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On the Wave-Front Approximation in Three-dimensional Gas Dynamics

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Summary

Since the differential equations governing steady supersonic flow of an inviscid gas are hyperbolic, the fluid acceleration must satisfy some compatibility relations along characteristic surfaces. These relations are here obtained and integrated for a characteristic surface bounding a region of uniform flow, and it is then shown that the same relations are satisfied to a first approximation in a region adjacent to the region of uniform flow. The singularities predicted in this manner are discussed, and an approximate method of solution, complementary to the Linearized Theory, is briefly explained.

1. Introduction

The theory of quasi-linear hyperbolic partial differential equations has been extensively developed in connection with problems involving two independent variables. But the theory is virtually undeveloped for problems involving three independent variables, and the present work is concerned with its extension to this class of problems. The example chosen here is that of steady, three-dimensional supersonic flow without entropy gradients.

The successful analytical methods for 2-variable problems were developed in two stages. The first was to study problems involving only progressive waves, such as those emitted by thin aerofoils and slender bodies of revolution. The second stage was the study of wave interactions, and this stage is still rather incomplete, for instance, for axially symmetrical flow. Accordingly, the present work is concerned only with progressive waves.

The approach also follows that found successful in 2-variable problems for the derivation of uniform first-order approximations by a wave-front approximation [*1*]. The procedure is briefly as follows. First, consider the characteristic surface forming the border between a region of uniform and a region of non-uniform flow. The fluid acceleration may be discontinuous at such a surface, and if so, the jump will satisfy a differential equation on the surface, that is, a differential equation in at most two variables. Next, a continuity argument is used to extend the same analysis to a suitable part of the region of non-uniform flow regardless of whether a discontinuity of the acceleration occurs or not. An approximation to the exact continuous solution of the governing differential equations is thereby obtained, but that solution may be multivalued. Finally, it is known how a weak shock wave can be fitted into the approximate solution to remove the multivaluedness in 2-variable problems, but this last part of the procedure is not studied in the present investigation.

The important role played by characteristic surfaces in the theory of hyperbolic equations makes it convenient to introduce a system of coordinates containing them as coordinate surfaces. It is impossible, however, to include them all in a system of coordinates. If only some characteristic surfaces are included, it is natural to construct a system of coordinates based on two families of characteristic surfaces and an additional family of surfaces which are not required to be characteristic. Because the bicharacteristic lines play a distinguished role, it would be desirable to take the third family of coordinate surfaces intersecting the characteristic surfaces of the other two families along bicharacteristics. In general, however, this is impossible. To avoid this difficulty, directional derivatives are used in what follows, but in order to profit from the properties of characteristic surfaces, a characteristic parameter will also be introduced.

The work reported here owes much to two earlier publications. M. SCHIFFER [2] has shown that in 3-variable problems any discontinuity of the acceleration must satisfy an ordinary differential equation along bicharacteristics, but his proof does not give this equation in a form in which it could be integrated, or even discussed. T. Y. THOMAS [3] has considered the unsteady motion of a gas in 3-dimensions, of which the steady problem here studied is a special case. He has derived and integrated the differential equation for the discontinuity of the acceleration for the case of a discontinuity occurring at the front of a wave spreading into gas at rest. But his method appears to be less suitable for the discussion of multivalued solutions or the construction of approximate solutions in the region of non-uniform motion behind the wave front, which is the aim of the present study.

In the following, the matter is therefore attacked *ab initio*. The first part of the work is devoted to the establishment of a suitable notation and of preliminary results required for the analysis. THOMAS' results are then obtained in the notation adopted here, and their significance is discussed. Finally, the analysis is extended to a uniform first-order approximation for a region behind the wave front.

2. Preliminary definitions and results

The motion of a gas devoid of viscosity and thermal conductivity which comes from a region of uniform flow is governed by the irrotationality condition,

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}, \tag{2.1 a}$$

and the equation of continuity,

$$\frac{\partial v_i}{\partial x_i} - c^{-2} v\, v_i \frac{\partial v}{\partial x_i} = 0, \tag{2.1 b}$$

where $v = (v_i v_i)^{\frac{1}{2}} > 0$ and $c = (\partial p/\partial \varrho)_S^{\frac{1}{2}} > 0$ is some known function of v.

When the velocity magnitude v exceeds the speed of sound c, the system (2.1) is hyperbolic and possesses characteristics. A surface is characteristic if and only if the normal component of the velocity at each of its points is c. A discontinuity of the derivatives of the velocity components can occur only across such surfaces, when the velocity components themselves are continuous.

Let R be a region next to and downstream from a region of uniform flow and such that the flow is non-uniform in R. It is assumed that the transition from uniform to non-uniform flow takes place by means of a break of the continuity of some derivative of the velocity, but that no shock is present in the flow, so that the velocity components are continuous. It is then clear that the "surface of transition" Σ_0, which separates R from the region of non-uniform flow, is characteristic. It is assumed, furthermore, that a family $\mathfrak{F}$ of characteristic surfaces is defined in R in such a manner that every point of R lies in one and only one member of $\mathfrak{F}$ (except at limit surfaces) and that Σ_0 belongs to $\mathfrak{F}$. In addition a function ξ is defined in R so as to be constant on the members of $\mathfrak{F}$, and to have non-vanishing gradient everywhere in R.

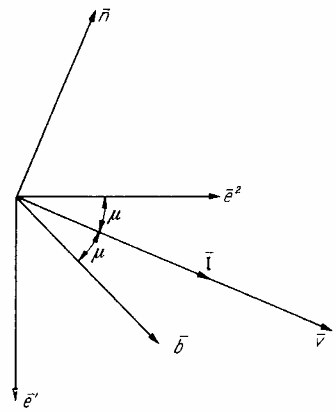

Fig. 1

Let P denote an arbitrary point of R, and let Σ be the member of $\mathfrak{F}$ which passes through P. At P, define the "ray vector" e_i^1 as the unit normal vector to Σ such that its scalar product with the velocity is positive; the "bicharacteristic vector" e_i^2 as the unit vector with the same direction and sense as the projection on Σ of the velocity vector; and the "transversal vector" e_i^3 by the vector product

$$\bar{e}^3 = \bar{e}^1 \times \bar{e}^2. \tag{2.2}$$

The ray, bicharacteristic and transversal vectors form an orthogonal system, he "characteristic system" at P. Thus

$$e_i^\alpha e_i^\beta = \delta_{\alpha\beta} \quad \text{and} \quad e_i^\alpha e_j^\alpha = \delta_{ij}. \tag{2.3}$$

The plane orthogonal to the transversal vector will be called the "principal lane" (see the figure). We define two orthogonal vectors lying on the principal lane, the "stream vector" l_i and the "normal vector" n_i, by

$$l_i = e_i^2 \cos\mu + e_i^1 \sin\mu, \qquad n_i = e_i^2 \sin\mu - e_i^1 \cos\mu \tag{2.4a}$$

o that

$$e_i^1 = l_i \sin\mu - n_i \cos\mu, \qquad e_i^2 = l_i \cos\mu + n_i \sin\mu \tag{2.4b}$$

here μ is the local Mach angle,

$$\mu = \sin^{-1} c/v, \qquad 0 < \mu < \pi/2. \tag{2.5}$$

he direction of the velocity is given by the stream vector, because

$$l_i = v_i/v. \tag{2.6}$$

he "Prandtl angle" ω, defined by

$$v\frac{d\omega}{dv} = \cot\mu, \qquad \omega = 0 \quad \text{when} \quad v = c, \tag{2.7}$$

often provides a useful measure of the velocity magnitude. The stream, normal and transversal vectors form a second orthonormal system, the "equipotential system" at P.

The "outer vector" b_i is the unit vector defined by

$$b_i = l_i \cos\mu - n_i \sin\mu \quad \text{or} \quad b_i = e_i^2 \cos 2\mu + e_i^1 \sin 2\mu. \tag{2.8}$$

The differential operators D_α, D_b and D_ξ are defined by

$$D_\alpha = e_i^\alpha \partial/\partial x_i, \qquad D_b = b_i \partial/\partial x_i, \qquad D_\xi = h_\xi D_b \tag{2.9}$$

where the "metric coefficient" h_ξ is given by

$$h_\xi = \frac{1}{D_b \xi}. \tag{2.10}$$

When they are applied to any function, they yield directional derivatives with respect to length, D_α and D_b, and with respect to the parameter ξ, D_ξ. Observe that (2.9) and (2.3) imply

$$\partial/\partial x_i = e_i^\alpha D_\alpha. \tag{2.11}$$

3. The characteristic equations

In this section the equations of motion (2.1) will be transformed into characteristic form. The introduction of the directional derivatives in these equations leads to

$$e_i^\beta D_\alpha v_i = e_i^\alpha D_\beta v_i, \tag{3.1}$$

$$e_i^\alpha D_\alpha v_i - c^{-2} v_i v_j e_i^\alpha D_\alpha v_j = 0.$$

The last equation is further reduced, by virtue of (2.6) and (2.4a), to

$$(e_i^1 - l_i \operatorname{cosec}\mu) D_1 v_i + (e_i^2 - l_i \cot\mu \operatorname{cosec}\mu) D_2 v_i + e_i^3 D_3 v_i = 0.$$

Since $e_i^2 D_1 v_i = e_i^1 D_2 v_i$, by (3.1), the derivative in the ray direction may be eliminated using (2.4a). This yields

$$(n_i - l_i \cot\mu) \operatorname{cosec}\mu D_2 v_i + e_i^3 D_3 v_i = 0.$$

When the velocity vector is expressed in terms of the stream vector (2.6) and of the Prandtl angle (2.7), this equation becomes

$$D_2\omega - n_i D_2 l_i = \sin\mu\, e_i^3 D_3 l_i, \tag{3.2}$$

a generalization of the well known characteristic relations for plane and axially symmetrical flows. Observe that the term $e_i^3 D_3 l_i = -l_i D_3 e_i^3$ has the geometrical interpretation of the normal curvature of the equipotential surface in the transversal direction.

One component of $D_2 l_i$ in the equipotential system is given by (3.2); the second,

$$e_i^3 D_2 l_i = e_i^2 D_3 l_i + \sin\mu D_3\omega,$$

follows from (3.1), if (2.6), (2.4a) and (2.7) are used, and the third vanishes because $l_i l_i = 1$. Hence

$$D_2 l_i = (D_2\omega - \sin\mu\, e_j^3 D_3 l_j) n_i + (e_j^2 D_3 l_j + \sin\mu D_3\omega) e_i^3. \tag{3.3}$$

This equation involves only the bicharacteristic, transversal and normal vectors, and the derivatives of the dynamical variables in the direction of the first two of these vectors. Now, starting with the outer vector b_i in the role of the bicharacteristic vector e_i^2, we can construct the vectors corresponding to the transversal and normal vectors, and they are $-\bar{e}^3$ and $-\bar{n}$. Since $e_k^3 D_3 \equiv e_k^3 e_j^3 \partial/\partial x_j$ is invariant under such a change of signs, it follows that corresponding to equation (3.3), we must also have

$$D_b l_i = -(D_b \omega - \sin\mu\, e_j^3 D_3 l_j)\, n_i + (b_j D_3 l_j + \sin\mu\, D_3 \omega)\, e_i^3, \tag{3.4}$$

whence

$$D_b \omega + n_i D_b l_i = \sin\mu\, e_i^3 D_3 l_i, \tag{3.5}$$

corresponding to (3.2). In terms of the intrinsic coordinate ξ, (3.4) and (3.5) read

$$D_\xi l_i = -(D_\xi \omega - h_\xi \sin\mu\, e_j^3 D_3 l_j)\, n_i + h_\xi (b_j D_3 l_j + \sin\mu\, D_3 \omega)\, e_i^3, \tag{3.6}$$

$$D_\xi \omega + n_i D_\xi\, l_i = h_\xi \sin\mu\, e_i^3 D_3 l_i. \tag{3.7}$$

4. The commutators

In general, $D_\alpha D_\beta \neq D_\beta D_\alpha$, $D_\alpha D_\xi \neq D_\xi D_\alpha$, and it will be necessary for our purpose to compute the differential operators

$$[2, 3] = D_2 D_3 - D_3 D_2, \quad [2, \xi] = D_2 D_\xi - D_\xi D_2, \quad [3, \xi] = D_3 D_\xi - D_\xi D_3. \tag{4.1}$$

The first result to be shown is

$$[2, 3] = \Omega' D_2 - \Omega D_3, \tag{4.2}$$

where

$$\Omega = e_i^3 D_3 e_i^2 \quad \text{and} \quad \Omega' = e_i^2 D_2 e_i^3. \tag{4.3}$$

It follows from the facts that

$$[2, 3] = (D_2 e_i^3 - D_3 e_i^2)\, \partial/\partial x_i = (e_i^\alpha D_2 e_i^3 - e_i^\alpha D_3 e_i^2)\, D_\alpha$$

by (2.11), $|\bar{e}^2| = |\bar{e}^3| = 1$ and

$$e_i^1 D_2 e_i^3 = e_i^1 D_3 e_i^2. \tag{4.4}$$

To prove (4.4), observe that

$$[2, 3]\xi = 0 \quad \text{and} \quad [2, 3]\xi = (e_i^1 D_2 e_i^3 - e_i^1 D_3 e_i^2) D_1 \xi.$$

Since $D_1 \xi \neq 0$, (4.4) follows.

We show next that $[2, \xi]$, and $[3, \xi]$ also, involve only differentiation in a direction tangential to the members of $\mathfrak{F}$. In order to be able to prove both results simultaneously, we adopt the convention that the index γ takes only the values 2 and 3 in the remainder of this section. The result will follow from

$$e_i^1 D_\gamma (h_\xi b_i) = e_i^1 D_\xi e_i^\gamma \tag{4.5}$$

because

$$[2, \xi] = \{D_2 (h_\xi b_i) - D_\xi e_i^2\}\, \partial/\partial x_i = \{e_i^\alpha D_2 (h_\xi b_i) - e_i^\alpha D_\xi e_i^2\}\, D_\alpha, \tag{4.6a}$$

$$[3, \xi] = \{D_3 (h_\xi b_i) - D_\xi e_i^3\}\, \partial/\partial x_i = \{e_i^\alpha D_3 (h_\xi b_i) - e_i^\alpha D_\xi e_i^3\}\, D_\alpha \tag{4.6b}$$

by (2.11). To prove (4.5) observe that $D_\xi \xi \equiv 1$, $D_\gamma \xi \equiv 0$ in R, by (2.9) and (2.10). Thus

$$[\gamma, \xi]\xi = 0 \quad \text{and} \quad [\gamma, \xi]\xi = \{e_i^1 D_\gamma (h_\xi b_i) - e_i^1 D_\xi e_i^\gamma\} D_1 \xi.$$

Since $D_1 \xi \neq 0$, (4.5) follows.

For later reference, note that

$$\sin 2\mu D_2 h_\xi = e_i^1 D_\xi e_i^2 - h_\xi e_i^1 D_2 b_i, \tag{4.7a}$$

$$\sin 2\mu D_3 h_\xi = e_i^1 D_\xi e_i^3 - h_\xi e_i^1 D_3 b_i \tag{4.7b}$$

follow from (4.5) by setting $\gamma=2$ and $\gamma=3$, respectively, and using (2.8).

5. The structure of the characteristic surfaces

The shape of the characteristic surfaces is governed by a system of ordinary differential equations along bicharacteristics which will now be obtained in our notation. Since $e_i^1 l_i = \sin\mu$,

$$l_i D_\alpha e_i^1 = -e_i^1 D_\alpha l_i + \cos\mu D_\alpha \mu,$$

and by (2.4a),

$$e_i^2 D_2 e_i^1 = -\sec\mu\, e_i^1 D_2 l_i + D_2\mu, \tag{5.1a}$$

$$e_i^2 D_3 e_i^1 = -\sec\mu\, e_i^1 D_3 l_i + D_3\mu = e_i^3 D_2 e_i^1, \tag{5.1b}$$

in view of (4.4). Since $|\bar{e}^1|=1$, (5.1a, b) give the non-zero components of $D_2 e_i^1$, and

$$D_2 e_i^1 = (D_2\mu - \sec\mu\, e_j^1 D_2 l_j)\, e_i^2 + (D_3\mu - \sec\mu\, e_j^1 D_3 l_j)\, e_i^3. \tag{5.2}$$

The corresponding relation for the bicharacteristic vector e_i^2,

$$D_2 e_i^2 = (\sec\mu\, e_j^1 D_2 l_j - D_2\mu)\, e_i^1 + \{\mathrm{tg}\,\mu D_3(\omega-\mu)\}\, e_i^3 \tag{5.3}$$

follows from (5.1a), because $e_i^2 e_i^1 = 0$, $|\bar{e}^2|=1$ and

$$e_i^3 D_2 e_i^2 = \mathrm{tg}\,\mu D_3(\omega-\mu). \tag{5.4}$$

To obtain (5.4), apply $e_i^3 D_2$ to the first of equations (2.4a) to get

$$e_i^3 D_2 e_i^2 = \sec\mu\, e_i^3 D_2 l_i - \mathrm{tg}\,\mu\, e_i^3 D_2 e_i^1,$$

use (3.3) and (5.1b) to eliminate $e_i^3 D_2 l_i$ and $e_i^3 D_2 e_i^1$, and reduce the result by the help of (2.4a).

Finally, for the transversal vector

$$D_2 e_i^3 = (\sec\mu\, e_j^1 D_3 l_j - D_3\mu)\, e_i^1 + \{\mathrm{tg}\,\mu D_3(\mu-\omega)\}\, e_i^2, \tag{5.5}$$

by (5.1b), (5.4) and the orthonormality of the characteristic system.

6. The acceleration on the transition surface

The equation (3.6) reduces to

$$D_\xi l_i = -n_i D_\xi \omega \quad \text{on } \Sigma_0 \tag{6.1}$$

because the velocity vector is constant there. To find the fluid acceleration at any point of Σ_0, it will therefore suffice to know $h_\xi^{-1} D_\xi\,\omega$ at that point. In this section we shall obtain ordinary first-order differential equations along bicharacteristics of Σ_0 for $D_\xi\omega$ and h_ξ, from which both may be computed at all points of Σ_0, if they are known along a curve intersecting every bicharacteristic of Σ_0.

Since the velocity is constant on Σ_0, the results of Section 4 imply that the operators D_2, D_3 and D_ξ commute on Σ_0 when applied to the dynamical variables, l_i, ω. Moreover (5.2), (5.3) and (5.5) imply that $D_2 e_i^1 = D_2 e_i^2 = D_2 e_i^3 = D_2 n_i = D_2 b_i = D_2 \omega = D_3 \omega = 0$ on Σ_0. In this section, we shall make repeated use of these facts without any further explicit mention.

The ordinary differential equation for $D_\xi \omega$,

$$D_2 D_\xi \omega + \frac{\Omega}{2} D_\xi \omega = 0 \quad \text{on } \Sigma_0, \tag{6.2}$$

is obtained by cross-differentiation of the characteristic equations (3.2) and (3.7), observing that

$$e_i^3 D_3 D_\xi l_i = -(e_i^3 D_3 n_i) D_\xi \omega \quad \text{on } \Sigma_0,$$

by (6.1), and that

$$e_i^3 D_3 n_i = \operatorname{cosec} \mu \, e_i^3 D_3 e_i^2 = \Omega \operatorname{cosec} \mu \quad \text{on } \Sigma_0,$$

by (2.4b), and using the definition (4.3) of Ω. To integrate (6.2) along bicharacteristics of Σ_0, observe that

$$D_2 \Omega = e_i^3 D_2 D_3 e_i^2 = -\Omega^2 \quad \text{on } \Sigma_0,$$

by (4.3) and (4.2). Hence

$$\Omega^{-1} = s \tag{6.3}$$

on an arbitrary bicharacteristic of Σ_0, where s is the oriented length along that bicharacteristic measured from the point (not necessarily in R) where $\Omega^{-1} = 0$. The equation (6.2) now integrates to

$$D_\xi \omega = \sigma |s|^{-\frac{1}{2}} \tag{6.4}$$

where σ is constant on bicharacteristics and the range of s must be restricted to be either positive or negative.

The equation for the metric coefficient is given by (4.7a), which may be transformed into

$$D_2 h_\xi = \operatorname{cosec} 2\mu \left(1 - \frac{d\mu}{d\omega}\right) D_\xi \omega \quad \text{on } \Sigma_0, \tag{6.5}$$

by the help of (6.1) and (2.4a), if use is made of the fact that

$$e_i^1 D_\xi e_i^2 = \sec \mu \, e_i^1 D_\xi l_i - D_\xi \mu \quad \text{everywhere in } R, \tag{6.6}$$

by (2.4a). Equations (6.4) and (6.5) imply

$$h_\xi = h_0 \pm 2 m \sigma |s|^{\frac{1}{2}}, \tag{6.7}$$

where h_0 is constant along bicharacteristics of Σ_0, m stands for the value of $\operatorname{cosec} 2\mu (1 - d\mu/d\omega)$ on Σ_0, and the sign is given by that of s.

On any bicharacteristic on which $\Omega = 0$ the equations (6.3), (6.4) and (6.7) must be replaced by

$$\Omega = 0, \tag{6.8a}$$

$$D_\xi \omega = \text{const.} = (D_\xi \omega)_0, \tag{6.8b}$$

$$h_\xi \omega = h_0 + m (D_\xi \omega)_0 s, \tag{6.8c}$$

where s is now measured from an arbitrary point.

7. Bicharacteristic equivalence and shock prediction

When the flow in R is axially symmetric, the results of the last section become

$$s = \pm r \operatorname{cosec} \mu, \tag{7.1 a}$$

$$\Omega^{-1} = \pm r \operatorname{cosec} \mu, \tag{7.1 b}$$

$$D_\xi \omega = \sigma r^{-\frac{1}{2}} (\operatorname{cosec} \mu)^{-\frac{1}{2}}, \tag{7.1 c}$$

$$h_\xi = h_0 \pm 2 m \sigma r^{-\frac{1}{2}} (\operatorname{cosec} \mu)^{-\frac{1}{2}}, \tag{7.1 d}$$

where r denotes distance from the axis of symmetry, the plus sign holds on the characteristics of one family, and the minus sign on those of the other family. Comparison with (6.3), (6.4) and (6.7) reveals the following

Equivalence Principle. *Given an arbitrary bicharacteristic on Σ_0, we can associate a locally equivalent axially symmetric flow, in the sense that the velocity, pressure, fluid acceleration and pressure gradient have the same distribution along the bicharacteristic as along a meridian characteristic of the axially symmetric flow.* Two-dimensional flows are included here as limiting cases, and they are correlated by this equivalence to bicharacteristics on Σ_0 on which $\Omega = 0$.

The results of Section 6 also elucidate the singularities of the fluid acceleration that can occur in the transition surface. Since the velocity components are constant on Σ_0, it follows from (6.1) that a singularity of the acceleration occurs at a point of Σ_0 if and only if, the outer derivative of ω is singular there, *i.e.* $(D_b \omega)^{-1} = h_\xi (D_\xi \omega)^{-1} = 0$. Consider first a point where $(D_\xi \omega)^{-1} = 0$. By (6.4), $s = 0$ there. In the associated axially symmetric flow, this point lies on the axis of symmetry. We shall accordingly call this type of singularity "center of symmetry". By (6.3), we see that this type of singularity of the fluid acceleration is induced by a geometrical singularity of the surface of transition and indeed will occur independently of conditions downstream, as long as $D_\xi \omega \neq 0$ on the bicharacteristic passing through the singular point. Observe that the centers of symmetry define a line on Σ_0.

The other type of singular points on the transition surface are those at which $h_\xi = 0$, that is, points of an envelope of the family $\mathfrak{F}$ of characteristic surfaces. These singularities define a line on Σ_0 which will usually be continued in R by a "limit surface" enveloping the family $\mathfrak{F}$. From the known results for two-dimensional and axially symmetric flows [*1*], it may be expected that such singularities are usually connected with the formation of shock waves.

8. Perturbation Assumptions

It is natural to expect that the relations along bicharacteristics of Section 6 ne satisfied to a first approximation, also in some region R of non-uniform flow bear the transition surface Σ_0. This will be proved under certain assumptions to be introduced and discussed now.

We shall consider continuous velocity fields. Such a field must possess a region $R(\delta)$, adjacent to and downstream of Σ_0, in which $|v_i - U_i| < \delta$, for arbitrary $\delta > 0$, if U_i denote the velocity components of the uniform flow upstream of Σ_0. In the following, δ will be taken small, so that the velocity in R differs from the incident velocity U_i by only a small perturbation. A similar statement concerning the fluid acceleration cannot be made in general, since continuity of the velocity has been seen above to be compatible with discontinuity of the acceleration, for instance across Σ_0. The assumption that the acceleration be as small as the velocity perturbation forms part of the basis of the Linearized Theory, but no such restriction will be made in the following. On the other hand, the acceleration may happen to be small, and a notation is desirable which exhibits the implications of such a state of affairs. A separate parameter ε will therefore be introduced which characterizes the fluid acceleration and which is not necessarily small.

This will be done in such a way, moreover, that singularities of the acceleration connected with limit surfaces are not excluded. It then becomes plausible, because it has been strictly confirmed for two-dimensional and axially symmetric flow [*1*], that the analysis of continuous velocity fields will possess validity and usefulness also for cases in which R contains a weak shock wave. The present analysis in such cases yields multi-valued continuous solutions which in two-dimensional and axially symmetric flow, can be converted into single-valued solutions with a discontinuity satisfying the weak shock equations, at least to the first order. A similar extension of the analysis given below is likely to be possible but is outside the scope of the present investigation. Similarly, strictly transonic and hypersonic flows will be excluded by the condition

$$0 < \gamma \leqq \mu \leqq \frac{\pi}{2} - \gamma \tag{8.1}$$

where γ is some positive number.

As before, the analysis will be based on the consideration of a family $\mathfrak{F}$ of characteristic surfaces Σ, which contains Σ_0 as a member and is labelled in a one-to-one manner by a parameter ξ such that $\xi = 0$ on Σ_0. Without undue restriction, from a practical point of view, of the boundary conditions downstream of Σ_0 which determine the flow in R, this flow may be assumed to satisfy some regularity conditions. More precisely, consideration will be restricted to the collection of sets $\mathscr{I}(\varepsilon, \delta)$ of flows defined as follows. A flow G (*i.e.* a solution v_i of (2.1)) belongs to $\mathscr{I}(\varepsilon, \delta)$ if, and only if, — for some choice of $\mathfrak{F}$ and ξ, and for some given positive numbers K, ε, δ, with $\delta < \varepsilon$ — the three conditions

$$|D_2\omega|, |D_2 l_i|, |e_i^3 D_2 D_3 l_i|, |e_i^1 D_3 e_i^2|, |h_\xi| < K, \tag{8.2a}$$

$$|D_\xi \omega| < \varepsilon, \tag{8.2b}$$

$$|n_i D_\xi e_i^3|, |D_\xi D_3 \omega|, |D_\xi D_3 l_i|, |D_\xi D_3^2 \omega|, |D D_3^2 l_i| < K\varepsilon \tag{8.2c}$$

are satisfied in the region $R(\varepsilon, \delta, G)$ which is the intersection of some fixed bounded region containing part of Σ_0 and the region where $0 \leqq \xi < \delta/\varepsilon$. Here K will be understood to be a fixed number; δ to be variable, subject to $0 < \delta < \delta_0$, where δ_0 is a small, fixed number; and $\varepsilon = \varepsilon(\delta)$ to be bounded and large compared with δ in the sense that a positive integer n exists such that

$$(\delta/\varepsilon)^n < K\delta \tag{8.2d}$$

for every δ, so that either ε is a fixed number independent of δ, or $\delta/\varepsilon \to 0$, if $\varepsilon(\delta) \to 0$ as $\delta \to 0$.

In accordance with this definition, a function $f(x_i, G)$ will be called $O(\delta)$ if a number M (independent of δ) exists such that, for all $\delta < \delta_0$,

$$|f(x_i, G)| < M\delta,$$

whenever $x_i \in R(\varepsilon, \delta, G)$ and $G \in \mathscr{I}(\varepsilon, \delta)$. Similarly, f will be called bounded if, for all $\delta < \delta_0$,

$$|f(x_i, G)| < M,$$

whenever $x_i \in R(\varepsilon, \delta, G)$ and $G \in \mathscr{I}(\varepsilon, \delta)$.

Assumptions (8.2) imply that $\omega - \omega_0 = O(\delta)$, and in Section 11, the velocity will be seen to be predicted by our analysis, except for errors $O(\delta^2/\varepsilon)$. Under the assumptions (8.2), the acceleration will be shown to be $O(\varepsilon)$, and it will be predicted except for errors $O(\delta)$. On the other hand, the thickness of R is $O(\delta/\varepsilon)$, while the errors in the prediction of position will be $O(\delta^2/\varepsilon)$. All this is independent of the relation between ε and δ but shows that the results become uninformative if we allow $\delta/\varepsilon \to 1$ as $\delta \to 0$. It will be shown in the next section that the derivatives of the velocity in any direction tangential to the members of $\mathfrak{F}$ are $O(\delta)$. On the other hand $D_\xi \omega$ is $O(\varepsilon)$, h_ξ is bounded and $D_b \omega = h_\xi^{-1} D_\xi \omega$, so that the derivative in the outer direction is much greater than the tangential derivatives to the members of $\mathfrak{F}$. Such a distinction in order of magnitude between the derivatives taken normally and tangentially to a family of characteristic surfaces is typical of a "progressive wave", in contrast to a wave interaction, and the main substance of (8.2) is indeed that the flow in the region R be a progressive wave in this sense. Such a region occurs where a uniform flow develops into a non-uniform one with streamline curvature much greater than the velocity perturbation. More generally, such a region can be expected to occur where a flow whose perturbations of the velocity and its derivatives of sufficiently high order are $O(\delta)$ develops into a flow with streamline curvature much greater than δ. The results here obtained do not generally yield the velocity field in the whole region adjacent to a thin wing, but they represent an essential step beyond the restriction to the transition surface Σ_0.

In two-dimensional flow the uniform first-order theory has been brought to the stage where it contains the linearized theory [*1*], and in axially symmetric flow this has been largely achieved [*1*], especially for progressive waves. The following analysis derives only a uniform first-order approximation of more limited scope which complements the linearized theory because it permits the treatment of flows with singular accelerations such as occur at limit surfaces. The continuous solutions of the non-linear equations of motion which contain limit surfaces are multi-valued, but as in two-dimensional and axially symmetric flow, when the region of multivaluedness occurs inside the region of flow, their analysis may be expected to represent the proper first step in the construction of physically acceptable solutions with shocks.

The assumption that $e_i^1 D_3 e_i^3$ is bounded excludes centers of symmetry. On a purely mathematical level, the exclusion of centers of symmetry is related to the condition (8.1) excluding the values 1 and 0 of the reciprocal of the local Mach number. All these conditions serve to exclude singularities of the governing differential equations, and it is well known from the theory of transonic flow and axially symmetrical flow that such singularities may introduce severe physical and mathematical difficulties. The exclusion of centers of symmetry is also related to the physical assumption that R is a region in which the flow is a progressive wave. Near centers of symmetry, wave interaction processes play necessarily a dominant role. In particular, if the region considered is that adjacent to a wing surface W, the admissible surface shapes must be restricted so as not to generate centers of symmetry in their neighborhood. For instance, the leading edge of W must not have certain types of corners.

With regard to (8.2c) observe that $D_\xi D_3\omega$, $D_\xi D_3 l_i$, $D_\xi D_3^2$ and $D_\xi D_3^2 l_i$ are determined by the manner in which the transversal derivatives of the velocity vary as we move into the region of non-uniform flow, and therefore it is natural to expect these quantities to be controlled by the boundary conditions. It is plausible that they are of the order of $D_\xi\omega$, unless $D_\xi\omega$ has strong wavy variations in the transversal direction.

The assumption $n_i D_\xi e_i^3 = O(\varepsilon)$ implies $D_\xi e_i^3 = O(\varepsilon)$, because $e_i^3 D_\xi e_i^3 = 0$, $l_i D_\xi e_i^3 = -e_i^3 D_\xi l_i$, and it will be shown later that $D_\xi l_i = O(\varepsilon)$. Therefore, since the range of ξ is $O(\delta/\varepsilon)$, any change of l_i and e_i^3 in the outer direction must be $O(\delta)$. This in turn implies that the same is true for n_i, e_i^1, e_i^2 and b_i. On the other hand, any surface of discontinuity for the derivatives of the velocity must belong to $\mathfrak{F}$. Therefore, in the case of a wing W, any line along which the normal curvature in the stream direction of W is discontinuous must be almost parallel to the leading edge.

9. Tangential derivatives in a velocity perturbation

It will be shown now that the assumptions (8.1) and (8.2) imply that

$$D_2\omega,\ D_2 l_i,\ D_3\omega \quad \text{and} \quad D_3 l_i \text{ are } O(\delta), \tag{9.1a}$$

$$[2,3]\omega,\ [2,3]l_i,\ [2,\xi]\omega,\ [2,\xi]l_i,\ [3,\xi]\omega \quad \text{and} \quad [3,\xi]l_i \text{ are } O(\delta). \tag{9.1b}$$

Recall that (8.1) implies $\operatorname{tg}\mu$, $\cot\mu$, $\sec\mu$ and $\operatorname{cosec}\mu$ are bounded; this will be used repeatedly without further mention. Since the range of ξ is $O(\delta/\varepsilon)$,

$$D_3\omega = O(\delta), \qquad D_3 l_i = O(\delta) \tag{9.2a}$$

$$D_3^2\omega = O(\delta), \qquad D_3^2 l_i = O(\delta) \tag{9.2b}$$

follows from (8.2). That $D_2\omega$ and $D_2 l_i$ are $O(\delta)$ follows, by (8.2d), from repeated application of the following

Lemma. *If $D_2\omega$, $D_2 l_i$ and $e_i^3 D_2 D_3 l_i$ are $O(\eta)$, where $\eta(\delta)$ is a bounded function, then they are in fact $O(\eta\delta/\varepsilon)$ or $O(\delta)$.*

To prove this lemma we first use (3.6) to show that

$$D_\xi l_i = O(\varepsilon) \tag{9.3}$$

by virtue of (9.2a) and (8.2). Secondly, we show that

$$D_\xi e_i^\alpha,\ D_\xi n_i,\ D_\xi b_i \quad \text{are } O(\varepsilon), \tag{9.4a}$$

$$D_2 e_i^\alpha,\ D_2 n_i,\ D_2 b_i \quad \text{are } O(\eta), \tag{9.4b}$$

$$D_3 e_i^\alpha,\ D_3 n_i,\ D_3 b_i \quad \text{are bounded.} \tag{9.4c}$$

The vectors e_i^α, n_i and b_i are determined by l_i and e_i^3 through the algebraic relations (2.2), (2.4) and (2.8), and in these relations the coefficients are functions of ω only. Now $D_\xi \omega$, $D_\xi l_i$ are $O(\varepsilon)$ by (8.2) and (9.3), $D_3\omega$, $D_3 l_i$ are $O(\delta)$ by (9.2), and we are assuming $D_2\omega$ and $D_2 l_i$ are $O(\eta)$, so (9.4) may be established by showing that $D_\xi e_i^3$ is $O(\varepsilon)$, $D_2 e_i^3$ is $O(\eta)$ and that $D_3 e_i^3$ is bounded. By (5.5) and (9.2a),

$$D_2 e_i^3 = O(\delta), \tag{9.5}$$

and since the equipotential system is a basis,

$$D_3 e_i^3 = (l_j D_3 e_j^3) l_i + (n_j D_3 e_j^3) n_i,$$
$$D_\xi e_i^3 = (l_j D_\xi e_j^3) l_i + (n_j D_\xi e_j^3) n_i.$$

Here $n_i D_\xi e_i^3$ and $l_i D_\xi e_i^3 = -e_i^3 D_\xi l_i$ are $O(\varepsilon)$, by (8.2) and (9.3), $l_i D_3 e_i^3 = -e_i^3 D_3 l_i$ is $O(\delta)$ by (9.2a), and (9.4) may therefore be established by showing that $n_i D_3 e_i^3$ is bounded. Indeed,

$$n_i D_3 e_i^3 = -e_i^3 D_3 n_i = -\operatorname{tg}\mu\, e_i^3 D_3 l_i + \sec\mu\, e_i^3 D_3 e_i^1,$$

by (2.4b), and since $e_i^3 D_3 e_i^1 = -e_i^1 D_3 e_i^3$ is bounded by (8.2), it follows that $n_i D_3 e_i^3$ is bounded.

Thirdly, we note that

$$D_2 h_\xi \quad \text{and} \quad D_3 h_\xi \quad \text{are bounded,} \tag{9.6}$$

which follows from (4.7) by virtue of (9.4) and the assumed boundedness of h_ξ. Fourthly, we show that

$$[2,3]\omega,\quad [2,3]l_i,\quad [2,\xi]\omega,\quad [2,\xi]l_i,\quad [3,\xi]\omega,\quad [3,\xi]l_i \quad \text{are} \quad O(\eta). \tag{9.7}$$

According to the results of Section 4, the commutators are linear combinations of derivatives in directions tangential to the members of $\mathfrak{F}$, and such derivatives of ω and l_i are bounded, by (8.2a) and (9.2a). Moreover, the coefficients in these linear combinations are bounded — for [2, 3] this follows from (4.2), (4.3), (9.4); and for [2, ξ] and [3, ξ], it follows from (4.6), (9.4), (8.2a) and (9.6) — and so (9.7) is established.

Observe that, since the quantities considered in (9.1b) are such linear combinations of those considered in (9.1a), and since they satisfy (9.7), the lemma will also serve to prove (9.1b) as a corollary of (9.1a).

Finally, we show from the characteristic equations (3.2) and (3.7) that $D_2\omega = O(\eta\,\delta/\varepsilon)$. Indeed, by (3.2) and (9.2a),

$$D_2\omega - n_i D_2 l_i = O(\delta). \tag{9.8}$$

On the other hand, if D_2 be applied to (3.7), there results

$$D_\xi(D_2\omega + n_i D_2 l_i) = D_2(h_\xi \sin\mu\, e_i^3 D_3 l_i) - [2,\xi]\omega - n_i[2,\xi]l_i +$$
$$+ (D_\xi n_i) D_2 l_i - (D_2 n_i) D_\xi l_i = O(\eta) + O(\varepsilon),$$

by (9.6), (8.2), (9.4), (9.7) and (9.3). Integrating this equation from Σ_0, we obtain

$$D_2\omega + n_i D_2 l_i = O(\eta\,\delta/\varepsilon) + O(\delta) = O(\eta\,\delta/\varepsilon),$$

and therefore by (9.8)

$$D_2\omega = O(\eta\,\delta/\varepsilon),$$

which also implies

$$D_2 l_i = O(\eta\,\delta/\varepsilon)$$

by (3.3) and (9.2a). This in turn implies that $[2, 3]\,l_i = O(\eta\,\delta/\varepsilon)$, and therefore applying $e_i^3 D_3$ to (3.3),

$$\begin{aligned} e_i^3 D_2 D_3 l_i &= e_i^3 D_3 D_2 l_i + O(\eta\,\delta/\varepsilon) \\ &= (D_2\omega - \sin\mu\, e_j^3 D_3 l_j)\, e_i^3 D_3 n_i + D_3(e_j^2 D_3 l_j + \sin\mu\, D_3\omega) \end{aligned}$$

because $n_i e_i^3 = 0$ and $e_i^3 e_i^3 = 1$; but $D_3 n_i$, $D_3 e_i^2$ are bounded by (9.4c), $D_3 l_j$, $D_3\omega$, $D_3^2 l_j$, $D_3^2\omega$ are $O(\delta)$ by (9.2), and we have just shown that $D_2\omega$ is $O(\eta\,\delta/\varepsilon)$, so that

$$e_i^3 D_2 D_3 l_i = O(\eta\,\delta/\varepsilon) + O(\delta),$$

and the proof of the Lemma is complete.

For later reference, we show that

$$D_2 e_i^\alpha, \quad D_2 n_i, \quad D_2 b_i \quad \text{are} \quad O(\delta) \tag{9.9}$$

by recalling that the vectors e_i^α, n_i, b_i, are given in terms of l_i and e_i^3 by means of a set of algebraic relations whose coefficients are functions of ω only. Since $D_2\omega$ and $D_2 l_i$ are $O(\delta)$ by (9.1) and $D_2 e_i^3$ is $O(\delta)$ by (9.5), (9.9) follows. From (2.4b) and (9.2), moreover,

$$D_3 n_i = \operatorname{cosec}\mu\, D_3 e_i^2 + O(\delta). \tag{9.10}$$

10. The acceleration on the members of $\mathfrak{F}$

The results of Section 6 will now be extended to the region of non-uniform flow. Corresponding to (6.1), we have

$$D_\xi l_i = -n_i D_\xi \omega + O(\delta) \tag{10.1}$$

by (3.6) and (9.1a), so that we shall again need to determine only $D_\xi\omega$ and h_ξ in order to know the acceleration, because the tangential derivatives of the velocity are $O(\delta)$ on the members of $\mathfrak{F}$.

First, apply D_2 to the characteristic equation (3.7), and interchange directional derivatives by means of (9.1b) to obtain

$$D_2 D_\xi \omega + n_i D_2 D_\xi l_i = h_\xi \sin\mu\, e_i^3 D_3 D_2 l_i + O(\delta),$$

in view of (9.9). This equation reduces to

$$D_2 D_\xi \omega + n_i D_2 D_\xi l_i = O(\delta), \tag{10.2}$$

because application of D_3 to (3.3) shows that $e_i^3 D_3 D_2 l_i = O(\delta)$, by (9.1a) and (9.2b). Next, apply D_ξ to (3.2), and interchange directional derivatives by means of (9.1b) to obtain

$$D_2 D_\xi \omega - n_i D_2 D_\xi l_i = \sin\mu\, e_i^3 D_3 D_\xi l_i + O(\delta)$$

by virtue of (9.1a). Then use (3.6) to show

$$D_2 D_\xi \omega - n_i D_2 D_\xi l_i = -\sin\mu\,(e_i^3 D_3 n_i)\, D_\xi l_i + O(\delta)$$

by (9.1 a) and (9.2b). This equation reduces to

$$D_2 D_\xi \omega - n_i D_2 D_\xi l_i = -\Omega D_\xi \omega + O(\delta), \tag{10.3}$$

by virtue of (9.10) and (4.3), and implies, together with (10.2),

$$D_2 D_\xi \omega + \frac{\Omega}{2} D_\xi \omega = O(\delta) \tag{10.4}$$

which corresponds to (6.2).

Moreover,

$$D_2 \Omega + \Omega^2 = O(\delta) \tag{10.5}$$

follows from the fact that (4.3), (9.9) and (4.2) imply

$$D_2 \Omega = e_i^3 D_2 D_3 e_i^2 + O(\delta) = e_i^3 D_3 D_2 e_i^2 - (\Omega')^2 - \Omega^2 + O(\delta),$$

where Ω' is $O(\delta)$ by (4.3) and (9.9), and $e_i^3 D_3 D_2 e_i^2$ is $O(\delta)$ by (5.3), in view of (9.1 a) and (9.2b). From (10.5) and (10.4),

$$\Omega^{-1} = s + O(\delta s_2), \tag{10.6}$$

$$D_\xi \omega = \sigma |s|^{-\frac{1}{2}} + O(\delta s_2), \tag{10.7}$$

where s_2 is the length along bicharacteristics measured from the point where the data are prescribed.

To compute the metric coefficient h_ξ, observe that (4.7a) reduces to

$$D_2 h_\xi = \operatorname{cosec} 2\mu\, e_i^1 D_\xi e_i^2 + O(\delta),$$

by (9.9), and it follows from (6.6), (10.1) and (2.4a) that

$$D_2 h_\xi = \operatorname{cosec} 2\mu (1 - d\mu/d\omega) D_\xi \omega + O(\delta). \tag{10.8}$$

By (10.7), therefore,

$$h_\xi = h_0 \pm 2m\sigma |s|^{\frac{1}{2}} + O(\delta s_2^2) \tag{10.9}$$

where the sign is again given by that of s, as in Section 6.

11. Conclusions

The Equivalence Principle of Section 7 holds, to a first approximation, along the bicharacteristics of the family $\mathfrak{F}$, by virtue of (10.6), (10.7), (10.9) and (10.10).

Equation (10.7) suggests that the centers of symmetry of Σ_0 may be continued by a surface of the same type of singularities in R. Since our theory breaks down near centers of symmetry, this point remains open. It is certainly true for some special types of flow, for instance axially symmetrical flows, for which the axis of symmetry is a line of singularities in any region of non-uniform flow.

At the points where $h_\xi = 0$, a limit surface must be expected, and the present scheme can be applied at such singularities if the family $\mathfrak{F}$ is properly chosen. The results of Section 10 can then be used, by letting h_ξ change sign, to compute a first approximation to the multivalued solution.

Perhaps the most interesting result of the present work is the first-order approximation to the flow which can be deduced. We shall illustrate the method by obtaining an approximation to the flow in a thin region next to, and downstream of a uniform flow when a stream surface W is prescribed. We treat first the case when the stream surface joins the uniform flow smoothly. Let C_0 denote the line on which the stream surface joins the uniform stream, and assume that C_0 is supersonic. Here as in what follows a curve is said to be supersonic if it makes an angle with the velocity greater than μ. This assumption is made in order to assure the existence of a characteristic surface passing through C_0. Assume also that a part of W adjacent to C_0 has non-small curvature, and restrict attention to a portion of it which lies within a short distance along streamlines from C_0. Let $\mathfrak{F}_0$ be a family of non-intersecting supersonic curves C covering W, and $\mathfrak{F}$ the family of characteristic surfaces passing through them. It is plausible, and the author has indeed shown, that h_ξ and $D_\xi \omega$ may be computed on W, except for terms $O(\delta)$, when the assumptions (8.1) and (8.2) are satisfied. The results of Section 10 then allow us to compute, h_ξ and $D_\xi \omega$ in R as a function of the length along bicharacteristics. To obtain the velocity, we can integrate $D_\xi \omega$ and (10.1) with respect to ξ in the outer direction. To obtain the corresponding position, we can integrate $D_\xi x_i = h_\xi b_j \partial x_i / \partial x_j = h_\xi b_i$ with respect to ξ observing that the change of b_i in the outer direction is $O(\delta/\varepsilon)$ by (8.2). Within this order of approximation, R may be decomposed in as many subregions as surfaces of discontinuity of the acceleration occur in R, and for integration in the outer direction, h_ξ and $D_\xi \omega$ need be recomputed only when we cross a surface of discontinuity. By proceeding in this manner, the acceleration is obtained except for terms of $O(\delta)$, the velocity except for terms of $O(\delta^2/\varepsilon)$ and the corresponding position except for terms of $O(\delta^2/\varepsilon^2)$.

When W does not join the uniform stream smoothly, C_0 is a limit line and $h_\xi = 0$ there. Corresponding to C_0, we then have infinitely many values of ξ, and the method described above can be used to compute the flow in the region next to the uniform flow, which in this case is similar to a Prandtl-Meyer expansion.

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