Bulletin of the Seismological Society of America. Vol. 54, No. 6, pp. 1875–1888. December, 1964

BODY FORCE EQUIVALENTS FOR SEISMIC DISLOCATIONS

By R. Burridge and L. Knopoff

ABSTRACT

An explicit expression is derived for the body force to be applied in the absence of a dislocation, which produces radiation identical to that of the dislocation. This equivalent force depends only upon the source and the elastic properties of the medium in the immediate vicinity of the source and not upon the proximity of any reflecting surfaces. The theory is developed for dislocations in an anisotropic inhomogeneous medium; in the examples isotropy is assumed. For displacement dislocation faults, the double couple is an exact equivalent body force.

1. Introduction

The interest in the "force equivalent" problem antedates its solution by a considerable number of years. The debates over "single couple" and "double couple" earthquake focal mechanisms have by now been well viewed and reviewed. For a discussion of the problem of earthquake source mechanisms see Stauder (1962).

It has been pointed out by several authors (Knopoff and Gilbert, 1960; Balakina, Shirokova, and Vvedenskaya, 1960) that the solution to the problem of the seismic radiation from a suddenly occurring earthquake in the earth's interior is likely to be connected with the solution to a "dislocation" problem, or, in the terminology of Baker and Copson (1950), to a "saltus" problem. In these problems the displacement field or stress field undergoes the more or less sudden creation of a discontinuity across the "fault" surface.

Perhaps the most complete solution to the dislocation problem has been given by Knopoff and Gilbert. These authors showed that a model of an earthquake can be represented as a linear combination of the solutions to a number of fundamental problems. The remaining problem, that of determining the appropriate linear combinations, was not attempted.

Although Knopoff and Gilbert were primarily interested in the radiation patterns from their various dislocation models they also provided the force equivalents that would produce the same first motions as each of their sources. In this paper we find the force equivalents for physically and mathematically reasonable dislocation models.

Nabarro (1951) obtained the radiation from a spreading dislocation by essentially the same method as Knopoff and Gilbert but in a less explicit form. He had in mind mainly metallurgical applications. Recently Maruyama (1963) has attempted to derive the force equivalents but his results are obscured by algebraic detail associated with the explicit expression of a certain Green's function.

In this paper we specifically exclude consideration of the radiation pattern. We are concerned with the body force which would have to be applied in the absence of the fault to produce the same radiation (in all respects, not only first motions) as a given dislocation. We find that the force equivalent depends only upon the source mechanism and the elastic properties of the medium in the immediate vicinity of the fault and not upon any reflecting surfaces or other inhomogeneities which may be present in the medium. The theory is developed for dislocations in an inhomogeneous, anisotropic medium although in the specific examples given at the end of the paper we have assumed isotropy.

We assume from the outset that the Green's function exists. This is physically reasonable since it is the response to an instantaneous body force concentrated at a point. We then derive a representation theorem in terms of this Green's function and interpret the surface integrals as representing certain surface distributions of body forces and "couples." Finally, specific examples are exhibited.

2. A Representation Theorem and Reciprocity Relation

The equations of motion for an inhomogeneous anisotropic elastic solid are

$$(c_{ijpq}(\mathbf{x})u_{p,q}(\mathbf{x}, t))_{,j} - \rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t) = -f_i(\mathbf{x}, t) \tag{1}$$

where $u_i(\mathbf{x}, t)$ is the i-component of the displacement vector, $c_{ijpq}(\mathbf{x})$ are the elastic constants of the medium and $f_i(\mathbf{x}, t)$ the i-component of the body force at the point $\mathbf{x} = (x_1, x_2, x_3)$ and time t. The coordinate system is rectangular cartesian. We have used the notation $F_{,i} = \partial F/\partial x_i$ and $\dot{F} = \partial F/\partial t$. The summation convention applies to letter subscripts. $c_{ijpq} = c_{jipq} = c_{pqij}$, and for an isotropic solid $c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$ where λ and μ are the Lamé constants of the medium and δ_{ij} the Kronecker delta.

Suppose $v_i(\mathbf{x}, t)$ is another motion due to body forces $g_i(\mathbf{x}, t)$. Then

$$(c_{ijpq}v_{p,q})_{,j} - \rho\ddot{v}_i = -g_i. \tag{2}$$

Equations (1) and (2) are satisfied in a volume V bounded by a surface S, and for all time. We assume that f_i and g_i vanish for $t < -T$, T a constant, and that u_i and v_i also vanish for $t < -T$. This last is a causality condition which guarantees that the disturbance does not start before the force which causes it.

On replacing t by $-t$ in equation (2) we have

$$[c_{ijpq}\bar{v}_{p,q}]_{,j} - \rho\ddot{\bar{v}}_i = -\bar{g}_i \tag{3}$$

where

$$\bar{v}_p(\mathbf{x}, t) = v_p(\mathbf{x}, -t), \bar{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}, -t)$$

and

$$\bar{v}_p = 0, t > T.$$

We now multiply (1) by $\bar{v}_i$, (3) by u_i, subtract, integrate over all of V, and further integrate with respect to t from $-\infty$ to ∞. This yields

$$\int_{-\infty}^{\infty} dt \int_V dV \Big\{(\bar{v}_i c_{ijpq} u_{p,q} - u_i c_{ijpq}\bar{v}_{p,q})_{,j} - \rho\frac{\partial}{\partial t}(\bar{v}_i\dot{u}_i - u_i\dot{\bar{v}}_i) - c_{ijpq}(\bar{v}_{i,j}u_{p,q} - u_{i,j}\bar{v}_{p,q})\Big\} = \int_{-\infty}^{\infty} dt \int_V dV(u_i\bar{g}_i - \bar{v}_i f_i). \tag{4}$$

The integrals over t are in fact finite since the integrands vanish outside the interval $(-T, T)$.

The last term on the left-hand side of (4) vanishes since $c_{ijpq} = c_{pqij}$. Since ρ is independent of t we may integrate the second term with respect to t and find that the result vanishes since u_i and $\dot{u}_i$ vanish for $t < -T$ and $\bar{v}_i$ and $\dot{\bar{v}}_i$ vanish for $t > T$. The first term on the left-hand side may be transformed by means of the divergence theorem so that we have the identity

$$\int_{-\infty}^{\infty} dt \int_V dV(u_i\bar{g}_i - \bar{v}_if_i) = \int_{-\infty}^{\infty} dt \int_S dSn_j(\bar{v}_ic_{ijpq}u_{p,q} - u_ic_{ijpq}\bar{v}_{p,q}), \qquad (5)$$

where n_i is an outward drawn unit vector normal to S.

If we now set $g_i(\mathbf{x}, t) = \delta_{ni}\,\delta(\mathbf{x}, t; \mathbf{y}, -s)$ where $\delta(\mathbf{x}, t; \mathbf{y}, -s) = \delta(x_1 - y_1)\delta(x_2 - y_2)\delta(x_3 - y_3)\delta(t + s)$, $\mathbf{y}$ is a point in V, and δ is the Dirac delta function, then $\bar{g}_i(\mathbf{x}, t) = \delta_{ni}\delta(\mathbf{x}, t; \mathbf{y}, s)$ and (5) becomes

$$\begin{aligned} u_n(\mathbf{y}, s) = & \int_{-\infty}^{\infty} dt \int_V G_{in}(\mathbf{x}, -t; \mathbf{y}, -s)f_i(\mathbf{x}, t)dV_x \\ & + \int_{-\infty}^{\infty} dt \int_S n_j\{G_{in}(\mathbf{x}, -t; \mathbf{y}, -s)c_{ijpq}(\mathbf{x})u_{p,q}(\mathbf{x}, t) \\ & - u_i(\mathbf{x}, t)c_{ijpq}(\mathbf{x})G_{pn,q}(\mathbf{x}, -t; \mathbf{y}, -s)\}\, dS_x, \end{aligned} \qquad (6)$$

where $G_{in}(\mathbf{x}; t; \mathbf{y}, s)$ is the displacement in the i-direction at $(\mathbf{x}, t)$ due to an instantaneous point force of unit impulse in the n-direction at $(\mathbf{y}, s)$.

If, in (6), u_i and G_{in} satisfy the same homogeneous boundary conditions on S then the surface integral will vanish. If, in addition, $f_i(\mathbf{x}, t) = \delta_{im}\delta(\mathbf{x}, t; \mathbf{y}', s')$, then (6) gives

$$G_{nm}(\mathbf{y}, s; \mathbf{y}', s') = G_{mn}(\mathbf{y}', -s'; \mathbf{y}, -s) \qquad (7)$$

where $G_{nm}(\mathbf{x}, t; \mathbf{y}', s')$ satisfies the same boundary conditions as $G_{mn}(\mathbf{x}, t; \mathbf{y}, s)$. Therefore (6) may be rewritten as

$$\begin{aligned} u_n(\mathbf{y}, s) = & \int_{-\infty}^{\infty} dt \int_V G_{ni}(\mathbf{y}, s; \mathbf{x}, t)f_i(\mathbf{x}, t)dV_x \\ & + \int_{-\infty}^{\infty} dt \int_S n_j\{G_{ni}(\mathbf{y}, s; \mathbf{x}, t)c_{ijpq}(\mathbf{x})u_{p,q}(\mathbf{x}, t) \\ & - u_i(\mathbf{x}, t)c_{ijpq}(\mathbf{x})G_{np,q'}(\mathbf{y}, s; \mathbf{x}, t)\}dS_x. \end{aligned} \qquad (8)$$

Here

$$G_{np,q'}(\mathbf{y}, s; \mathbf{x}, t) = \frac{\partial}{\partial x_q} G_{np}(\mathbf{y}, s; \mathbf{x}, t)$$

where q' indicates that the subscript refers to the second set of arguments of G_{np} and the summation convention still applies.

Equation (7) is the reciprocal theorem. It was examined by Knopoff and Gangi (1959). It is a special case of Helmholtz's reciprocal theorem in generalized mechanics (see Whittaker, 1944; Lamb, 1889). Equation (5) is Betti's reciprocal theorem applied to u_i and $\bar{v}_i$ and integrated with respect to t. (Love, 1944, pp. 173–174) Equation (8) is the representation theorem which we shall use in the next section. De Hoop (1958) derives a similar theorem for an isotropic homogeneous medium and our notation has followed his. The first representation theorem of this type was derived by Knopoff (1956).

Let it be assumed that we wish to find the radiation from prescribed discontinuities in the displacement and its derivatives across a surface Σ imbedded in V (Figure 1). Let $\boldsymbol{\nu}$ be the unit normal to Σ and let $[u_i](\mathbf{x}, t)$ and $[u_{p,q}](\mathbf{x}, t)$ be the discontinuities in u_i and $u_{p,q}$ across Σ and in the direction of $\boldsymbol{\nu}$ at the point $\mathbf{x}$ at time t. Assume u_i and G_{ij} satisfy the same homogeneous boundary conditions on S and apply

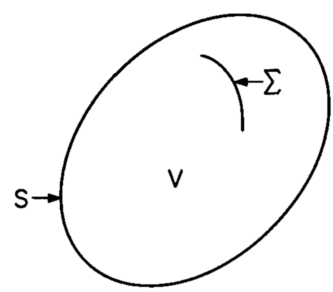

FIG. 1. Schematic diagram of surfaces and volume used in the integration.

equation (8) to the region bounded internally by Σ and externally by S. G_{mi} does not have prescribed discontinuities on Σ. Then the surface integral over S vanishes and we are left with the surface integral over Σ only:

$$
\begin{aligned}
u_m(\mathbf{y}, s) = &\int_{-\infty}^{\infty} dt \int_V G_{mi}(\mathbf{y}, s; \mathbf{x}, t) f_i(\mathbf{x}, t)\, dV_x \\
&+ \int_{-\infty}^{\infty} dt \int_\Sigma \nu_j \{[u_i](\xi, t) c_{ijpq}(\xi) G_{mp,q'}(\mathbf{y}, s; \xi, t) \\
&\qquad - G_{mi}(\mathbf{y}, s; \xi, t) c_{ijpq}(\xi)[u_{p,q}](\xi, t)\}\, d\Sigma_\xi .
\end{aligned} \tag{9}
$$

Equation (9) is a representation of u_m in terms of the prescribed discontinuities in $\mathbf{u}$ and its derivatives across Σ.

3. Equivalent Forces

By means of the properties of the delta function and its derivatives we shall introduce volume integrals into the second term on the right hand side of (9).

We note that

$$G_{mi}(\mathbf{y}, s; \boldsymbol{\xi}, t) = \int_V \delta(\mathbf{x}, \boldsymbol{\xi}) G_{mi}(\mathbf{y}, s; \mathbf{x}, t) dV_x .$$

and

$$-G_{mi,q'}(\mathbf{y}, s; \boldsymbol{\xi}, t) = \int_V \delta_q(\mathbf{x}; \boldsymbol{\xi}) G_{mi}(\mathbf{y}, s; \mathbf{x}, t) dV_x ,$$

where

$$\delta(\mathbf{x}; \boldsymbol{\xi}) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2)\delta(x_3 - \xi_3)$$

and

$$\delta_q(\mathbf{x}; \boldsymbol{\xi}) = \frac{\partial}{\partial x_q} \delta(\mathbf{x}; \boldsymbol{\xi}).$$

If these expressions are substituted in (9) we get

$$u_m(\mathbf{y}, s) = \int_{-\infty}^{\infty} dt \int_V G_{mp}(\mathbf{y}, s; \mathbf{x}, t) \Big[f_p(\mathbf{x}, t) - \int_\Sigma d\Sigma_\xi \nu_j \{ [u_i](\boldsymbol{\xi}, t) c_{ijpq}(\boldsymbol{\xi}) \delta_q(\mathbf{x}, \boldsymbol{\xi}) + [u_{i,q}](\boldsymbol{\xi}, t) c_{pjiq}(\boldsymbol{\xi}) \delta(\mathbf{x}; \boldsymbol{\xi}) \} \Big] dV_x .$$

But $f_p(x, t)$ represents a body force; since the surface integral within the square bracket is involved in the same way as f_p, we conclude that the effect of the prescribed discontinuities across Σ is the same as the effect of introducing extra body forces $e_p(\mathbf{x}, t)$, given by

$$e_p(\mathbf{x}, t) = -\int_\Sigma d\Sigma_\xi \nu_j \{ [u_i](\boldsymbol{\xi}, t) c_{ijpq}(\boldsymbol{\xi}) \delta_q(\mathbf{x}; \boldsymbol{\xi}) + [u_{i,q}](\boldsymbol{\xi}, t) c_{pjiq}(\boldsymbol{\xi}) \delta(\mathbf{x}; \boldsymbol{\xi}) \}, \quad (10)$$

into an unfaulted medium.

Equation (10) holds for any inhomogeneity and anisotropy of the elastic medium provided only that the equations of motion are given by (1), and $c_{ijpq} = c_{pqij}$ at each point of V.

4. Mathematically Consistent Discontinuities

We cannot assign values to $[u_i]$ and to $[u_{p,q}]$ completely arbitrarily. For instance, if Σ is part of the plane $x_3 = 0$, then $[u_{p,1}] = [u_p]_{,1}$ and $[u_{p,2}] = [u_p]_{,2}$ so that the discontinuities in these tangential derivatives are determined when $[u_p]$ is specified. The discontinuities in the normal derivatives $[u_{p,3}]$ however, may be specified independently. This feature was not considered by Knopoff and Gilbert.

The discontinuity in the normal traction across $x_3 = 0$ is given by

$$[T_i] = [\tau_{3i}] = c_{3ipq}[u_{p,q}] = [\mathfrak{I}_i] + c_{3ip3}[u_{p,3}]$$

where τ_{ij} is the stress tensor and $[\mathfrak{I}_i]$ is the part of the discontinuity $[T_i]$ that is determined when $[u_i]$ is specified. Clearly $[T_i]$ may be given arbitrary values by specifying $[u_{p,3}]$ suitably if det $c_{3ip3} \neq 0$. This condition is certainly satisfied by an isotropic solid since only the diagonal terms are non-zero, and $c_{3113} = c_{3223} = \mu$, $c_{3333} = \lambda + 2\mu$ where λ and μ are the Lamé constants of the medium. It is not satisfied, however, by a liquid since $\mu = 0$.

From now on we shall assume that we may choose the six quantities $[u_i]$ and $[T_i] = \nu_j[\tau_{ij}]$ independently. We rewrite (10), accordingly, as

$$e_p(\mathbf{x}, t) = -\int_\Sigma \{[u_i](\xi, t)\nu_j c_{ijpq}(\xi)\delta_q(\mathbf{x}, \xi) + [T_p](\xi, t)\delta(\mathbf{x}, \xi)\} d\Sigma_\xi . \qquad (11)$$

The quantities $[T_i]$ measure the failure of the two sides of the discontinuity to obey the law of equal and opposite action and reaction, that is that sources of stress are introduced on Σ.

We note that the equivalent force depends only on the local properties of the medium in the immediate vicinity of the surface Σ.

5. Equivalent Total Forces When the Tractions are Continuous

In this case the total equivalent force obtained by integrating over V is given by

$$\int_V e_p(\mathbf{x}, t) dV = -\int_\Sigma c_{ijpq}(\xi)\nu_j[u_i](\xi, t) \left\{\int_V \delta_q(\mathbf{x}, \xi) dV_x\right\} d\Sigma_\xi .$$

It is seen to be zero since

$$\int_V \delta_q(\mathbf{x}, \xi) dV_x = \int_S n_q \delta(\mathbf{x}, \xi) dS_x = 0$$

for ξ on Σ since S and Σ have no common point.

Also the total moment about any coordinate axis is zero since the moment M about any axis may be written as

$$M = \int_V \{x_r e_p(\mathbf{x}, t) - x_p e_r(\mathbf{x}, t)\} dV.$$

This is

$$M = -\int_\Sigma \nu_j[u_i](\xi, t) \left\{c_{ijpq}(\xi) \int_V x_r \delta_q(\mathbf{x}, \xi) dV_x - c_{ijrq}(\xi) \int_V x_p \delta_q(\mathbf{x}, \xi) dV_x\right\} d\Sigma.$$

But

$$-c_{ijpq} \int_V x_r \delta_q(\mathbf{x}, \xi) dV + c_{iirq} \int_V x_p \delta_r(\mathbf{x}, \xi) dV = -c_{ijpq}(\xi)\delta_{rq} + c_{ijrq}\delta_{pr}$$

$$= -c_{ijpr} + c_{ijrp} = 0$$

since $c_{ijpr} = c_{ijrp}$.

Hence, in an anisotropic, inhomogeneous medium the application of a displacement dislocation leads to force equivalents that have zero total moment and zero resultant force at all instants of time.

6. Some Specific Examples

We now consider only faulting in a neighborhood in which the medium is isotropic. First we consider some cases in which the normal tractions are continuous, i.e. $[T_p] = 0$. Let $x_3 = 0$ be the surface Σ across which the displacements are discontinuous. Thus $\nu_1 = 0$, $\nu_2 = 0$, $\nu_3 = 1$.

Non-propagating faults

(i) $$[u_1] = \delta(x_1)\delta(x_2)H(t),\ [u_2] = 0,\ [u_3] = 0.$$

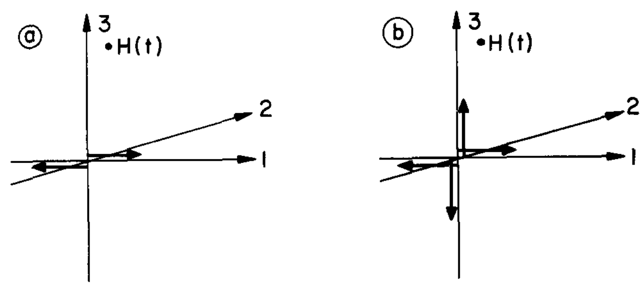

Fig. 2. a) Displacement dislocation at the focus for model i. b) Equivalent double couple in an unfaulted medium for model i.

In this model a tangential displacement of the material in $x_3 > 0$ occurs relative to that in $x_3 < 0$. The displacement has a sudden onset at $t = 0$; thereafter it remains constant. The formal statement above implies that the area over which the faulting occurs has been shrunk in the limit to the origin.

In (11) the only non-zero terms are those for which $(ijpq)$ is (1, 3, 1, 3) and (1, 3, 3, 1).

Thus

$$e_1(\mathbf{x}, t) = -\mu\delta(x_1)\delta(x_2)\delta'(x_3)H(t)$$

$$e_2(\mathbf{x}, t) = 0$$

$$e_3(\mathbf{x}, t) = -\mu\delta'(x_1)\delta(x_2)\delta(x_3)H(t).$$

This is the familiar "double couple" force equivalent which begins to act at $t = 0$ and thereafter is held at a constant level. It may be represented schematically as shown in Figure 2.

(ii) $$[u_1] = 0,\ [u_2] = \delta(x_1)\delta(x_2)H(t),\ [u_3] = 0.$$

This is similar to (i). The x_2-axis replaces the x_1-axis.

$$e_1(\mathbf{x}, t) = 0$$

$$e_2(\mathbf{x}, t) = -\mu\delta(x_1)\delta(x_2)\delta'(x_3)H(t)$$

$$e_3(\mathbf{x}, t) = -\mu\delta(x_1)\delta'(x_2)\delta(x_3)H(t).$$

Again the result is a double couple.

(iii) $$[u_1] = 0,\ [u_2] = 0,\ [u_3] = \delta(x_1)\delta(x_2)H(t).$$

This represents a sudden separation of the two sides of the fault as for example, an explosion in a plane crack. For this case the non-zero terms in (i, j, p, q) are $(3, 3, 1, 1)$, $(3, 3, 2, 2)$ and $(3, 3, 3, 3)$.

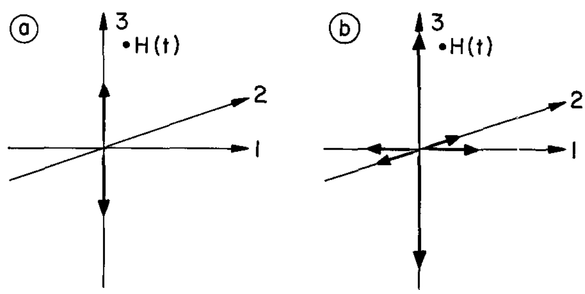

FIG. 3. a) Displacement dislocation at the focus for model 3. b) Equivalent doublets in an unfaulted medium for model 3.

From (11) we get

$$e_1(\mathbf{x}, t) = -\lambda\delta'(x_1)\delta(x_2)\delta(x_3)H(t)$$

$$e_2(\mathbf{x}, t) = -\lambda\delta(x_1)\delta'(x_2)\delta(x_3)H(t)$$

$$e_3(\mathbf{x}, t) = -(\lambda + 2\mu)\delta(x_1)\delta(x_2)\delta'(x_3)H(t).$$

The force equivalent consists of three linear doublets aligned along the three coordinate axes (Figure 3).

Unilateral moving faults

(iv) $$[u_1] = H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2),\ [u_2] = 0,\ [u_3] = 0.$$

In this case matter in $x_3 > 0$ between $x_1 = 0$ and $x_1 = vt$ on the line $x_2 = x_3 = 0$ has a permanent tangential displacement in the x_1-direction relative to the matter adjacent to it in $x_3 < 0$. The dislocation stops at $x_1 = a$ where a is the length of the fault.

As in (1) the only two terms in (11) are $(i, j, p, q) = (1, 3, 1, 3)$ and $(1, 3, 3, 1)$.

Hence (11) gives

$$e_1(\mathbf{x}, t) = -\mu \iint H(vt - \xi)\{H(\xi) - H(\xi - a)\}\delta(\eta)\delta(x_1 - \xi)\delta(x_2 - \eta) \cdot \delta'(x_3)d\xi d\eta$$

$$e_2(\mathbf{x}, t) = 0$$

$$e_3(\mathbf{x}, t) = -\mu \iint H(vt - \xi)\{H(\xi) - H(\xi - a)\}\delta(\eta)\delta'(x_1 - \xi)\delta(x_2 - \eta) \cdot \delta(x_3)d\xi d\eta.$$

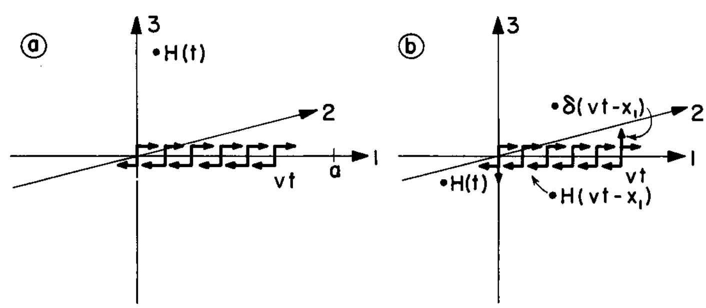

FIG. 4. a) Propagating displacement dislocation in the seismic source for model iv. b) Equivalent propagating double couple in an unfaulted medium for model iv.

This shows that in this case we have an extending line of double couples starting at $t = 0$ and extending between $x \neq 0$ and vt until $vt = a$, when no further propagation takes place.

It follows that

$$e_1(\mathbf{x}, t) = -\mu H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)\delta'(x_3)$$

$$e_2(\mathbf{x}, t) = 0$$

$$e_3(\mathbf{x}, t) = -\mu\{\delta(x_1) - \delta(x_1 - \min(a, vt))\}\delta(x_2)\delta(x_3)H(t).$$

This is shown schematically in figure 4. We note that the line of double couples gives a line of single couples e_1 together with two isolated single forces e_3 when the integrations are carried out.

(v) $[u_1] = 0$, $[u_2] = H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)$, $[u_3] = 0$.

This represents a relative tangential displacement over the same region as in model (iv) but in the x_2-direction

$$e_1(\mathbf{x}, t) = 0$$

$$e_2(\mathbf{x}, t) = -\mu H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)\delta'(x_3)$$

$$e_3(\mathbf{x}, t) = -\mu H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta'(x_2)\delta(x_3).$$

This is shown schematically in figure 5 as an extending line of double couples.

(vi) $[u_1] = 0,\ [u_2] = 0,\ [u_3] = H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2).$

This represents the opening of a growing lenticular cavity. Formula (11) now gives

$$e_1(\mathbf{x}, t) = -\lambda\{\delta(x_1) - \delta(x_1 - \min(vt, a))\}\delta(x_2)\delta(x_3)H(t)$$

$$e_2(\mathbf{x}, t) = -\lambda H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta'(x_2)\delta(x_3)$$

$$e_3(\mathbf{x}, t) = -(\lambda + 2\mu)H(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)\delta'(x_3).$$

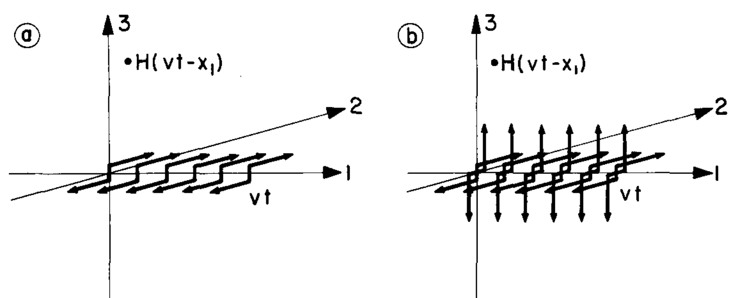

FIG. 5. a) Propagating displacement dislocation on the seismic source for model v. b) Equivalent propagating double couple in an unfaulted medium for model v.

This may be represented as in Figure 6. The solution consists of a set of propagating doublets in the three cardinal directions. The doublets in the x_1-direction integrate into two single forces at the ends of the fault.

Before considering the discontinuous tractions we calculate the equivalent forces in cases (i), (ii) and (iii) when the axes are rotated.

Let us take the fault plane as $x_3 = x_2 \tan \phi$; hence $\nu_1 = 0$, $\nu_2 = -\sin \phi$, $\nu_3 = \cos \phi$. Let ξ and η be coordinates in the fault plane, $d\Sigma = d\xi d\eta$ and

$$\delta(\mathbf{x}, \boldsymbol{\xi}) = \delta(x_1 - \xi)\delta(x_2 - \eta \cos \phi)\delta(x_3 - \eta \sin \phi).$$

(vii) $[u_1] = \delta(\xi)\delta(\eta)H(t),\ [u_2] = 0,\ [u_3] = 0.$

This is like case (i), but a rotation through an angle ϕ about the x_1-axis has been performed.

Formula (11) is, in this case,

$$e_p(\mathbf{x}, t) = -\int \{\delta(\xi)\delta(\eta)H(t)\nu_j c_{1jpq}\delta_q(\mathbf{x}, \boldsymbol{\xi})\}d\xi d\eta$$

$$= -\nu_j c_{ijpq}\delta_q(\mathbf{x})H(t)$$

The non-zero terms are those for which (p, j, q) are $(1, 2, 2)$, $(1, 3, 3)$, $(2, 2, 1)$ and $(3, 3, 1)$. Collecting these results we have

$$\begin{pmatrix} e_1(\mathbf{x}, t) \\ e_2(\mathbf{x}, t) \\ e_3(\mathbf{x}, t) \end{pmatrix} = \cos\phi \begin{pmatrix} -\mu\delta_1(x_1)\delta(x_2)\delta'(x_3)H(t) \\ 0 \\ -\mu\delta'(x_1)\delta(x_2)\delta(x_3)H(t) \end{pmatrix} - \sin\phi \begin{pmatrix} -\mu\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \\ -\mu\delta'(x_1)\delta(x_2)\delta(x_3)H(t) \\ 0 \end{pmatrix}.$$

Thus, in this case, the equivalent force appears as a linear combination of the force equivalents for a fault in the plane $x_3 = 0$ and one in the plane $x_2 = 0$. The coefficients are the x_2 and x_3 components of the normal to the fault plane.

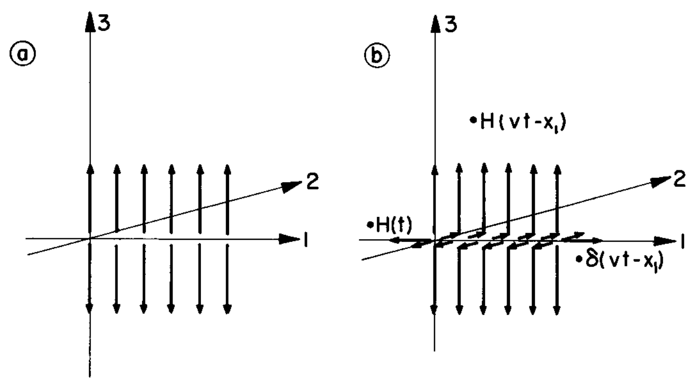

FIG. 6. a) Propagating displacement dislocation on the seismic source for model vi. b) Equivalent propagating doublets in an unfaulted medium for model vi.

(viii) $[u_1] = 0$, $[u_2] = \delta(\xi)\delta(\eta)H(t)\cos\phi$, $[u_3] = \delta(\xi)\delta(\eta)H(t)\sin\phi$.

This is a tangential displacement perpendicular to that in (vii) and represents a rotation of model (ii) about the x_1-axis through an angle ϕ. Formula (11) gives

$$e_p(\mathbf{x}, t) = -\int [u_i]\nu_j c_{ijpq}\delta_q(\mathbf{x}, \xi)\,d\xi\,d\eta$$

For a non-zero contribution $i \neq 1$, $j \neq 1$.
Collecting the contributions from the various terms, we have

$$\begin{pmatrix} u_1(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) \\ u_3(\mathbf{x}, t) \end{pmatrix} = \cos 2\phi \begin{pmatrix} 0 \\ -\mu\delta(x_1)\delta(x_2)\delta'(x_3)H(t) \\ -\mu\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \end{pmatrix} + \sin 2\phi \begin{pmatrix} 0 \\ \mu\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \\ -\mu\delta(x_1)\delta(x_2)\delta'(x_3)H(t) \end{pmatrix}$$

The diagram in figure 7 shows the force equivalent for $\phi = 45°$ in which case $\cos 2\phi = 0$ and $\sin 2\phi = 1$.

For $\phi = 0$ this force equivalent looks somewhat different from that of example (ii), but by rotating the axes through an angle ϕ, resolving the forces along the new axes, and then using relations like

$$\delta(x_2)\delta'(x_3) = \delta(x_3') \frac{\partial x_2'}{\partial x_1} \delta(x_2') + \delta(x_2') \frac{\partial x_3'}{\partial x_2} \delta'(x_3')$$

where x_2 is one of the old coordinates and x_2' and x_3' are new ones, it can be seen that (ii) and (viii) are but different descriptions of the same set of force equivalents. The same remark may be made for models (i) and (vii).

(ix) $[u_1] = 0, [u_2] = -\sin\phi\delta(\xi)\delta(\eta)H(t), [u_3] = \cos\phi\delta(\xi)\delta(\eta)H(t).$

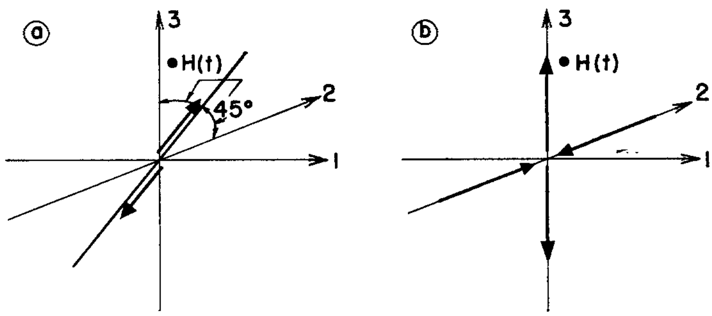

FIG. 7. a) Displacement dislocation at the focus for model (viii). b) Equivalent doublets in an unfaulted medium for model (viii).

This corresponds to model (iii) with a rotation of axes. The results, which are not as elegant as those for (vii) and (viii) are

$$\begin{pmatrix} e_1(\mathbf{x}, t) \\ e_2(\mathbf{x}, t) \\ e_3(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} -\lambda\delta'(x_1)\delta(x_2)\delta(x_3)H(t) \\ -\lambda\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \\ -\lambda\delta(x_1)\delta(x_2)\delta'(x_3)H(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -2\mu\sin^2\phi\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \\ -2\mu\cos^2\phi\delta(x_1)\delta(x_2)\delta'(x_3)H(t) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \mu\sin 2\phi\delta(x_1)\delta(x_2)\delta'(x_3)H(t) \\ \mu\sin 2\phi\delta(x_1)\delta'(x_2)\delta(x_3)H(t) \end{pmatrix}.$$

Equivalent forces when the displacements are continuous but the tractions are not. The basic equation (11) reduces in this case to

$$e_p(\mathbf{x}, t) = -\int_\Sigma [T_p](\xi, t)\delta(\mathbf{x}, \xi)d\Sigma_\xi.$$

If we again specialize to $x_3 = 0$ for Σ, then $[T_p] = [\tau_{p3}]$ where τ_{ij} is the stress tensor.

(x). Non-propagating fault

$$[T_p] = t_p\delta(x_1)\delta(x_2)H(t).$$

where t_p is a constant vector

$$e_p(\mathbf{x}, t) = -t_p\delta(x_1)\delta(x_2)\delta(x_3)H(t).$$

This is an isolated force acting at the point (0, 0, 0) in the opposite direction to that of the discontinuity in traction and with the same magnitude.

(xi). Unilateral propagating fault

$$[T_p] = t_pH(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)$$

$$e_p(\mathbf{x}, t) = -t_pH(vt - x_1)\{H(x_1) - H(x_1 - a)\}\delta(x_2)\delta(x_3).$$

Here, the equivalent force is a line distribution of force propagating along the line of faulting. The force always agrees in magnitude and direction with $-[T_p]$, the negative of the discontinuity in the traction.

(xii). Bilateral faults

Bilateral faulting can be considered for all these models. The mathematics in all cases is relatively simple since the solutions for the unilateral prototypes have already been given. We replace the unilateral propagator $H(vt - x_1)H(t)$ by the bilateral propagator $\{H(vt + x_1) - H(-vt + x_1)\}H(t)$. No new features are introduced into the solutions already obtained. The results obtained in the earlier models are simply modified to take into account that the dislocations also propagate in the opposite direction.

7. General Remarks

Although the body forces equivalent to a given dislocation are supposed to act in an unfaulted medium and therefore cannot in any sense represent real forces acting on the real medium, these force equivalents may nevertheless provide a useful theoretical tool. Their usefulness lies in the fact that if two dislocations have the same equivalent force they also emit the same radiation. Thus, when we wish to find the radiation from a given dislocation, it is sometimes possible to find, by means of the equivalent forces, a second dislocation giving the same radiation as the original source; the radiation from the second source may be simpler to treat analytically. Burridge, Lapwood, and Knopoff (1964) have used this technique to compute the radiation from a dislocation in the presence of a plane free surface.

As another example we mention a method of finding the source function in spherical polar coordinates corresponding to an isolated point body force acting at the point (b, θ, ϕ). In this method the point force is expressed as a series of vector harmonic surface distributions over the sphere $r = b$ (Morse and Feshbach, 1953, p. 1898). The radiation corresponding to each term of the series may then be found by imposing discontinuities in traction across $r = b$ in accordance with our formula (11) when $[u_i] = 0$. This type of simulation of point sources was also used by Lamb (1904) for the half space.

The results of the present calculations show that for propagating or non-propagating displacement dislocation faults, the propagating or non-propagating double couple model is appropriate, whether boundaries are present or not.

Acknowledgment

This work was supported by Grant No. AF-AFOSR-26-63 of the Air Force Office of Scientific Research as part of the Advanced Research Projects Agency project VELA.

References

Balakina, L. M., Shirokova, H. I., and Vvedenskaya, A. V., Study of stresses and ruptures in earthquake foci with the help of dislocation theory, *Publications of the Dominion Observatory, Ottawa*, 24, 321–327, 1960.

Baker, B. B. and Copson, E. T., *The Mathematical Theory of Huygens' Principle*, second edition (Oxford, Clarendon Press, 1950).

Burridge, R., Lapwood, E. R., and Knopoff, L., First motions from seismic sources near a free surface, *Bull. Seis. Soc. Amer.*, 54, 1875–1899.

De Hoop, A. T., *Representation theorems for the displacement in an elastic solid and their application to elastodynamic diffraction theory.* D.Sc. thesis, Technische Hogeschool, Delft, 1958.

Knopoff, L., Diffraction of elastic waves, *J. Acoust. Soc. Am.*, 28, 217–229, 1956.

Knopoff, L. and Gangi, A. F., Seismic reciprocity, *Geophysics*, 24, 681–691, 1959.

Knopoff, L. and Gilbert, F., First motions from seismic sources, Bull. *Seis. Soc. Am.*, 50: 117–134, 1960.

Lamb, H., On reciprocal theorems in dynamics, *Lond. Math. Soc.*, 19, 144–151, 1889.

Lamb, H., On the propagation of tremors over the surface of an elastic solid, *Phil. Trans. Roy. Soc.* (London) A, 203, 1–42, 1904.

Love, A. E. H., *A treatise on the mathematical theory of elasticity*, fourth edition (New York: Dover Publications, 1944).

Maruyama, T., On the force equivalents of dynamical elastic dislocations with reference to the earthquake mechanism, *Bull. Earthquake Res. Inst.*, Tokyo Univ., 41, 467–486, 1963.

Morse, P. M. and Feshbach, H., *Methods of Theoretical Physics* (New York: McGraw Hill Book Co., 1953).

Nabarro, F. R. N., The synthesis of elastic dislocation fields, *Phil. Mag.*, 42, 1224–1231, 1951.

Stauder, W., The focal mechanism of earthquakes, *Advances in Geophysics*, 9, 1–76, 1962.

Whittaker, E. T., *A treatise on the analytical dynamics of particles and rigid bodies*, fourth edition (New York: Dover Publications, 1944).

Manuscript received August 14, 1964.