldsymbol{\eta}, \tau) G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\eta}, 0)\, dV(\boldsymbol{\eta})$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ \Big[u_i(\boldsymbol{\xi}, \tau) c_{ijpq} \nu_j \partial G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\xi}, 0)/\partial \xi_q \Big]$$
$$- \Big[G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\xi}, 0) T_p(\mathbf{u}(\boldsymbol{\xi}, \tau), \boldsymbol{\nu}) \Big] \Big\}\, d\Sigma.$$

Suppose that Σ is transparent to G (i.e., G satisfies the equation of motion everywhere and is continuous across Σ as well as its derivatives). In the absence of body forces for u, if slip arises across Σ then [u] is nonzero, and since tractions should be continuous across the fault when rupture happens (i.e. [T(u,ν)] = 0), then

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_\Sigma \big[u_i(\boldsymbol{\xi}, \tau) \big]\, c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\xi}, 0)\, d\Sigma.$$

Representation Theorem for an Internal Surface

Representation Theorem for a Faulting Source (repeated)

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} [u_i(\xi, \tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \xi, 0)\, d\Sigma.$$

This representation formula for displacements, which has been used by many seismologists to evaluate the wavefield radiated from earthquakes, has the following outstanding properties:

1. Slip in the fault $[u_n]$ is enough to determine displacements everywhere.
2. No boundary conditions on Σ are needed for the Green function G_{np}.
3. Fault motion, which may be intricate and may complicate the determination of the slip function $[u_n(\xi,t)]$, is completely independent of the Green function.

Body-Force Equivalents

Making no assumptions about [u] and [T(u,ν)] across Σ, we have

$$u_n(x,t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta,\tau) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ [u_i(\xi,\tau)]\, c_{ijpq} \nu_j G_{np,q}(\mathbf{x}, t-\tau; \xi, 0)$$
$$- \big[T_p(\mathbf{u}(\xi,\tau), \nu)\big] G_{np}(\mathbf{x}, t-\tau; \xi, 0) \Big\}\, d\Sigma(\xi).$$

Traction Discontinuity: The body-force distribution of a traction discontinuity across Σ is [T] δ(η-ξ) dΣ as η varies throughout V. Thus, the contribution to displacement of such a discontinuity is

$$\int_{-\infty}^{\infty} d\tau \iiint_V \left\{ -\iint_\Sigma \big[T_p(\mathbf{u}(\xi,\tau),\nu)\big]\, \delta(\eta-\xi)\, d\Sigma \right\} G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV.$$

From the representation theorem above we see that the body-force equivalent of a traction discontinuity on Σ is given by $\mathbf{f}^{[T]}$, where

$$\mathbf{f}^{[\mathbf{T}]}(\eta,\tau) = -\iint_\Sigma [\mathbf{T}(\mathbf{u}(\xi,\tau),\nu)]\, \delta(\eta-\xi)\, d\Sigma(\xi).$$

Body-Force Equivalents

$$u_n(x,t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta,\tau) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ [u_i(\xi,\tau)]\, c_{ijpq} \nu_j G_{np,q}(\mathbf{x}, t-\tau; \xi, 0)$$
$$- \big[T_p(\mathbf{u}(\xi,\tau), \nu)\big] G_{np}(\mathbf{x}, t-\tau; \xi, 0) \Big\}\ d\Sigma(\xi).$$

Displacement Discontinuity: We use the following property of the delta-function derivative to localize points of Σ within V:

$$\frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t-\tau; \xi, 0) = -\iiint_V \frac{\partial}{\partial \eta_q} \delta(\eta - \xi) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta),$$

so that the displacement discontinuity contributes the displacement with

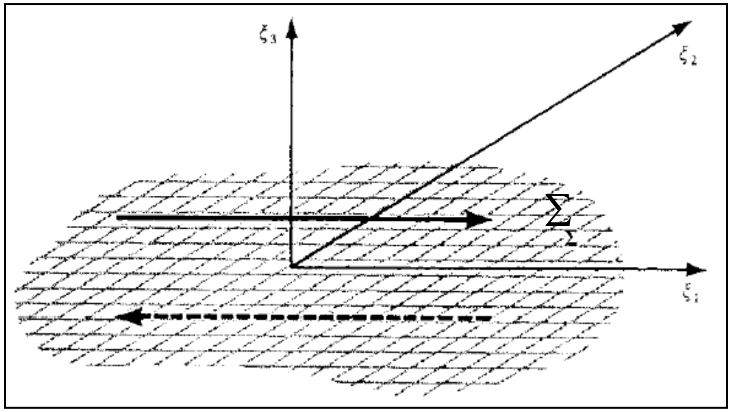
$$\int_{-\infty}^{\infty} d\tau \iiint_V \left\{ -\iint_\Sigma [u_i(\xi,\tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\Sigma \right\} G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV$$

Body-Force Equivalents

From the representation theorem before we see that the body-force equivalent of a displacement discontinuity on Σ is given by $f^{[u]}$, where

$$f_p^{[u]}(\eta, \tau) = -\iint_\Sigma [u_i(\xi, \tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\Sigma.$$

The seismic waves set up by fault slip are the same as those set up by a distribution of certain forces on the fault with canceling moment.



The body-force distribution is not unique but in a isotropic medium it can always be chosen as a surface distribution of double couples.

Homework:
Study and write a report of the paper:
Burridge and Knopof, BSSA, 1964