Journal of the
ENGINEERING MECHANICS DIVISION
Proceedings of the American Society of Civil Engineers

ON A KIND OF HYSTERETIC DAMPING

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SYNOPSIS

Slightly nonlinear materials and structures with purely hysteretic damping, subjected to harmonic disturbances are studied. It is shown that a load-deformation or stress-strain relationship $P = A\,x^{\alpha}$ on first loading (in which P is the stress, x denotes strain and A and α are material constants) supplemented by a rule to obtain the curves on unloading and reloading, leads to a behavior very near that of a linear viscoelastic material, but the equivalent damping ratio does not depend on frequency or amplitude. The relations between P and x in unloading and reloading can be obtained from that for first loading in accordance with a rule developed by Masing and studied by Housner and Jennings. It is contended that some materials thought of as viscoelastic actually possess this type of hysteretic behavior. Expressions are included for equivalent viscous damping and for the period of free vibration.

Note.—Discussion open until January 1, 1965. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 90, No. EM4, August, 1964.

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INTRODUCTION

It is known that for a large class of materials and structures, over a wide range of frequencies, the degree of damping depends little on the frequency and on the amplitude of vibrations.[4,5,6,7,8] Their behavior in this range is known as that of constant Q. The load-deflection curves of these materials and structures are independent of the rate of loading, within the range in question, and can be determined from static tests.

Many materials behave, apparently, as linearly viscoelastic over a wide range of strains. However, independence of the degree of damping from the frequency implies a creep function inversely proportional to time. This is inadmissible since it can be shown that this function in turn implies that deformation of an initially undisturbed material introduces a nonzero state of stress in the past.

On the other hand, purely hysteretic materials under harmonic disturbance absorb an amount of energy, per cycle, that depends only on the amplitude, not on the frequency. In general, the corresponding equivalent degree of viscous damping is also a function of amplitude, but a relationship can be found, as done herein, for which the degree of equivalent viscous damping is independent of the amplitude of vibration. It is natural, therefore, to advance the assumption that the actual behavior of these apparently linear viscoelastic materials is, in fact, mildly hysteretic and of the type treated herein.

It is the purpose of this paper to establish mildly nonlinear load-deformation or stress-strain curves that give an equivalent degree of damping which is independent of amplitude and of frequency under steady-state sinusoidal oscillations. With suitable restrictions the principle of superposition will give a satisfactory approximation to the behavior of the present type of constant-Q materials and structures, as the degree of nonlinearity is small over a wide range of deformations.

Results derived here apply to continuous as well as to discrete systems with any number of degrees of freedom provided certain restrictions are met. For the sake of simplicity the theory will be developed for systems having a single degree of freedom, and generalizations will be described subsequently.

The main object in establishing load-deformation curves of this type is to allow the characteristics of slightly nonlinear structures that have constant-Q damping to be set down in a simple manner so that their responses to a number of disturbances can be computed, for example in a Monte Carlo

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3 Mindlin, R. D., Stubner, F. N., and Cooper, H. L., "Response of Damped Elastic Systems to Transient Disturbances," Bell System Technical Publication Monograph B-1561, 1945.

4 Alford, J. L., and Housner, G. W., "A Dynamic Test of a Four-Story Reinforced Building," Bulletin of the Seismological Society of America, Vol. 43, No. 1, January, 1953.

5 Knopoff, L., "The Seismic Pulse in Materials Possessing Solid Friction, 1: Plane Waves," Bulletin of the Seismological Society of America, Vol. 46, No. 3, July, 1956, pp. 175-183.

6 Hunter, S. C., "Viscoelastic Waves," Progress in Solid Mechanics, edited by I. N. Sneddon, Vol. 1, Chapter 1, North Holland Publishing Co., Amsterdam, 1960.

7 Herrera, I., Rosenblueth, E., and Rascón, O. A., "Earthquake Spectrum Prediction for the Valley of Mexico," 3rd World Conf. on Earthquake Engineering, New Zealand, 1964.

8 Kimball, A. L., and Lovell, D. E., Physical Reviews, Vol. 30, 1927, pp. 948-960.

type of analysis. It is true that essentially the same can be achieved by stipulating the correct dashpot constants, and methods are available to compute the constants required to give a specified set of damping ratios.[9] But such methods are not simple and must be applied anew in every specific instance.

Notation.--Letter symbols adopted for use in this paper are defined where they first appear and listed alphabetically in the Appendix.

TYPE OF LOAD-DEFORMATION CURVES

Following Masing[10] it shall be assumed that the load-deformation curves of all flexible elements to be dealt with are given by

$$\frac{P_0 - P}{2} \quad F\left(\frac{x_0 - x}{2}\right) \qquad (1)$$

in unloading, and by

$$\frac{P - P_0}{2} = F\left(\frac{x - x_0}{2}\right) \qquad (2)$$

in reloading, in which $P = F(x)$ is the load-deflection relation for first loading and x_0, P_0 are the displacement and load at which the direction of loading was reversed. Inconsistencies are obviated by specifying, as done by Jennings,[11] that if a curve defined by Eq. 1 or Eq. 2 intersects the curve described by a previous loading or unloading process, respectively, the P(x) curve follows one defined in the earlier process (see Fig. 1).

These relations closely resemble many experimental results and include several common idealizations (such as linear and elastoplastic behavior) as special cases. The curves are characterized by the fact that the initial stiffnesses on first loading, on unloading, and on reloading are equal to each other.

THE NONLINEAR SYSTEM

The problem of specifying a linear system equivalent to a nonlinear one will be approached by treating the latter as quasilinear: It shall be assumed that under an external force which varies in proportion to sin ω_1 t, where ω_1 is the undamped natural circular frequency of the linear system and t = time,

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[9] Berg, F. V., "Finding System Properties from Experimentally Observed Modes of Vibration," Primeras Jornadas Argentinas de Ingenieria Antisismica, San Juan, Argentina, April, 1962.

[10] Tananbashi, R., and Kaneta, K., "On the Relation Between the Restoring Force Characteristics of Structures and the Pattern of Earthquake Ground Motion," Proceedings of Japan, Natl. Symposium on Earthquake Engineering, Tokyo, Japan, 1962.

[11] Jennings, P. C., "Periodic Response of a General Yielding Structure," Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM2, Proc. Paper 3871, April, 1964, pp. 131-166.

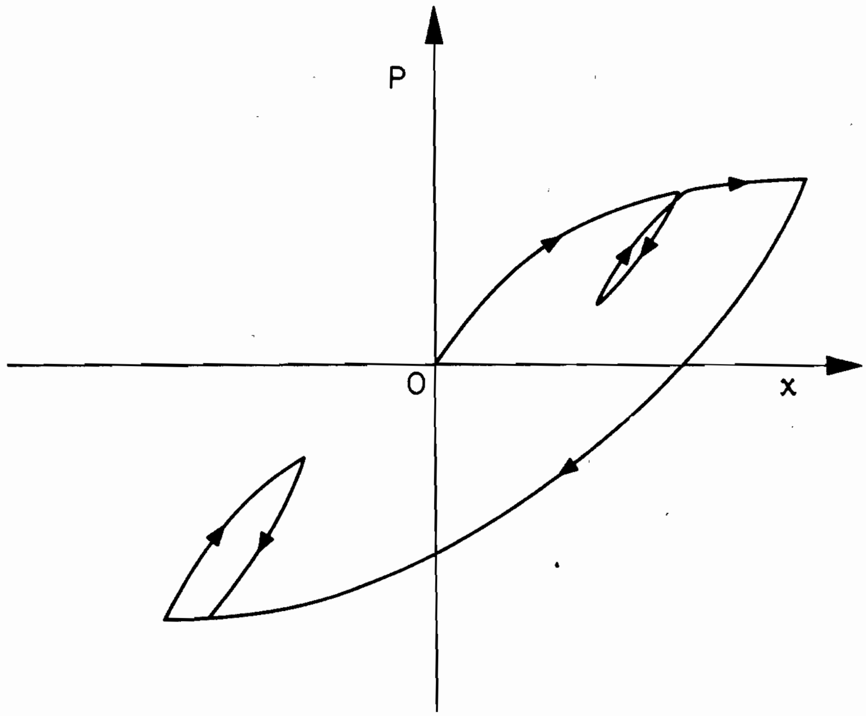

FIG. 1.—GENERAL LOAD-DEFORMATION RELATIONSHIP

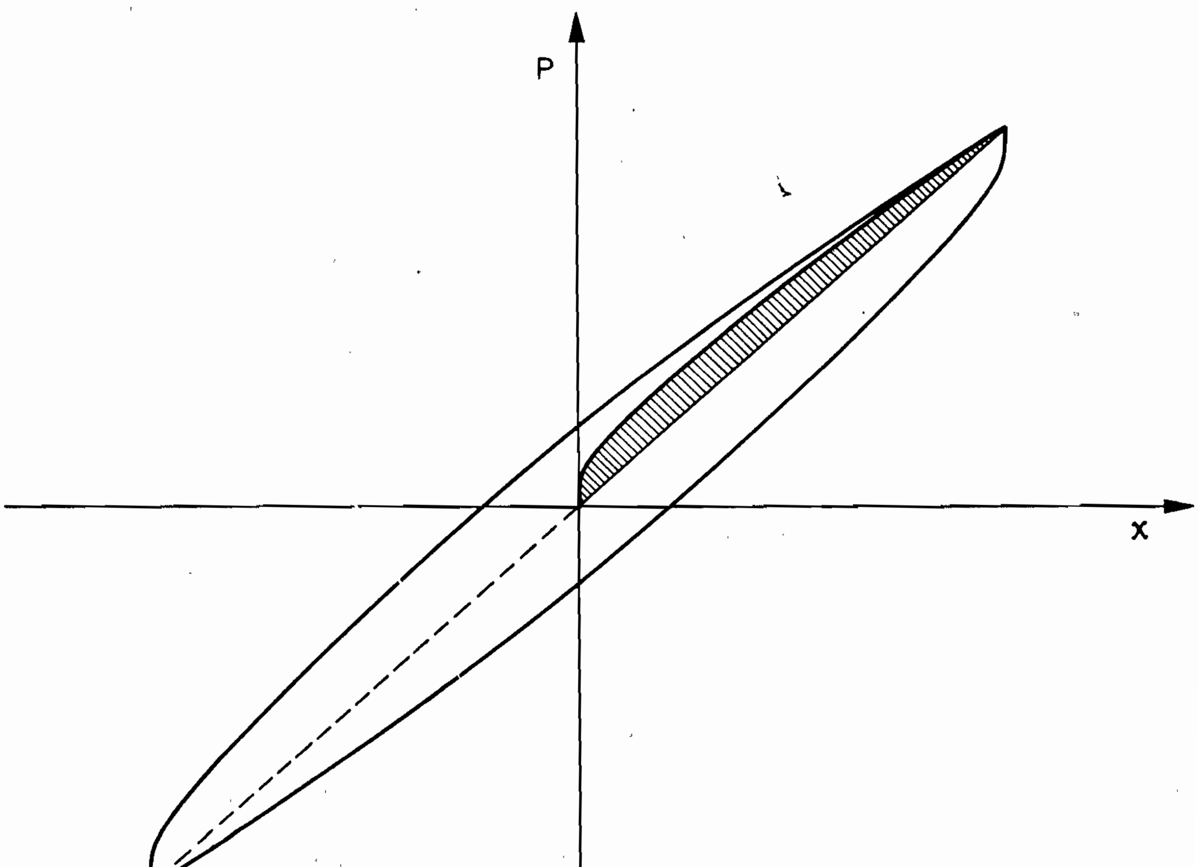

FIG. 2.—CALCULATION OF ENERGY DISSIPATED PER CYCLE

the response of the nonlinear system is proportional to $\sin \omega_1 (t - t_1)$, in which t_1 is a time shift.

In steady state harmonic oscillations of frequency ω_1 the energy absorbed per cycle by a viscously damped linear system is[12]

$$W = 2 \pi \zeta x^2 \omega_1 \sqrt{K M} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

in which ζ is its percentage of damping, x its maximum displacement, K the stiffness, and M the mass. The equivalent stiffness and viscous damping of the nonlinear system may be defined in a variety of ways. The equivalent stiffness will be defined in such a manner that the corresponding load-deformation relation of the linear system, under static loading, pass through the extreme points of the load-deformation graph of the nonlinear system. The equivalent viscous damping is such that it minimizes the mean squared error deficiency term in the equation of motion when a solution of the form $x \cos \omega t$ is assumed. Then, the absorbed energy may be put in the form

$$W = 2 \pi \zeta x P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

when $\omega_1^2 = K/M$, because $K = P/x$.

The foregoing definition of equivalent viscous damping implies that the energy lost per cycle by the linear system equals the energy lost per cycle by the nonlinear one. Now, $\omega_1^2 = K/M$ so that the absorbed energy may be put in the form of Eq. 4.

The last expression for W also gives the energy lost per unit volume in a constant-Q viscoelastic solid (one having the same degree of damping at all frequencies of vibration) if P is replaced with stress and x with strain. The generalization from single-degree systems to viscoelastic materials will be treated later more fully.

In the nonlinear system the energy lost per cycle will be the hysteresis area. Under the assumptions made this area is eight times the shaded area in Fig. 2, or

$$W = 8 \int_0^x P(x')\, dx' - 4 Px \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Equating both expressions yields

$$(2 + \pi \zeta)\, x P(x) = 4 \int_0^x P(x')\, dx' \quad \ldots\ldots\ldots\ldots\ldots \quad (6)$$

Differentiating with respect to x, a differential equation results whose solution is

$$P = A x^{\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

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12 Timoshenko, S., "Vibration Problems in Engineering," D. Van Nostrand Co. Inc., New York, N. Y., 1956, 3rd Edition.

in which A is a constant and

$$\alpha = \frac{2 - \pi \zeta}{2 + \pi \zeta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

This expression has meaning only if $\zeta < 0.636$. To obviate difficulties that occur when α is not equal to the ratio of two odd integers, Eq. 7 should be read

$$|P| = A|x|^{\alpha}, \ \mathrm{sig}\ P = \mathrm{sig}\ x \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

When α is near unity Eq. 7 may be approximated by

$$P \cong A\,x(1 - \beta \log x) \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

in which $\beta = 1 - \alpha$. In view of this equation it is clear that when β is small the force-deformation relation is nearly linear for a wide range of x.

The solution is independent of M, while the equivalent stiffness and A are related by

$$A = K\,x^{1 - \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

Fig. 3 shows hysteresis loops for various percentages of damping. Very accurate measurements taken in a large triaxial machine show that confined sands, gravels, and broken rock behave as described by Eqs. 1, 2 and 7 almost up to failure if P is replaced with stress and x with unit strain.[13] (Of course the origin corresponds to a finite axial compression for these materials.) Marsal has deduced Eq. 7 from statistical mechanics of cohesionless granular materials.[13]

It is not difficult to show that whenever hysteretic behavior arises entirely from Coulomb friction between constituent portions of a material or structure the overall stress-strain or force-deformation unloading and reloading curves follow Masing's criterion and can be constructed from the curve for first loading, as described in this paper, provided the behavior of individual consistuent elements is linearly elastic; these elements may be soil grains, crystals, or structural members with Coulomb friction at their connections.

Such is very nearly the case with a sphere pressed against a flat surface, for example, and subjected to tangential forces if the material of which the sphere is made has negligible internal damping as compared with the hysteretic damping due to sliding at the surfaces of contact. Since the contact-stress distribution is nonuniform and the contact pressure is radially symmetric, sliding takes place along an annulus if the tangential forces applied are smaller than the normal force times the coefficient of friction. A theory describing this phenomenon is available[14] and has been well substantiated by tests with a steel possessing negligible internal damping.[15]

Force-displacement curves for a steady-state harmonic disturbance do show the type of hysteretic damping anticipated but the shape of these curves

[13] Marsal, R. J., Report to the Inst. of Engrg. Natl. Univ. of Mexico, Mexico City, Mexico, 1963.

[14] Mindlin, R. D., and Dresceiwicz, H., "Elastic Spheres in Contact under Varying Oblique Forces," Journal of Applied Mechanics, Vol. 20, 1953.

[15] Goodman, L. E., and Brown, C. B., "Energy Dissipation in Contact Friction: Constant Normal and Cyclic Tangential Loading," Journal of Applied Mechanics, Vol. 29, 1962.

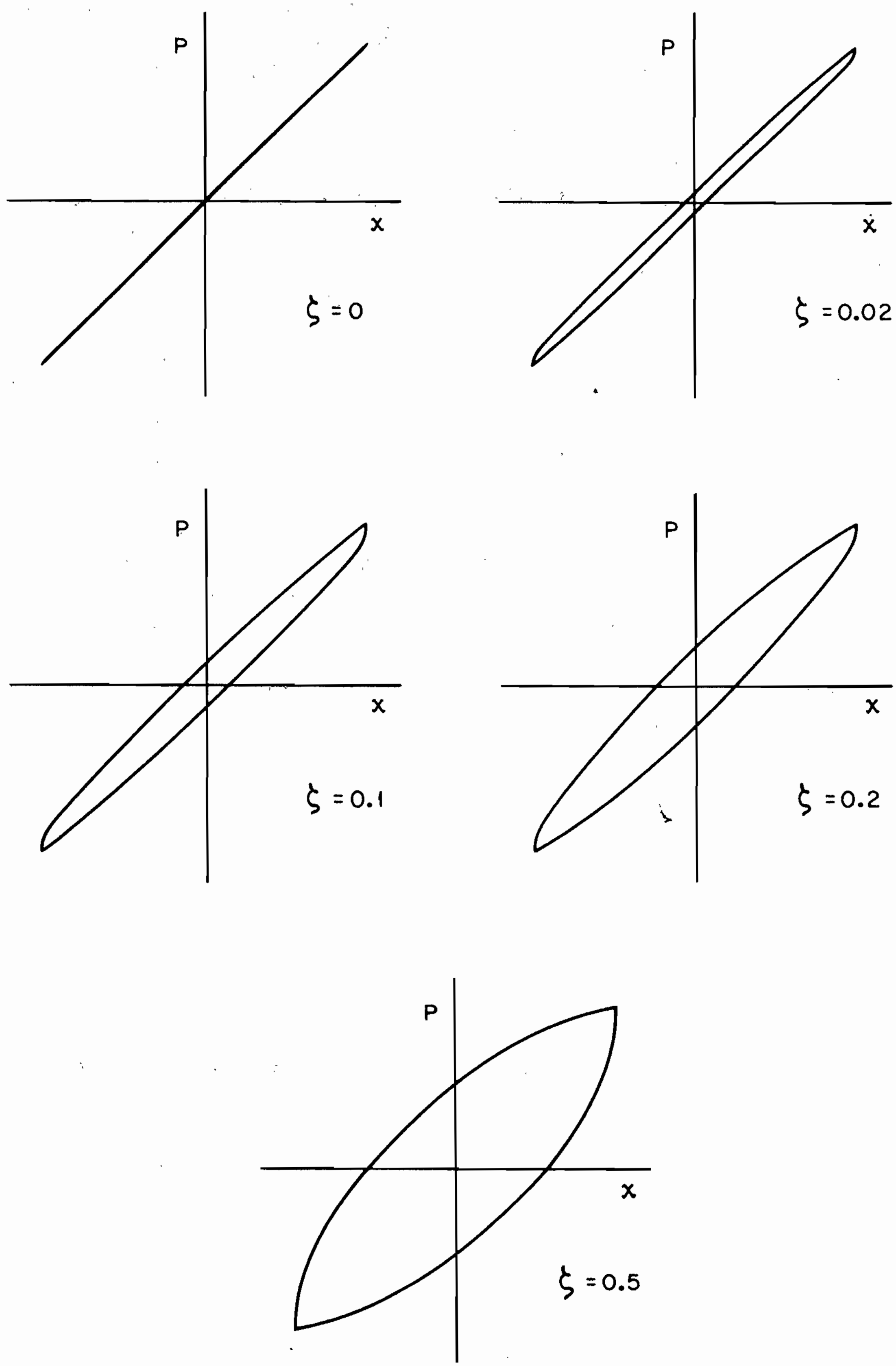

FIG. 3.—HYSTERESIS LOOPS

differs from the exponential relationship involved in Eq. 7. The energy dissipated per cycle in those tests, divided by the product Px, increases rapidly with the amplitude of oscillation in accordance with the theoretical prediction[14] while it would be independent of the amplitude if Eq. 7 were applicable. In similar tests with glass spheres the dependence of W/Px on P was less pronounced, apparently due to the influence of internal damping[15] but it cannot be stated that Eq. 7 described these phenomena adequately.

Under other conditions certain experiments have substantiated a specific damping function of the exponential type for some materials.[16] Other test results[3,4,5,6,7] lend support to the assumption that under a wide range of parameters many materials exhibit a behavior that is adequately described by the stress-strain curves treated in this paper. A much wider class of materials, including granular soils, have a different stress-strain curve on first loading and have a different behavior in what concerns damping, but their unloading and reloading curves can still be contracted satisfactorily in accordance with the same rules.

The infinite slope of the P(x) curve at $x = 0$ is objectionable on many counts, including the infinite velocity of small amplitude waves that it implies. Clearly, Eq. 7 can only be accepted as an idealization valid within a certain range of variables and for the calculation of certain phenomena. If the P(x) curve is changed in the neighborhood of the origin, say in the interval $|x| \leq x_1$, ζ is still found to be independent of ω_1 and of the amplitude for amplitudes greater than x_1 if Eq. 7 is retained for $|x| > x_1$. Smaller degrees of damping are found at smaller amplitudes.

It is of interest to write explicit formulas for the case of free vibration. Let the system be given a deformation $x_0 (> 0)$ and released. Eqs. 1 and 7 state that its equation of motion will be

$$M\ddot{x} = P_0 - 2^{1-\alpha} A \left(x_0 - x\right)^{\alpha} \quad \ldots \ldots \ldots \ldots \ldots \quad (12)$$

Let $y = x_0 - x$ and integrate between zero and some value of y

$$-M\dot{y}^2 = 2\,P_0\,y - 2^{2-\alpha} A\,(1+\alpha)^{-1} y^{1+\alpha} \quad \ldots \ldots \ldots \quad (13)$$

The system reverses its direction of motion when $\dot{y} = 0$ and $y = y_1$, say. Let P_1 denote the corresponding spring force. Then

$$P_0 = 2^{1-\alpha} A (1+\alpha)^{-1} y_1^{\alpha} \quad \ldots \ldots \ldots \ldots \ldots \quad (14)$$

But according to Eq. 1

$$P_1 = P_0 - 2^{1-\alpha} A\, y_1^{\alpha} = -\alpha\, P_0 \quad \ldots \ldots \ldots \ldots \quad (15)$$

Inductively, then, the $\underline{n}^{th}$ reverse in motion takes place when the spring force is

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16 Lazan, B. J., "Damping and Resonant Fatigue Behaviour of Materials, Conf. on Fatigue of Metals, Institution of Mech. Engrs. and Amer. Soc. of Mech. Engr., 1956.

$$P_n = (-\alpha)^n \, P_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

This result should be compared with that of a quasilinear treatment, according to which the deformations as well as the spring forces in successive half cycles are in the ratio $-\exp\left[-\pi\,\zeta\left(1-\zeta^2\right)^{1/2}\right]$ rather than in the ratio $^-\alpha$. The two ratios are nearly equal when $\zeta << 1$, as they differ only by terms of the order of ζ^3. For $\zeta = 0.1$ for example, the viscous-damping decrement ratio is 0.731 while $\alpha = 0.7285$.

Now, from Eq. 14

$$y_1 = \left[\frac{(1+\alpha)\, P_0}{2^{1-\alpha}\, A}\right]^{1/\alpha}$$

$$= 2\, x_0 \left(\frac{1+\alpha}{2}\right)^{1/\alpha} \quad \ldots\ldots\ldots\ldots\ldots \quad (17)$$

so that the deformation at the $\underline{n}^{th}$ reverse in motion is

$$x_n = x_0 \left[1 - 2\left(\frac{1+\alpha}{2}\right)^{1/\alpha} \sum_{i=0}^{n-1} \left(-\alpha^{1/\alpha}\right)^n\right] \quad \ldots\ldots \quad (18)$$

Hence, when the system comes to rest it has a permanent deformation

$$x_\infty = x_0 \left[1 - \frac{2}{1+\alpha^{1/\alpha}} \left(\frac{1+\alpha}{2}\right)^{1/\alpha}\right] \quad \ldots\ldots\ldots\ldots \quad (19)$$

For $\zeta << 1$ Eq. 19 gives

$$x_\infty / x_0 \doteq \left[\frac{(1-\alpha)}{2}\right]^3 \doteq \left(\frac{\pi\,\zeta}{2}\right)^3 \quad \ldots\ldots\ldots\ldots \quad (20)$$

With $\zeta = 0.0425$, $\alpha = 7/8$, the asymptotic expression gives $x_\infty / x_0 = 0.00030$ while the exact value is 0.00036.

Integrating Eq. 13 and writing P_n for P_0, the $\underline{n}^{th}$ half period of free vibrations is found to be

$$\frac{T_n}{2} = \int_0^{y_{n+1} - y_n} \left[\frac{2\, P_n\, y}{M} - \frac{2^{2-\alpha}\, A\, y^{1+\alpha}}{(1+\alpha)\, N}\right]^{-1/2} dy \quad \ldots\ldots \quad (21)$$

Through a suitable change of variables it is easily shown that

$$\frac{T_n}{T_0} = \alpha^{(n/2)(1/\alpha - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

Combining with Eq. 16 it is seen that when oscillations have damped down to the extent that the absolute value of the spring force has decreased from P_0 to P, the natural half period has decreased in the ratio $(P/P_0)^{(1/2)(1/\alpha-1)}$. If for example, $P/P_0 = 1/2$ and $\zeta = 0.1$, the half period of free vibrations has decreased to 0.88 of its original value, while with $\zeta = 0.02$ it decreases to 0.978 of the original period.
From Eq. 22 the time required for the structure to come to rest is

$$t = \left(\frac{T_0}{2}\right) \sum_{n=1}^{\infty} \alpha^{(n/2)(1/\alpha-1)}$$

$$= \frac{\frac{T_0}{2}}{1 - \alpha^{(1/2)(1/\alpha-1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

When $\zeta << 1$ Eq. 23 may be written

$$t \doteq T_0 (\pi \zeta)^{-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

If $\zeta = 0.1$ for example, Eq. 23 gives $t = 8.8\ T_0$ whereas the asymptotic approximation gives $t = 10.1\ T_0$. With $\zeta = 0.02$, $t = 254\ T_0$.

GENERALIZATIONS

It is easily shown that the conclusions derived herein apply also to multidegree systems provided all the corresponding spring elements have the same parameter α and provided the system possesses natural modes of vibration in the usual or classical sense.

The results presented also apply to the uniaxial vibrations of continuous media of materials whose behavior is apparently viscoelastic with a percentage of damping independent of vibration. In adapting the expressions derived to these media, P must be replaced with the generalized force for which the medium vibrates uniaxially (torque or longitudinal force in the case of slender bars, shear in a horizontally stratified semispace, etc.), x will represent the corresponding generalized deformation, K the stiffness or appropriated modulus, and A a constant having the proper dimensions. The equivalence does not then require that the forced vibrations take place in one of the natural modes of the system.

The assumption of slightly nonlinear behavior has the advantage over the linear treatment with a creep function 1/t that it dispenses with the spurious strains which the latter introduces for negative time, besides being in better accord with experimental evidence for several materials. Such uniaxial vibrations occur in the longitudinal oscillations and torsion of slender bars and in the transverse oscillations of taut strings, shear beams, and stratified soil formations.

Finally, the same results apply to the three-dimensional vibrations of any continuous system, in that the steady-state behavior of a constant-Q linearly viscoelastic material is similar to that of a mildly nonlinear one having a tensorial stress-strain relation given by Eqs. 1, 2, 7 and 8 with the proper substitutions in nomenclature.

CONCLUSIONS

The following point has been proved herein.

Let $\zeta << 1$ and $\alpha = (2 - \pi\zeta)/(2 + \pi\zeta)$. Consider a system formed by a number of rigid masses interconnected by springs whose load-deformation law on first loading is $P = F(x) = A\,x^{\alpha}$. Each spring may have different A but they all have the same α. On unloading the law is given by Eq. 1 and on reloading it is given by Eq. 2, but if the unloading or the reloading curve intersects that from an earlier cycle in the same sense, the P(x) curve follows the one of the earlier cycle. Define equivalent stiffness as the ratio of maximum spring force to maximum deformation. Then this system, when subjected to steady-state forced vibrations, behaves in nearly linear manner with an equivalent viscous damping equal to ζ in all its natural modes independently of the natural frequency in question and of the amplitude of vibration. Moreover, no other type of function F satisfies these criteria.

Generalization of these results to continuous media of constant-Q materials under steady-state vibrations of arbitrary frequency shows that the nonlinear treatment yields essentially the same behavior as the assumption of linear viscoelastic behavior with creep function $1/t$ but dispenses with inconsistencies that the latter introduces for $t < 0$.

It has also been shown that the periods of successive cycles in free vibration are progressively shorter and the process ends after a finite length of time, leaving a permanent deformation. For $\zeta << 1$, the permanent deformation is approximately equal to the initial deflection times $(\pi\zeta/2)^3$ and the duration of motion is approximately $(\pi\zeta)^{-2}$ times the initial period.

ACKNOWLEDGMENTS

The authors express their gratitude to T. K. Caughey, California Institute of Technology, Pasadena, Calif. for having read the manuscript and made valuable suggestions.

APPENDIX.—NOTATION

The following letter symbols have been adopted for use in this paper:

A = constant in Eq. 3 $(\text{kg cm}^{-\alpha})$;

C = dashpot constant $(\text{kg cm}^{-1}\text{ sec})$;

e = base of natural logarithms (dimensionless);

F = function of deformation (kg);

i = integer (dimensionless);

K = stiffness (kg cm^{-1});

M = mass (kg cm^{-1} sec^2);

n = integer (dimensionless);

P = force (kg) or generalized force;

T_1 = initial natural period of vibration (sec);

T = superscript signifying transposed;

t = time (sec);

W = energy lost per cycle of forced vibrations;

x = deformation (cm) or generalized deformation;

α = exponent in Eq. 3 (dimensionless);

ζ = damping expressed as a fraction of critical (dimensionless); and

ω = natural circular frequency (sec^{-1}).