Body-Force Equivalents for Seismic Sources

1. Point source representation
2. Body force equivalents
3. Case of a buried fault
4. Seismic Moment

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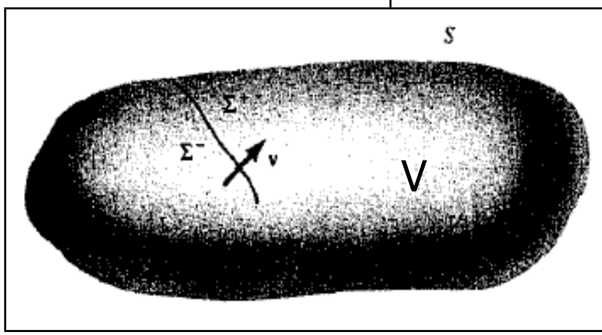
Representation Theorem for an Internal Surface

Without assuming any boundary condition on Σ, our first representation theorem reads (repeated):



$$u_n(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\boldsymbol{\eta},\tau)\, G_{np}(\mathbf{x}, t-\tau; \boldsymbol{\eta}, 0)\, dV(\boldsymbol{\eta})$$

$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \left\{ \left[u_i(\xi,\tau)\, c_{ijpq}\, \nu_j\, \partial G_{np}(\mathbf{x}, t-\tau; \xi, 0)/\partial \xi_q \right] \right.$$

$$\left. - \left[G_{np}(\mathbf{x}, t-\tau; \xi, 0)\, T_p(\mathbf{u}(\xi,\tau), \boldsymbol{\nu}) \right] \right\} d\Sigma.$$

Suppose that Σ is transparent to G (i.e. G satisfies the equation of motion everywhere and is continuous across Σ as well as its derivatives). In the absence of body forces for u, if slip arises across Σ then [u] is nonzero, and since tractions are continuous across the fault when rupture propagates spontaneously (i.e. [T(u,ν)] = 0), then

$$u_n(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \iint_\Sigma \left[u_i(\boldsymbol{\xi},\tau) \right] c_{ijpq}\, \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t-\tau; \boldsymbol{\xi}, 0)\, d\Sigma.$$

Representation Theorem for an Internal Surface

Representation Theorem for a Faulting Source (repeated)

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \left[u_i(\xi, \tau)\right] c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \xi, 0)\, d\Sigma.$$

This representation formula for displacements, which has been used by many seismologists to evaluate the wavefield radiated from earthquakes, has the following outstanding properties:

1. Slip in the fault $[u_n]$ is enough to determine displacements everywhere.

1. No boundary conditions on Σ are needed for the Green function G_{np}.

1. Fault motion, which may be extremely intricate and may complicate the determination of the slip function $[u_i(\xi,t)]$, is completely independent from the Green function determination.

Body-Force Equivalents

Making no assumptions about [u] and [T(u,ν)] across Σ, we have

$$u_n(x,t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\boldsymbol{\eta},\tau) G_{np}(\mathbf{x}, t-\tau; \boldsymbol{\eta}, 0)\, dV(\boldsymbol{\eta})$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ [u_i(\boldsymbol{\xi},\tau)]\, c_{ijpq} \nu_j G_{np,q}(\mathbf{x}, t-\tau; \boldsymbol{\xi}, 0)$$
$$- \big[T_p(\mathbf{u}(\boldsymbol{\xi},\tau), \boldsymbol{\nu})\big] G_{np}(\mathbf{x}, t-\tau; \boldsymbol{\xi}, 0) \Big\}\, d\Sigma(\boldsymbol{\xi}).$$

Traction Discontinuity: The body-force distribution of a traction discontinuity across Σ is [T] δ(η-ξ) dΣ as η varies throughout V. Thus, the contribution to displacement of such a discontinuity is

$$\int_{-\infty}^{\infty} d\tau \iiint_V \left\{ - \iint_\Sigma \big[T_p(\mathbf{u}(\boldsymbol{\xi},\tau), \boldsymbol{\nu})\big]\, \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, d\Sigma \right\} G_{np}(\mathbf{x}, t-\tau; \boldsymbol{\eta}, 0)\, dV.$$

From the representation theorem above we see that the body-force equivalent of a traction discontinuity on Σ is given by $\mathbf{f}^{[T]}$, where

$$\mathbf{f}^{[\mathbf{T}]}(\boldsymbol{\eta},\tau) = - \iint_\Sigma [\mathbf{T}(\mathbf{u}(\boldsymbol{\xi},\tau), \boldsymbol{\nu})]\, \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, d\Sigma(\boldsymbol{\xi}).$$

Body-Force Equivalents

$$u_n(x,t) = \int_{-\infty}^{\infty} d\tau \iiint_V f_p(\eta,\tau) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta)$$
$$+ \int_{-\infty}^{\infty} d\tau \iint_\Sigma \Big\{ [u_i(\xi,\tau)]\, c_{ijpq} \nu_j G_{np,q}(\mathbf{x}, t-\tau; \xi, 0)$$
$$- \big[T_p(\mathbf{u}(\xi,\tau), \nu)\big] G_{np}(\mathbf{x}, t-\tau; \xi, 0) \Big\}\, d\Sigma(\xi).$$

Displacement Discontinuity: We use the following property of the delta-function derivative to localize points of Σ within V:

$$\frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t-\tau; \xi, 0) = -\iiint_V \frac{\partial}{\partial \eta_q} \delta(\eta - \xi) G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV(\eta),$$

so that the displacement discontinuity contributes the displacement with

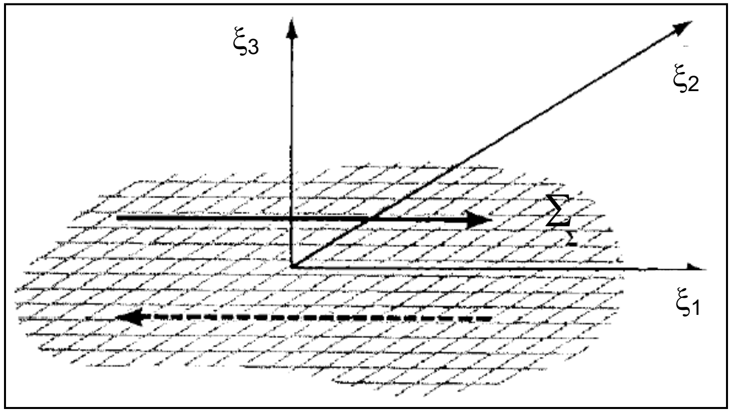
$$\int_{-\infty}^{\infty} d\tau \iiint_V \left\{ -\iint_\Sigma [u_i(\xi,\tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\eta - \xi)\, d\Sigma \right\} G_{np}(\mathbf{x}, t-\tau; \eta, 0)\, dV$$

Body-Force Equivalents
(Displacement Discontinuity)

From the representation theorem before, we see that the body-force equivalent of a displacement discontinuity on Σ is given by $f^{[u]}$, where

$$f_p^{[\mathrm{u}]}(\boldsymbol{\eta}, \tau) = -\iint_\Sigma [u_i(\boldsymbol{\xi}, \tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, d\Sigma.$$

The seismic waves set up by fault slip are the same as those set up by a distribution of certain forces on the fault with canceling moment and net force.



The body-force distribution is not unique but in an isotropic medium it can always be chosen as a surface distribution of double couples.

Assume that the fault Σ lies in the plane $\xi_3 = 0$ and that the slip [u] is parallel to the ξ_1-direction so that $[u_2] = [u_3] = 0$. Then the body force equivalent reduces to

$$f_p(\boldsymbol{\eta}, \tau) = -\iint_\Sigma [u_1(\boldsymbol{\xi}, \tau)]\, c_{13pq} \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, d\xi_1\, d\xi_2$$

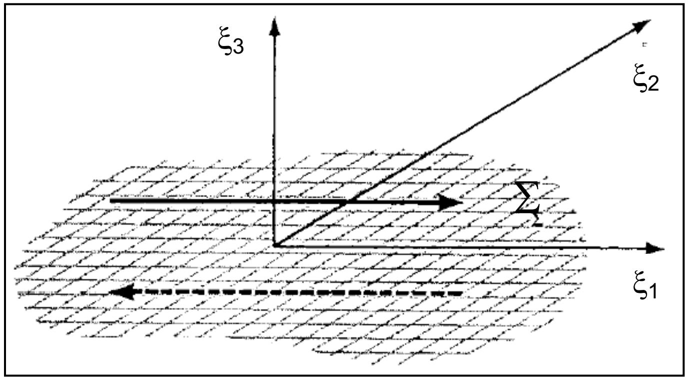
Body-Force Equivalents
(Displacement Discontinuity)

Body force equivalent to the displacement discontinuity shown in the figure below (repeated):

$$f_p(\boldsymbol{\eta}, \tau) = -\iint_\Sigma [u_1(\boldsymbol{\xi}, \tau)]\, c_{13pq} \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, d\xi_1\, d\xi_2$$

In isotropic heterogeneous media, from

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$



we find that all c_{13pq} vanish except c_{1313} = c_{1331} = μ. Hence the body-force equivalent distribution over Σ becomes:

$$f_1(\boldsymbol{\eta}, \tau) = -\iint_\Sigma \mu(\boldsymbol{\xi})\, [u_1(\boldsymbol{\xi}, \tau)]\, \delta(\eta_1 - \xi_1)\delta(\eta_2 - \xi_2) \frac{\partial}{\partial \eta_3} \delta(\eta_3)\, d\xi_1\, d\xi_2$$

$$f_2(\boldsymbol{\eta}, \tau) = 0,$$

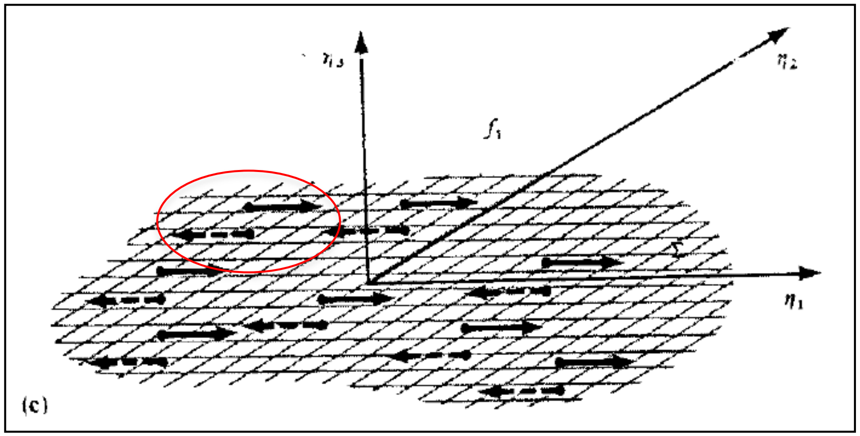
$$f_3(\boldsymbol{\eta}, \tau) = -\iint_\Sigma \mu\, [u_1] \frac{\partial}{\partial \eta_1} \delta(\eta_1 - \xi_1)\delta(\eta_2 - \xi_2)\delta(\eta_3)\, d\xi_1\, d\xi_2.$$

Force Equivalent for a Buried Fault

(Displacement Discontinuity)

Integrating the first force component we obtain

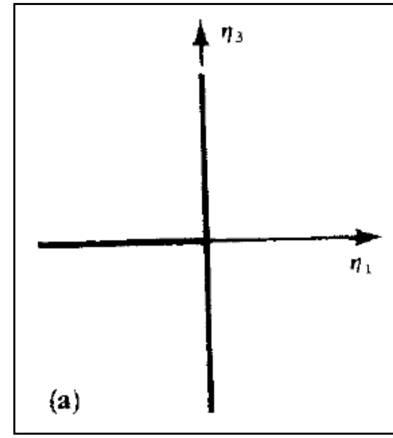
$$f_1(\eta,\tau) = -\mu(\eta)\,[u_1(\eta,\tau)]\,\frac{\partial}{\partial \eta_3}\delta(\eta_3).$$



The body force (f_1,0,0) is proportional to the derivative of the spike (Dirac) function. It thus represents a system of single couples of forces acting in the +/- η_1-direction with moment along the η_2-direction. The total moment is not zero and equal to

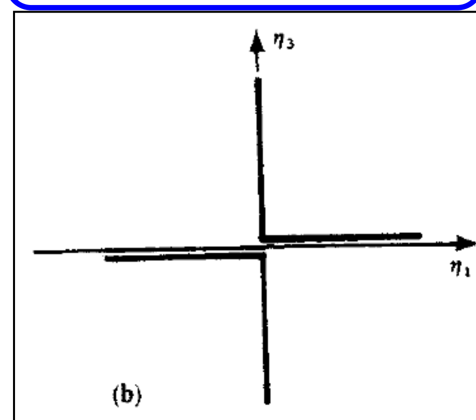
$$\iint_{\Sigma} \mu\,[u_1(\xi,\tau)]\; d\Sigma.$$

Spike as a function of η_3



$$(-\delta(\eta_3), 0, 0)$$

Spike derivative as a function of η_3



$$((-\partial/\partial\eta_3)\delta(\eta_3), 0, 0)$$

Force Equivalent for a Buried Fault

(Displacement Discontinuity)

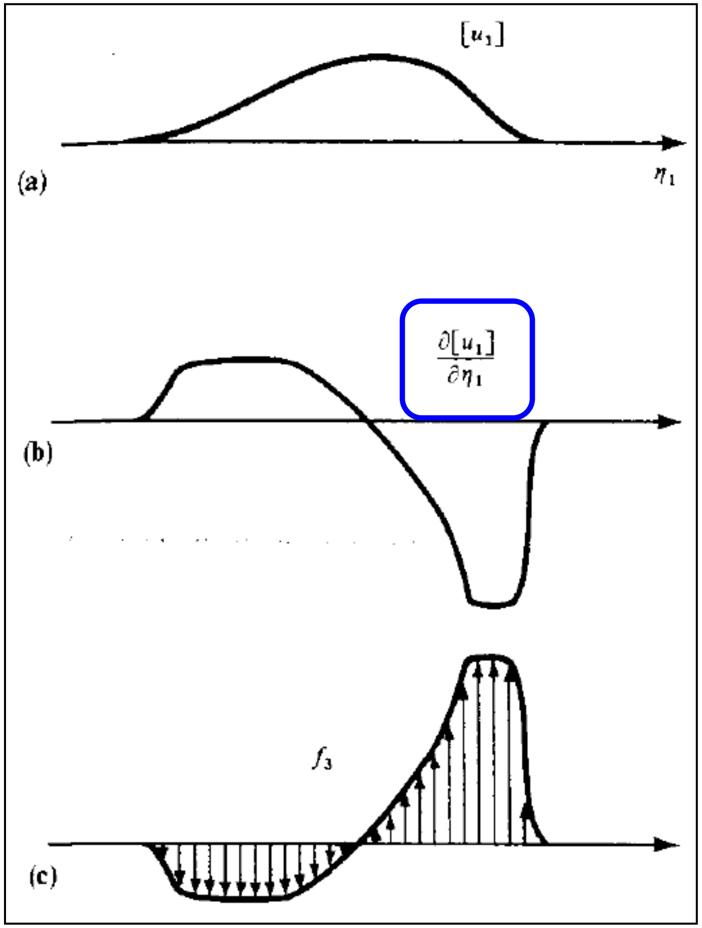
Using properties of the delta derivative, the third force component becomes

$$f_3(\eta, \tau) = -\frac{\partial}{\partial \eta_1} \{\mu [u_1(\eta, \tau)]\} \delta(\eta_3).$$

The body force $(0,0,f_3)$ is a single force proportional to the derivative of the slip function. It thus represents a force distribution in the +/- η_3-direction yielding a net couple with moment along the -η_2-direction. The total moment is not zero and equal to

$$-\iint_\Sigma \mu[u_1]\, d\xi_1\, d\xi_2.$$

which is the same, but with opposite sign, as the total moment due to the force equivalent f_1. It can be shown that such moment is given by $\mu \bar{u} A$ where A is the total fault area. Moments from both body-force equivalents cancel out.



Force Equivalent for a Buried Fault

(Displacement Discontinuity)

Body force equivalents to a given fault slip are not unique. The single force distribution f_3 shown in last figure is also equivalent to a distribution of single couples. To see this, take the representation

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} [u_i(\xi, \tau)]\, c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \xi, 0)\, d\Sigma.$$

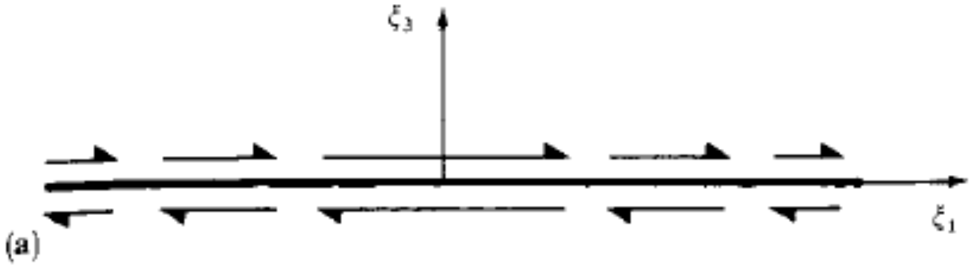
and particularize it for the chosen fault plane and slip direction:

$$u_n(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} \mu\, [u_1] \left\{ \frac{\partial G_{n1}}{\partial \xi_3} + \frac{\partial G_{n3}}{\partial \xi_1} \right\} d\Sigma.$$

The first term in the curl brackets is the limit

$$\frac{G_{n1}(\mathbf{x}, t - \tau; \xi + \varepsilon\hat{\xi}_3, 0) - G_{n1}(\mathbf{x}, t - \tau; \xi - \varepsilon\hat{\xi}_3, 0)}{2\varepsilon}$$

as ε --> 0. This is the following single-couple distribution (notice the moment units, Nw*m, of the equation):



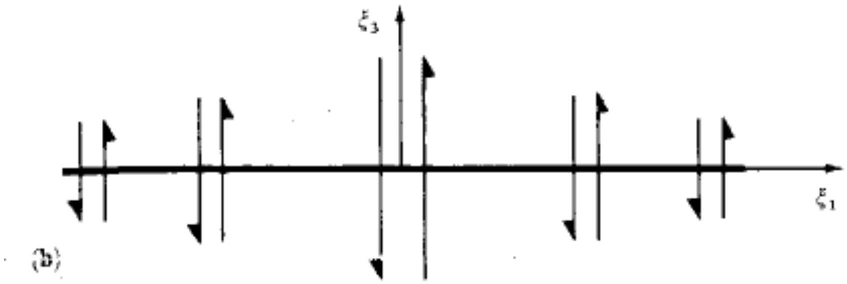
Force Equivalent for a Buried Fault

(Displacement Discontinuity)

The second term involves the limit

$$\frac{G_{n3}(\mathbf{x}, t-\tau; \xi + \varepsilon\hat{\xi}_1, 0) - G_{n3}(\mathbf{x}, t-\tau; \xi - \varepsilon\hat{\xi}_1, 0)}{2\varepsilon},$$

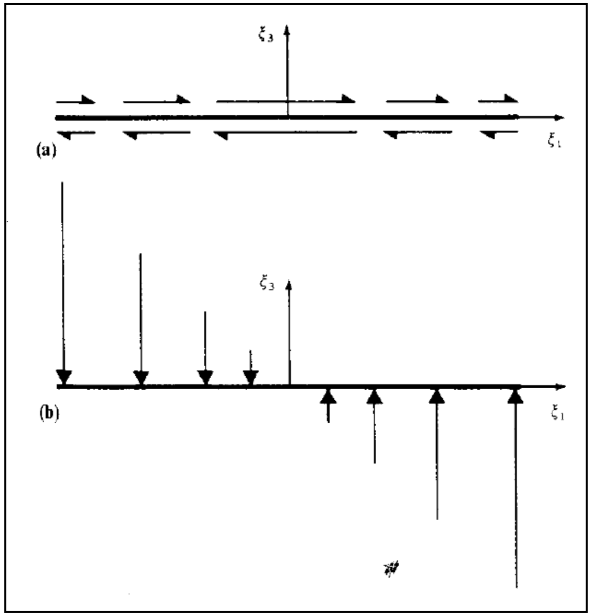
as ε --> 0, which represents the following single-couple distribution:



Force Equivalent for a Buried Fault

The complete body-force equivalent to fault slip consists of two parts with both canceling moment and net force:

1. Single couples (f_1) made up of forces pointing in the fault slip direction, and
2. A distribution of a fault-normal single forces over Σ (f_3) with total moment cancelling the one due to f_1.



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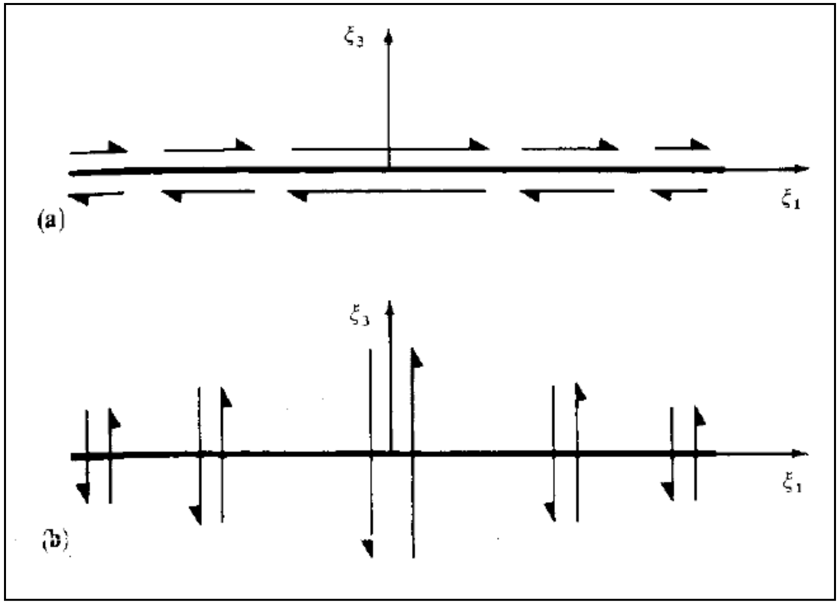


The radiation from these two distributions is the same as the radiation from slip on the fault. In this sense, these two single-couple distributions, taken together, are equivalent to fault slip.

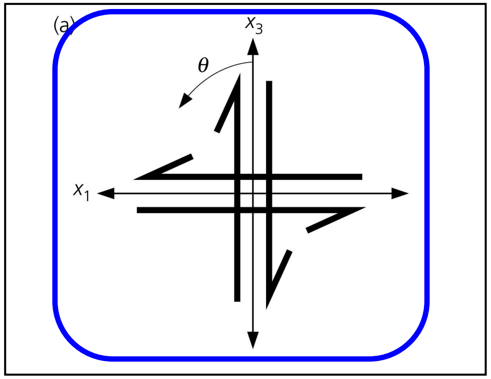
Force Equivalent for a Buried Fault

The complete body-force equivalent to fault slip consists of two parts with both canceling moment and net force:

1. Single couples (f_1) made up of forces pointing in the fault slip direction, and
2. A distribution of a fault-normal single forces (f_3) over Σ with total moment cancelling the one due to f_1.

The absolute value of the total moment associated with each contribution is equal to $\mu \bar{u} A$

Double Couple: Point dislocation body-force equivalent



At great distances from the fault, wavelengths of seismic waves are much greater than the linear dimension of Σ, and their periods much longer than the source duration. The slip thus becomes localized in space and time $\bar{\mathbf{u}} A \delta(\xi_1)\delta(\xi_2) H(\tau)$ and then:

$$f_1(\eta, \tau) = -M_0 \delta(\eta_1)\delta(\eta_2) \frac{\partial}{\partial \eta_3}\delta(\eta_3) H(\tau)$$

$$f_3(\eta, \tau) = -M_0 \frac{\partial}{\partial \eta_1}\delta(\eta_1)\delta(\eta_2)\delta(\eta_3) H(\tau)$$

where M_0 is called the seismic moment:

$$M_0 = \mu \bar{u} A = \mu \times \text{ average slip } \times \text{ fault area.}$$