Solution for the Elastodynamic Green Function

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Solution for the Elastodynamic Green Function

Let us derive our first solution for the elastodynamic Green function. Recall that this function represents the displacement field u(x,t) due to a body force f(x,t) applied impulsively (i.e., a spike), in space and time, to a given particle at position x = ξ and time t = τ.

The equation of motion in an isotropic medium may be expressed in terms of u(x,t) as: $\rho \ddot{u}_i = f_i + (\lambda + \mu) u_{j,ji} + \mu u_{i,jj}$

which in vectorial notation reads:

$$\rho \ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})$$

Helmholts Potentials:

For the vector field $\mathbf{Z} = \mathbf{Z}(\mathbf{x})$ there always are Helmholts potentials X and $\mathbf{Y}$ such that $\mathbf{Z} = \nabla X + \nabla \times \mathbf{Y}$ with $\nabla \cdot \mathbf{Y} = 0$. Given $\mathbf{Z}$, to construct X and $\mathbf{Y}$ it is enough to solve the vector Poisson equation $\nabla^2 \mathbf{W} = \mathbf{Z}$ so that thanks to the identity $\nabla^2 \mathbf{W} = \nabla(\nabla \cdot \mathbf{W}) - \nabla \times (\nabla \times \mathbf{W})$ we can choose potentials $X = \nabla \cdot \mathbf{W}$ and $\mathbf{Y} = -\nabla \times \mathbf{W}$. The solution for the Poisson equation is:

$$\mathbf{W}(\mathbf{x}) = -\iiint_V \frac{\mathbf{Z}(\xi)}{4\pi |\mathbf{x} - \xi|} dV(\xi).$$

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Lamé's Theorem:

If the displacement u=u(x,t) satisfies the equation of motion

$$\rho\ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u})$$

and if the body-force, the initial values for u and its time derivative are expressed in terms of the Helmholtz potentials via

$$\mathbf{f} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}; \qquad \dot{\mathbf{u}}(\mathbf{x},0) = \nabla A + \nabla \times \mathbf{B}; \qquad \mathbf{u}(\mathbf{x},0) = \nabla C + \nabla \times \mathbf{D},$$

with $\nabla \cdot \boldsymbol{\Psi}, \quad \nabla \cdot \mathbf{B}, \quad \nabla \cdot \mathbf{D} \quad$ all zero,

there exist potentials ϕ and ψ for u with all of the following four properties:

$$\mathbf{u} = \nabla\phi + \nabla \times \psi \qquad \nabla \cdot \psi = 0,$$

$$\ddot{\phi} = \frac{\Phi}{\rho} + \alpha^2\nabla^2\phi \qquad \left(\text{with } \alpha^2 = \frac{\lambda + 2\mu}{\rho}\right),$$ Wave equation for ϕ

$$\ddot{\psi} = \frac{\Psi}{\rho} + \beta^2\nabla^2\psi \qquad \left(\text{with } \beta^2 = \frac{\mu}{\rho}\right)$$ Wave equation for ψ

Solution for the Elastodynamic Green Function

The first step is to find potentials Φ and Ψ for the body-force f_i applied in the x_1-direction such that

$$X_0(t)\,\delta(\mathbf{x})\,\hat{\mathbf{x}}_1 = \mathbf{f} = \nabla\Phi + \nabla \times \Psi \qquad \text{and} \qquad \nabla \cdot \Psi = 0.$$

Since these are Helmholtz potentials for f(x,t), they may be constructed from

$$\mathbf{W} = -\frac{X_0(t)}{4\pi}\iiint_V (1,0,0)\frac{\delta(\xi)\,dV}{|\mathbf{x}-\xi|} = -\frac{X_0(t)}{4\pi|\mathbf{x}|}\hat{\mathbf{x}}_1,$$

in the following manner:

$$\Phi(\mathbf{x},t) = \nabla \cdot \mathbf{W} = -\frac{X_0(t)}{4\pi}\frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|}$$

$$\Psi(\mathbf{x},t) = -\nabla \times \mathbf{W} = \frac{X_0(t)}{4\pi}\left(0, \frac{\partial}{\partial x_3}\frac{1}{|\mathbf{x}|}, -\frac{\partial}{\partial x_2}\frac{1}{|\mathbf{x}|}\right).$$

Solution for the Elastodynamic Green Function

The second step in finding displacements (i.e., the Green function) is to solve the wave equations for the Lamé's potentials ϕ and ψ which, after substitution of the body-force potentials we obtained, are given by:

$$\ddot{\phi} = -\frac{X_0(t)}{4\pi\rho}\frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|} + \alpha^2\nabla^2\phi$$ (wave equation for ϕ)

and

$$\ddot{\psi} = \frac{X_0(t)}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\frac{1}{|\mathbf{x}|}, -\frac{\partial}{\partial x_2}\frac{1}{|\mathbf{x}|}\right) + \beta^2\nabla^2\psi.$$ (wave equation for ψ)

Solutions for these equations are, respectively (see book eq. 4.5 and 4.6):

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi\rho}\left(\frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\alpha}\tau X_0(t-\tau)\,d\tau$$

$$\psi(\mathbf{x}, t) = -\frac{1}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\frac{1}{|\mathbf{x}|}, -\frac{\partial}{\partial x_2}\frac{1}{|\mathbf{x}|}\right)\int_0^{|\mathbf{x}|/\beta}\tau X_0(t-\tau)\,d\tau.$$

Solution for the Elastodynamic Green Function

The third and final step in finding displacements due to body-force $X_0(t)$ (i.e. the Green function) applied at the origin in the x_1-direction is forming such a field u_i by using its first property settled by Lamé's theorem:

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}$$

so that, after substitution of potentials, we obtain the representation:

$$u_i(\mathbf{x}, t) = \frac{1}{4\pi\rho}\left(\frac{\partial^2}{\partial x_i \partial x_1}\frac{1}{r}\right)\int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau)\,d\tau$$

$$+\frac{1}{4\pi\rho\alpha^2 r}\left(\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)X_0\left(t-\frac{r}{\alpha}\right)+\frac{1}{4\pi\rho\beta^2 r}\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)X_0\left(t-\frac{r}{\beta}\right).$$

where $r = |\mathbf{x}|$.

To generalize this formula for any direction x_j of the body-force $X_0(t)$, we use the direction cosines γ_i for the vector x_i, so that $\gamma_i = x_i/r = \partial r/\partial x_i$ and substitute 1 by j.

$$\frac{\partial^2}{\partial x_i\,\partial x_j}\frac{1}{r} = \frac{3\gamma_i\gamma_j - \delta_{ij}}{r^3}.$$

Solution for the Elastodynamic Green Function

To generalize this formula for any direction x_j of the body-force $X_0(t)$, we use the direction cosines γ_i for the vector x_i, so that $\gamma_i = x_i / r = \partial r / \partial x_i$ and substitute 1 by j to derive the general representation of displacement

$$u_i(\mathbf{x}, t) = \frac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\frac{1}{r^3}\int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau)\,d\tau + \frac{1}{4\pi\rho\alpha^2}\gamma_i\gamma_j\frac{1}{r}X_0\left(t-\frac{r}{\alpha}\right) - \frac{1}{4\pi\rho\beta^2}\left(\gamma_i\gamma_j - \delta_{ij}\right)\frac{1}{r}X_0\left(t-\frac{r}{\beta}\right).$$

that we know, thanks to Betti's theorem, corresponds to $u_i(\mathbf{x}, t) = X_0 * G_{ij}$.

An equivalent formula for displacements was first found by Stokes in 1849. It represents one of the most important solutions in elastic wave radiation and we next examine its main properties:

- The amplitude of different terms depends on distance r between source and receiver.
- For small r, the term dominating the ground motion behaves as $1/r^2$ and is called the near-field term.
- For larger r, both terms decreasing as 1/r dominate the ground motion so they are called the far-field terms.

Solution for the Elastodynamic Green Function

Properties of the Far-Field P-wave

We introduce the far-field P-wave, which from our Green function representation before has the displacement u_i^P given by

$$u_i^P(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha^2}\gamma_i\gamma_j\frac{1}{r}X_0\left(t - \frac{r}{\alpha}\right).$$

Along a given direction γ_i from the source, which is a point force applied in the *j* direction, this wave

1. attenuates as 1/r ;
2. travels with the P-wave speed α so that its arrival time is given by $t - r / \alpha$;
3. has a waveform that is proportional to the applied force at retarded time; and
4. has a direction of displacement at x_i that is parallel to the direction γ_i from the source (i.e. radial movement).

Homework: Demonstrate that director cosines determine displacements in the radial and transverse direction for the P and S waves respectively.

Solution for the Elastodynamic Green Function

Properties of the Far-Field S-wave

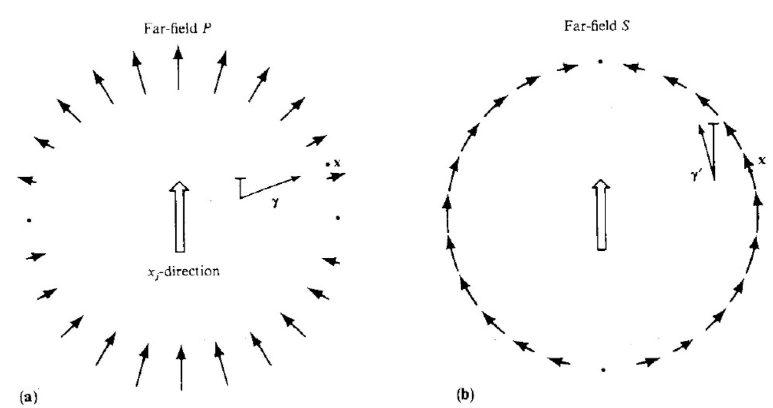
We introduce the far-field S-wave, which has the displacement u_i^S given by

$$u_i^S(x,t) = \frac{1}{4\pi\rho\beta^2}(\delta_{ij} - \gamma_i\gamma_j)X_0\left(t - \frac{r}{\beta}\right)\frac{1}{r}$$

Along a given direction γ_i from the source, this wave

1. attenuates as 1/r ;
2. travels with the S-wave speed β since its arrival time is given by $t - r/\beta$;
3. has a waveform that is proportional to the applied force at retarded time; and
4. has a direction of displacement at x_i that is perpendicular to the direction γ_i from the source (i.e. transverse movement).

Radiation patterns and amplitudes for displacements u_i^P and u_i^S associated with both the P- and S-waves excited by the force $X_0(t)$.



Solution for the Elastodynamic Green Function

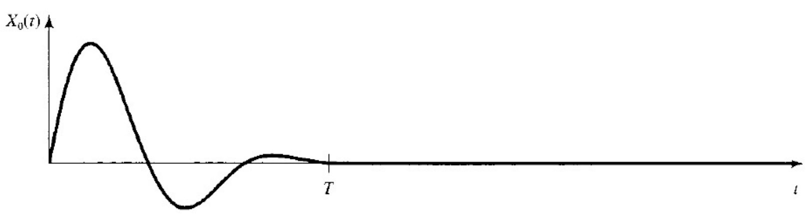
Properties of the Near-Field Term

We define the near-field displacement u_i^N , which is given by

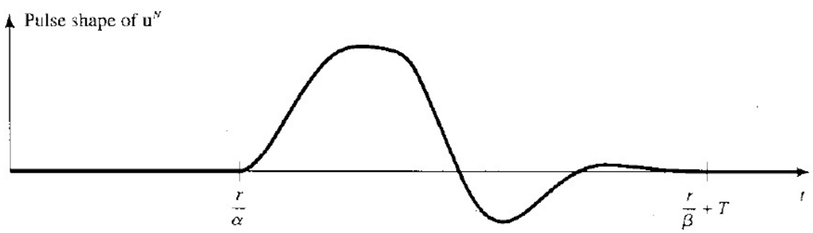
$$u_i^N(\mathbf{x}, t) = \frac{1}{4\pi\rho}(3\gamma_i\gamma_j - \delta_{ij})\frac{1}{r^3}\int_{r/\alpha}^{r/\beta} \tau X_0(t-\tau)\, d\tau.$$

Along a given direction γ_i from the source, this wave has contributions from both the P- and S-waves that are difficult to separate. The near-field is nonzero only between r/α and r/β, the arrival time of the P- and S-waves respectively.

Body-force time function $X_0(t)$



Near-field pulse shape $\mathbf{u}^N$

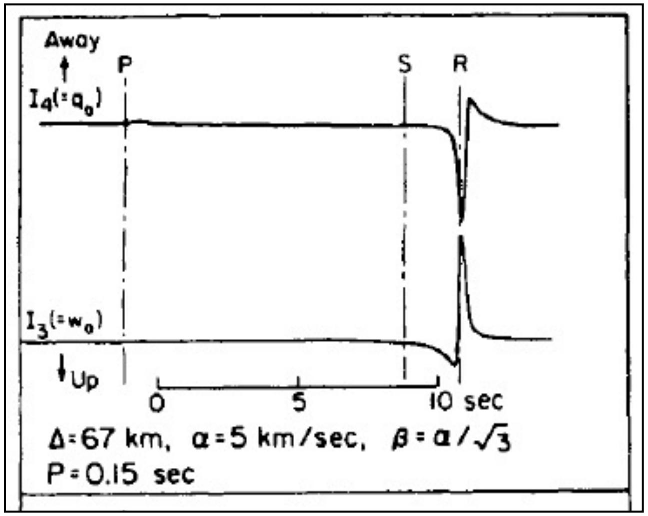


Point-Force Sources in Nature

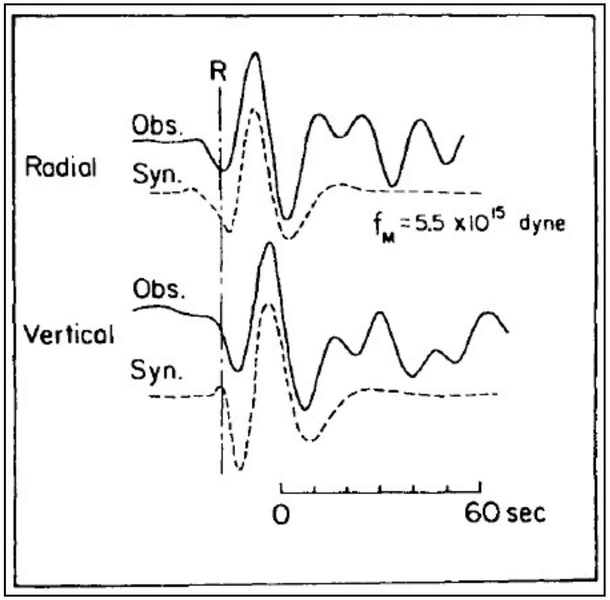
Several natural phenomena have been found to be well explained by single body-force sources, on the basis of observed and compatible radiation patterns. Among them there are some volcanic explosions and landslides.

The wavefield excited by some volcanic explosions essentially corresponds to that excited by a body-force applied to the Earth's surface. The solution for this problem was first found by Lamb (1904) and consists, within a homogeneous halfspace, in two body waves (P and S-waves) and a Rayleigh pulse.

The Lamb's Pulse



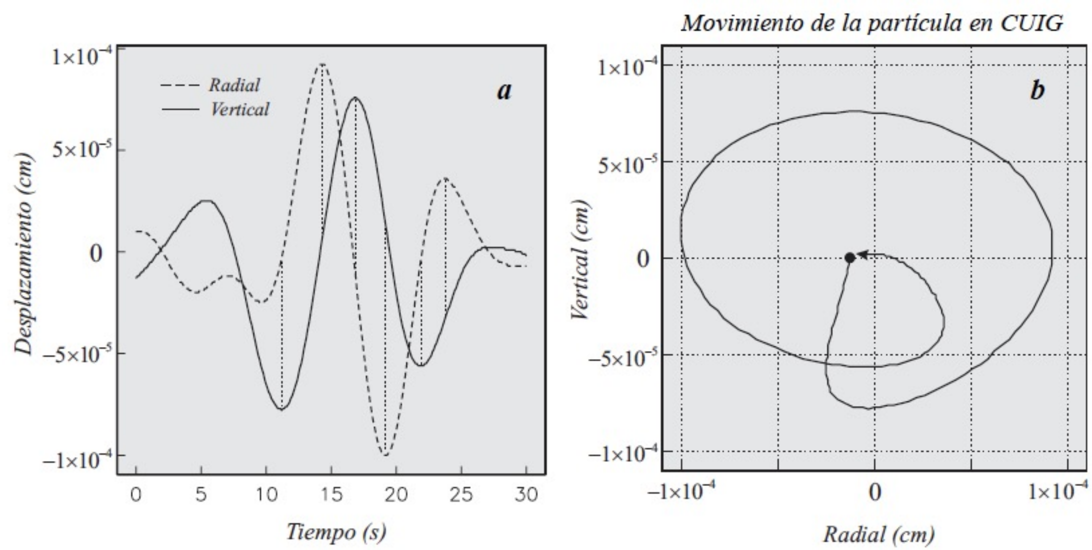
Observed seismograms due to the Mont St. Helens eruption (1980) (solid) and synthetics computed for a vertical body-force (dash). The strength of the explosion was estimated to be of 5.5×10^{15} dyne (Kanamori and Given, 1983)



Point-Force Sources in Nature

The Popocatépetl volcano, Mexico, experience regular explosions where ejecta is suddenly released. The dynamic reaction may be modeled as a body-force applied to the ground surface.

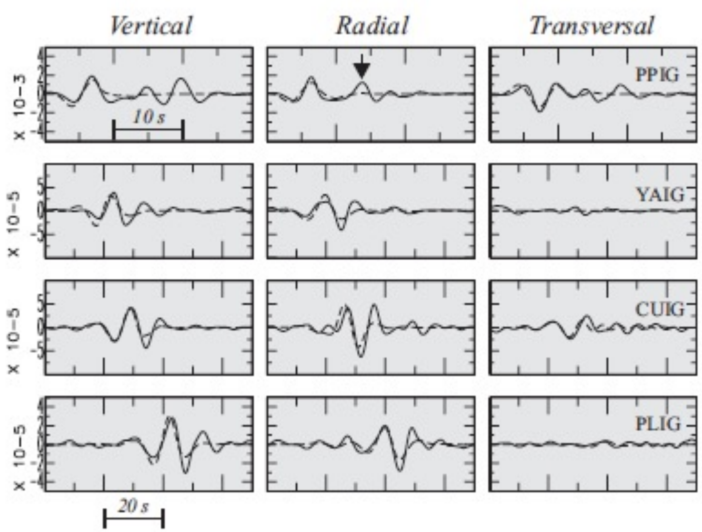
Radial and vertical observed seismograms in Mexico City due to a Popocatépetl explosion (1997). The odogram clearly shows that recorded seismograms correspond to a Rayleigh (Lamb's) pulse (Cruz-Atienza, MSc Thesis,2001)

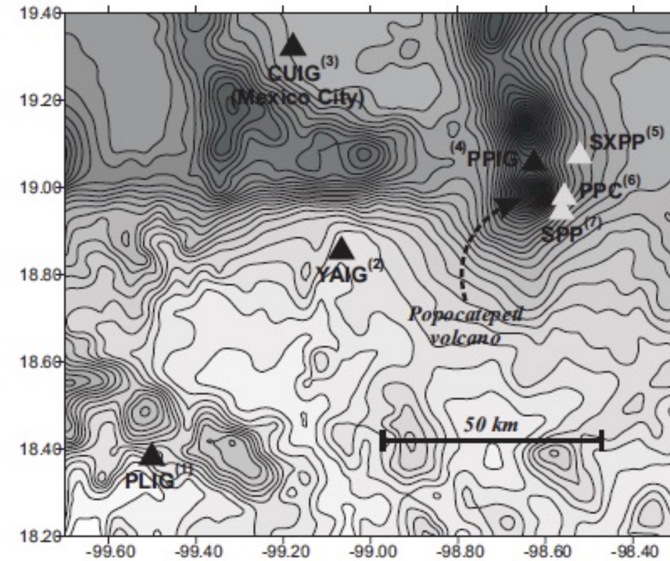


Point-Force Sources in Nature

Broadband seismic stations around the Popocatépetl volcano that recorded the seismic wavefield excited by several explosions from 1997-2000.

Wavefield Modelling





Linear relationship between the force magnitude and source duration that leads to a magnitude scale in terms of the impulse K (Cruz-Atienza et al., GRL, 2001)

$$M_k = \frac{2}{3}\log K - 4.71.$$