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ON A METHOD TO OBTAIN A GREEN'S FUNCTION FOR A MULTI-LAYERED HALF SPACE

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ABSTRACT

In this paper the surface wave terms of the Green's function for a two-dimensional multi-layered half space are obtained. The method used is new and remarkable by its simplicity. It is based on the integral representation theorems for elastodynamics. The orthogonality properties of surface waves are generalized to include not only Love waves but Rayleigh waves as well.

Introduction

The Green's function technique (Knopoff 1956 & de Hoop 1958) has recently come to to play a central role in the theory of elastic wave transmission (Hudson and Knopoff 1964, Knopoff and Hudson 1964). On the other hand the author (Herrera 1964) is developing a linearized theory of wave transmission. In this theory, the solution to any problem is given in terms of integrals of known quantities by the Green's function corresponding to the region which is being perturbed. In many cases of geophysical interest the region which is perturbed is a multi-layered half-space. It is therefore very important to have simple means of computing the Green's function for a multi-layered half space.

When one is concerned with problems of surface wave transmission it is usually enough to know the surface wave terms of the Green's function. For the three dimensional problem, Harkrider (1964) has recently given a method to obtain them. Here a method which is very simple, is presented for the two-dimensional problem. All the methods thus far used are based on the evaluation of the residues at the poles of the expressions giving the Green's function. The method presented here has a completely different basis, it is based on the Green's function representation theorem for elastodynamics (Knopoff, 1956, de Hoop, 1958). These theorems give any solution of the equations of elasticity in terms of the Green's functions. When the Green's function is known it allows us to compute any solution in terms of the boundary data. It is shown here, that when the Green's function is unknown but some solutions are known, the integral representation theorem may be used to obtain information regarding the possible shape of the Green's function. In particular when a surface wave solution is known, its contribution to the Green's function is determined directly by the integral representation theorem.

In another paper (Herrera 1964) and in some papers to come the author is presenting a linearized theory of wave transmission. The results presented here allow us to solve any two-dimensional problem of surface wave transmission for which linearized theory is applicable, because for surface waves the results of that theory depend only on the surface wave terms of the Green's function.

In the process of obtaining the main result of this paper an orthogonality property for surface waves is obtained which is a generalization of the known properties, because it includes not only Love waves but Rayleigh waves as well.

The simplicity and ease with which the results are obtained show the power of the type of arguments used in this paper and point towards the possibility of extending these methods to a more general class of problems.

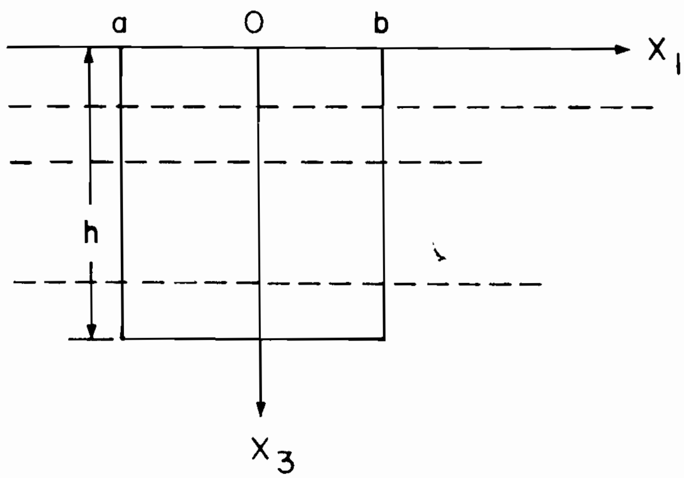
THE SET OF SOLUTIONS CONSIDERED

We shall be concerned with the quasi-steady motion of elastic bodies which is governed by

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho\omega^2 u_i = 0 \tag{1a}$$

$$\tau_{ij} = C_{ijpq} \frac{\partial u_p}{\partial x_q} \tag{1b}$$

where C_{ijpq} is the elasticity tensor, the other symbols having the usual meaning and summation convention has been adopted. We shall consider only the two-dimen-



----- INTERFACES BETWEEN LAYERS (PLANES OF DISCONTINUITY)

FIG. 1. Multilayered half space and integration contour.

sional problems, i.e., we assume that all the quantities in (1) are independent of X_2. The tensor C_{ijpq} will be assumed independent of x_1, but in general it will be assumed a piecewise continuous and differentiable function of x_3 with at most a finite number of jump discontinuities.

The displacements **u** are assumed to have continuous second derivatives except at points where C_{ijpq} or its first derivatives are discontinuous. There

$$[u_i] = 0 \tag{2a}$$

$$[\tau_{3i}] = 0 \tag{2b}$$

where the brackets stand for the jumps when crossing the discontinuities from the upper to the lower side.

We shall consider only solutions of (1a) in the half space $0 \leqq x_3 < \infty$ such that its boundary is stress free, i.e.

$$\tau_{i3} = 0 \quad \text{at} \quad x_3 = 0$$

and which satisfy the additional condition

$$\int_0^\infty |u_i|\, dx_3 < \infty; \qquad \int_0^\infty |\tau_{ij}|\, dx_3 < \infty.$$

On Green's Theorem for Unbounded Regions

Let now **u** and **v** be two solutions of (1) satisfying the conditions stated in the last section. Then it is easily shown (see Appendix) that if R is a bounded region with boundary S

$$\int_S \{v_i \tau_{ij}(\mathbf{u}) - u_i \tau_{ij}(\mathbf{v})\} n_j \, dS = 0 \tag{3}$$

where n_j is the normal vector to S.

Apply (3) to the rectangle (figure 1)

$$a \leqq x_1 \leqq b; \qquad h \geqq x_3 \geqq 0$$

where h is any positive real number, to get

$$\int_0^h \{v_i \tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{v})\}_{x_1=a}\, dx_3$$
$$= \int_0^h \{v_i \tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{v})\}_{x_1=b}\, dx_3 + \int_a^b \{v_i \tau_{i3}(\mathbf{u}) - u_i \tau_{i3}(\mathbf{v})\}_{x_3=h}\, dx_1$$

where the fact that the normal stresses vanish at $x_3 = 0$ has been used.

We now let $h \to \infty$ to get

$$\int_0^\infty \{v_i \tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{v})\}_{x_1=a}\, dx_3 = \int_0^\infty \{v_i \tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{v})\}_{x_1=b}\, dx_3 \tag{4}$$

Since a and b are arbitrary, equation (4) shows that

$$\int_0^\infty \{v_i \tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{v})\}\, dx_3$$

depends only on the couple of solutions **u** and **v** but is independent of the line of integration.

We next compute this integral for the case in which **u** and **v** are surface waves. Thus, assume **u** and **v** are of the form

$$u_i(\mathbf{x}) = U_i^{(n)}(x_3) e^{ik_n x_1}; \qquad v_i(\mathbf{x}) = U_i^{(m)}(x_3) e^{ik_m x_1}$$

where $U_i^{(n)}$ and $U_i^{(m)}$ are function of x_3 only. Then by virtue of (1b)

$$\tau_{ij}(\mathbf{u}) = T_{ij}^{(n)} e^{ik_n x_1}$$

$$\tau_{ij}(\mathbf{v}) = T_{ij}^{(m)} e^{ik_m x_1}$$

where $T_{ij}^{(n)}$ and $T_{ij}^{(m)}$ are only functions of x_3. Therefore

$$\int_0^\infty \{v_i \tau_{ij}(\mathbf{u}) - u_i \tau_{ij}(\mathbf{v})\} n_j \, dx_3 = e^{i(k_n + k_m)x_1} \int_0^\infty \{U_i^{(m)} T_{i1}^{(n)} - U_i^{(n)} T_{i1}^{(m)}\} \, dx_3 \quad (5)$$

Now the integrals of the first and second members are independent of x_1. Therefore, for surface waves,

$$\int_0^\infty \{v_i \tau_{ij}(\mathbf{u}) - u_i \tau_{ij}(\mathbf{v})\} n_j \, dx_3$$
$$= \int_0^\infty \{U_i^{(m)} T_{ij}^{(n)} - U_i^{(n)} T_{ij}^{(m)} n_j \, dx_3 = 0 \quad \text{if} \quad k_n + k_m \neq 0 \quad (6)$$

The representation theorems for elastodynamics given by de Hoop (1958) have been extended by Hudson and Knopoff (1964) to cover the case of a multilayered medium. The extension of this theorem to a medium with not-necessarily homogeneous layers is straightforward (assuming the existence of the Green's function) and a proof is given in the Appendix. It is

$$u_k(x_1, x_3) = \int_S \{G_{ki}(\xi, \mathbf{x}) \tau_{ij}(\mathbf{u}) - u_i(\xi) \tau_{ij}(\mathbf{G}_k)\} n_j \, dS \quad (7)$$

where $\mathbf{G}_k$ is the tensor Green's function $\mathbf{G}_k = (G_{k1}, G_{k2}, G_{k3})$ which satisfies for every k, the equation

$$\frac{\partial}{\partial x_i} \tau_{ij}(\mathbf{G}_k) + \rho\omega^2 G_{kj}(\mathbf{x}, \xi) = -\delta_{jk} \delta(x_1 - \xi_1) \delta(x_3 - \xi_3) \quad (8)$$

together with the continuity conditions (2) at the interfaces.

It is important for the discussion which follows to have information regarding the behavior of $\mathbf{G}_k$ as $x_1 \to \pm\infty$. It is generally thought (although it apparently has never been proved) that the displacements due to a source in a multilayered half space tend to a linear combination of surface waves as $x_1 \to \pm\infty$.

More precisely, we shall assume that if the source is located somewhere in the x_3-axis then

$$\mathbf{G}_k(\mathbf{x}, 0, \xi_3) = \mathbf{B}_k(\mathbf{x}, 0, \xi_3) + \begin{cases} \sum_n a_{nk} \mathbf{U}^{(n)} e^{ik_n x_1} + \sum_n b_{nk} \bar{\mathbf{U}}^{(n)} e^{-ik_n x_1}; & x_1 > 0 \\ \sum_n C_{nk} \mathbf{U}^{(n)} e^{ik_n x_1} + \sum_n d_{nk} \bar{\mathbf{U}}^{(n)} e^{-ik_n x_1}; & x_1 < 0 \end{cases}$$

where $\mathbf{B}_k(\mathbf{x}, 0, \xi_3)$ stands for body waves which are assumed such that they, together with their derivatives, tend to zero uniformly as $x_1 \to \pm\infty$ and the surface waves have been represented by

$$\mathbf{U}^{(n)}(x_3) e^{ik_n x_1}$$

and $\bar{\mathbf{U}}^{(n)}$ is the complex conjugate of $\mathbf{U}^{(n)}$, a_{nk}, b_{nk}, c_{nk}, and d_{nk} are coefficients which are functions of the depth ξ_3 of the source and the sum extends over all possible surface wave solutions (Rayleigh and Love wave solutions). It is assumed above that $k_n \geqq 0$. Observe that if $\mathbf{U}^{(n)}e^{ik_nx_1}$ is a surface wave, then its complex conjugate $\bar{\mathbf{U}}^{(n)}e^{-ik_nx_1}$ is also a surface wave in view of the fact that (1) is a linear equation with real coefficients and the boundary conditions are homogeneous and therefore, given any solution, its complex conjugate is also a solution. However, the first represents a wave moving towards the right while the second represents a wave moving towards the left.

Now

$$\sum_n b_{nk}\bar{\mathbf{U}}^{(n)}e^{-ik_nx_1} + \sum_n c_{nk}\mathbf{U}^{(n)}e^{ik_nx_1}$$

is a linear combination of solutions and therefore it is a regular solution in the whole half space we are considering. Thus, if we subtract this from $\mathbf{G}_k$, we shall obtain a new function which will be itself a Green's function for the half space. This permits us to modify the definition of the Green's function $\mathbf{G}_k$ to be

$$\mathbf{G}_k(\mathbf{x}, 0, \xi_3) = \mathbf{B}_k(\mathbf{x}, 0, \xi_3) + \begin{cases} \sum_n A_{nk}\mathbf{U}^{(n)}e^{ik_nx_1}; \; x_1 > 0 \\ \sum_n C_{nk}\bar{\mathbf{U}}^{(n)}e^{-ik_nx_1}; \; x_1 < 0 \end{cases} \tag{9}$$

where

$$A_{nk} = a_{nk} - c_{nk}; \qquad C_{nk} = d_{nk} - b_{nk} .$$

This Green's function that we shall call the symmetric Green's function, has the advantage of being more symmetrical because to the right of the source there are only advancing surface waves and to the left there are only receding surface waves.

Let now $u_k(x_1, x_3)$ be a surface wave and apply (7) in the rectangle shown in figure 1 for which $a < 0 < b$ and such that contains the point $(0, x_3)$ in its interior. Repeating the reasoning that led previously to equation (4), we obtain now

$$\begin{aligned} u_k(0, x_3) = \int_0^\infty \{G_{ki}(b, \xi_3; 0, x_3)\tau_{i1}(\mathbf{u}) - u_i\,\tau_{i1}(\mathbf{G}_k)\}\, d\xi_3 \\ - \int_0^\infty \{G_{ki}(a, \xi_3, 0, x_3)\tau_{i1}(\mathbf{u}) - u_i\,\tau_{i1}(\mathbf{G}_k)\}\, d\xi_3 \end{aligned} \tag{10}$$

Because of (9) we can write

$$\begin{aligned} u_k(0, x_3) = & \int_0^\infty \{B_{ki}(b, \xi_3; 0, x_3)\tau_{i1}(\mathbf{u}) - u_i\,\tau_{i1}(\mathbf{B}_k)\}\, d\xi_3 \\ & - \int_0^\infty \{B_{ki}(a, \xi_3; 0, x_3)\tau_{i1}(\mathbf{u}) - u_i\,\tau_{i1}(\mathbf{B}_k)\}\, d\xi_3 \\ & + \sum_n A_{nk}(x_3) \int_0^\infty \{\mathbf{U}_i^{(n)}(\xi_3)\tau_{i1}(\mathbf{u}) - u_i\, T_{i1}^{(n)}\}_{\xi_1=b}\, e^{ik_nb}\, d\xi_3 \\ & - \sum_n C_{nk}(x_3) \int_0^\infty \{\bar{\mathbf{U}}_i^{(n)}(\xi_3)\tau_{i1}(\mathbf{u}) - u_i\, \bar{T}_{i1}^{(n)}\}_{\xi_1=a} e^{-ik_na}\, d\xi_3 . \end{aligned} \tag{11}$$

Observe now that although $\mathbf{G}_k$ is not a regular solution of (1) in the whole plane, it is a regular solution for $x_1 > 0$ and for $x_1 < 0$. On the other hand, surface waves are regular solutions on the whole half space and therefore $\mathbf{B}_k(\mathbf{x}, 0, \xi_3)$ is the difference of two regular solutions so that it is regular for $x_1 > 0$ and for $x_1 < 0$. This shows by (4) that the value of the first two integrals is not altered if we let $b \to +\infty$ and $a \to -\infty$, a fact we are going to use to show that both vanish when $\mathbf{u}$ is a surface wave. By the definition of a surface wave, $|\mathbf{u}|$ and $|\tau_{ij}|$ are independent of x_1 and they are integrable from 0 to ∞. Thus, there is a constant $M > 0$, such that

$$\int_0^\infty |u_i| \, d\xi_3 < M \quad \text{and} \quad \int_0^\infty |\tau_{ij}(\mathbf{u})| \, d\xi_3 < M. \tag{12}$$

On the other hand, since $\mathbf{B}_k$ together with its derivatives tend to zero uniformly as $x_1 \to +\infty$, for every $b > 0$ there exists a constant $K(b) > 0$, such that

$$|\tau_{i1}(\mathbf{B}_k)| < K \tag{13a}$$

$$|B_{ki}| < K \tag{13b}$$

and $K(b) \to 0$ as $b \to +\infty$.

By virtue of (12) and (13) we have that for every $b > 0$

$$0 \leqq \left| \int_0^\infty \{B_{ki}(b, \xi_3, 0, x_3)\tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{B}_k)\} \, d\xi_3 \right| \leqq 6MK(b)$$

Since the integral is independent of b and the last term in the inequality tends to zero as $x_1 \to +\infty$, it follows that

$$\int_0^\infty \{B_{ki}(b, \xi_3, 0, x_3)\tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{B}_k)\} \, d\xi_3 = 0 \tag{14}$$

In a similar manner it follows that

$$\int_0^\infty \{B_{ki}(a, \xi_3, 0, x_3)\tau_{i1}(\mathbf{u}) - u_i \tau_{i1}(\mathbf{B}_k)\} \, d\xi_3 = 0 \tag{15}$$

Therefore when $\mathbf{u}$ is a surface wave (11) reduces to

$$\begin{aligned} u_k(0, x_3) = {} & \sum_n A_{nk}(x_3) \int_0^\infty \{U_i^{(n)}(\xi_3)\tau_{i1}(\mathbf{u}) - u_i T_{i1}^{(n)}\}_{\xi_1=b} e^{ik_n b} \, d\xi_3 \\ & - \sum_n C_{nk}(x_3) \int_0^\infty \{\bar{U}_i^{(n)}(\xi_3)\tau_{i1}(\mathbf{u}) - u_i \bar{T}_{i1}^{(n)}\}_{\xi_1=a} e^{-ik_n a} \, d\xi_3 \end{aligned} \tag{16}$$

If in particular

$$u_i(\mathbf{x}) = U_i^{(m)}(x_3) e^{ik_m x_1}$$

then, in view of (6) equation (16) reduces to

$$U_k^{(m)}(x_3) = -\sum_{i=1}^{3} C_{mk}(x_3) \int_0^\infty \{\bar{U}_i^{(m)} T_{i1}^{(m)} - U_i^{(m)} \bar{T}_{i1}^{(m)}\} \, d\xi_3 \tag{17}$$

In this equation as well as in those to follow we drop summation convention.

Let us define J_m by

$$J_m = \sum_{i=1}^{3} \int_0^\infty \{U_i^{(m)} \bar{T}_{i1}^{(m)} - \bar{U}_i^{(m)} T_{i1}^{(m)}\} \, d\xi_3 \tag{18}$$

then equation (17) shows that J_m cannot vanish because $U_k^{(m)}$ is not identically zero. Therefore we can divide by J_m to obtain

$$C_{mk}(x_3) = \frac{1}{J_m} U_k^{(m)}(x_3) \tag{19a}$$

In a similar manner we obtain

$$A_{mk}(x_3) = \frac{1}{J_m} \bar{U}_k^{(m)}(x_3) \tag{19b}$$

Substituting these expressions in (9), we get

$$\mathbf{G}_k(\mathbf{x}, 0, \xi_3) = \mathbf{B}_k(\mathbf{x}, 0, \xi_3) + \begin{cases} \sum_n \frac{1}{J_n} \bar{U}_k^{(n)}(\xi_3) \mathbf{U}^{(n)}(x_3) e^{ik_n x_1} & \text{if} \quad x_1 > \xi_1 \\ \sum_n \frac{1}{J_n} U_k^{(n)}(\xi_3) \bar{\mathbf{U}}^{(n)}(x_3) e^{-ik_n x_1} & \text{if} \quad x_1 < \xi_1 \end{cases}$$

Or more generally in view of the invariance of the equations of motion and boundary conditions under translations along the x_1-axis

$$\mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{B}_k(\mathbf{x}, \boldsymbol{\xi}) + \begin{cases} \sum_n \frac{1}{J_n} U^{(n)}(\xi_3) \bar{\mathbf{U}}^{(n)}(x_3) e^{ik_n (x_1 - \xi_1)} & \text{if} \quad x_1 > \xi_1 \\ \sum_n \frac{1}{J_n} U^{(n)}(\xi_3) \bar{\mathbf{U}}^{(n)}(x_3) e^{-ik_n (x_1 - \xi_1)} & \text{if} \quad x_1 < \xi_1 \end{cases} \tag{20}$$

where summation is not understood over repeated indices.

Equation (20) shows that when a surface wave $U_i^{(m)} e^{ik_m x_1}$ is known, in order to determine its contribution to the Green's function it is enough to compute the normalizing factor

$$J_m = \sum_{i=1}^{3} \int_0^\infty \{U_i^{(m)} \bar{T}_{i1}^{(m)} - \bar{U}_i^{(m)} T_{i1}^{(m)}\} \, d\xi_3 \tag{21}$$

For Love waves

$$u_2(\mathbf{x}) = L^{(n)}(x_3) e^{ik_n x_1}; \qquad u_1 \equiv u_3 \equiv 0$$

equation (21) adopts a form specially simple

$$J_n = -2i \int_0^\infty (L^{(n)})^2 \, dx_3 \tag{22}$$

Conclusions

The contribution of surface waves to the symmetric Green's function has been obtained by a method which seems to be new and which is very simple. It has been shown that whenever a surface wave solution is known for a half space, then its contribution to the Green's function can be determined computing some sort of normalization factor given by (21). This is true regardless of the way in which the properties of the materials vary with depth, as long as all the surface wave solutions give rise to displacements and stresses which are absolutely integrable.

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Appendix

Let **u** and **v** be two solutions of (1) and multiply (1) by v_i to obtain

$$v_i \frac{\partial}{\partial x_j}\left(C_{ijpq} \frac{\partial u_p}{\partial x_q}\right) + \rho\omega^2 u_i v_i = 0 \tag{A1}$$

Interchange the role of **u** and **v** in this equation and subtract the resulting equation from (A1) to obtain

$$v_i \frac{\partial}{\partial x_j}\left(C_{ijpq} \frac{\partial u_p}{\partial x_q}\right) - u_i \frac{\partial}{\partial x_j}\left(C_{ijpq} \frac{\partial v_p}{\partial x_q}\right) = 0 \tag{A2}$$

From (A2), using the fact that

$$C_{ijpq} = C_{pqij}$$

obtain

$$\frac{\partial}{\partial x_j}\left(v_i C_{ijpq} \frac{\partial u_p}{\partial x_q}\right) - \frac{\partial}{\partial x_j}\left(u_i C_{ijpq} \frac{\partial v_p}{\partial x_q}\right) = 0$$

and integrate over the region R whose boundary is S, to obtain, using divergence theorem and (1.b)

$$\int_S \{v_i \tau_{ij}(\mathbf{u}) - u_i \tau_{ij}(\mathbf{v})\} n_j \, dS = 0 \tag{A3}$$

which is formula (3) of the paper. In this formula set

$$v_i(\mathbf{x}) = G_{ki}(\mathbf{x}, \xi) \tag{A4}$$

where ξ is an interior point of R, to obtain

$$\int_{S+S_\epsilon} \{G_{ki}\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS = 0 \tag{A5}$$

where S_ϵ is a circle of radius ϵ around the point ξ and $\mathbf{G}_k(\mathbf{x}, \xi)$ are the displacements produced by a concentrated force of unit magnitude acting in the k-direction and located at the point ξ, i.e. $\mathbf{G}_k(\mathbf{x}, \xi)$ is for every ξ a solution of (1) which is regular everywhere except at the point ξ and there

$$\lim_{\epsilon\to 0}\int_{S_\epsilon} \tau_{ij}(\mathbf{G}_k)n_j\, dS = \delta_{ik} \tag{A6}$$

Observe now that since

$$\tau_{ij}(\mathbf{G}_k) = C_{ijpq}\frac{\partial G_{kp}}{\partial x_q}$$

equation (A6) implies that the derivatives of $\mathbf{G}_k$ are $0(1/r)$ and therefore $\mathbf{G}_k$ is $0(\mathrm{Ln}\, r)$, where $r = \sqrt{(x_1 - \xi_1)^2 + (x_3 - \xi_3)^2}$.

Let now $\epsilon \to 0$ in equation (A5) using the above observation to get

$$u_k(\xi) = \int_S \{G_{ki}\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS \tag{A7}$$

A slight modification of the above argument shows that if $\mathbf{u}$ is a solution of

$$\frac{\partial\tau_{ij}}{\partial x_j} + \rho\omega^2 u_i = -F_i$$

then for every interior point ξ of the region R, we have

$$u_k(\xi) = \int_R F_i G_{ik}(\mathbf{x}, \xi)\, dV(\mathbf{x}) + \int_S \{G_{ki}(\mathbf{x}, \xi)\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS(\mathbf{x}) \tag{A8}$$

REFERENCES

De Hoop, A. T.

1958. Representation theorems for the displacement in an elastic solid and their application to elastodynamic diffraction theory, *Sc.D. thesis*, Technische Hogeschool, Delft.

Harkrider, D. G.

1964. Surface waves in multi-layered elastic media, I: Rayleigh and Love waves from buried sources in a multi-layered elastic half space, *Bull. Seism. Soc. of Am.*, S4: 627–679.

Herrera, I.
1964. A perturbation method for elastic wave propagation. I: non-parallel boundaries (to be published)

Hudson, J. A. and L. Knopoff
1964. "Transmission and Reflection of Surface waves at a corner, I and II", *J. Geophys. Res.*, *69*, 275–290.

Knopoff, L. and J. A. Hudson
1964. "Transmission of Love waves past a continental margin" (to be published in *J. Geophys. Res.*)

Knopoff, L.
1956. "Diffraction of Elastic Waves", *J. Acoust. Soc. Am.*, *28*, 217–229.

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