A Perturbation Method for Elastic Wave Propagation[1]

1. Nonparallel Boundaries

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Abstract. This is the first of a series of papers in which a small-perturbation theory for elastic wave propagation is presented. Using classical perturbation techniques, we obtain boundary conditions for the perturbation of the displacement field which, by means of the integral representation theorems of elastodynamics, permit us to express the solutions to the problems as integrals of known quantities. In this paper the method is formulated for mediums with boundaries which are slightly nonparallel. It is then applied to a study of the propagation of Love waves through a crustal layer.

Introduction. The study of wave propagation in elastic mediums with nonparallel boundaries is important to geophysicists because knowledge in this field will help them to understand and predict the seismic behavior at continental margins, mountain roots, etc.

Some progress has recently been made in this field. *Hudson and Knopoff* [1964] and *Knopoff and Hudson* [1964] have obtained approximate solutions to several problems of this type using a method based on a representation theorem for elastic wave propagation. Although some other techniques have been used in dealing with problems of this kind [*Kane and Spence*, 1963], the method based on the representation theorem seems to be the most promising because these representation theorems reduce the problems of wave propagation to quadratures when the Green's functions corresponding to the given regions and boundary conditions are known. However, as is usual when applying Green's function techniques, the problem is then to find the Green's functions suitable for the given regions because they are known only for some simple types of regions.

In geophysics we frequently encounter regions which are close to some simple type of region such as a layered half-space, for which the Green's function is known. It is therefore natural to attempt to solve the problem using classical perturbation methods which will permit transforming the given region into one for which the Green's function is known.

In this paper, using the representation theorems given by *Knopoff* [1956] and *de Hoop* [1958] and classical perturbation techniques, we formulate an approximate method of solution which is applicable to a great variety of problems of practical interest. The representation theorems mentioned above are extended to cases where the displacements and stresses are discontinuous. The method is then applied to a study of the transmission of Love waves through a crustal layer of variable thickness.

Symbols.

i, j, k, p, q, indices whose range is 1, 2, 3 unless otherwise stated, and for which the summation convention holds (i is also used for the square root of -1, but in every case the meaning is clear from the context).

n, m, indices whose range is not necessarily 1, 2, 3 and for which the summation convention does not hold.

ϵ, perturbation parameter.

$\mathbf{x}$ $= (x_1, x_2, x_3)$, Cartesian coordinates of a point.

$\boldsymbol{\xi}$ $= (\xi_1, \xi_2, \xi_3)$, Cartesian coordinates of a point.

$\mathbf{u}(\mathbf{x}, \epsilon)$ $= (u_1, u_2, u_3)$, displacement field

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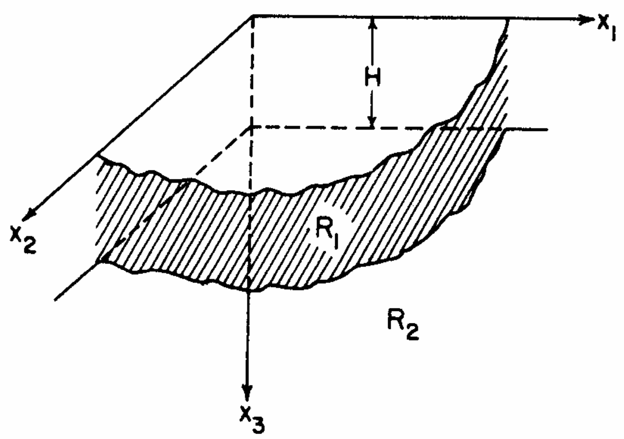

Fig. 1. Half-space with overlying layer.

corresponding to the value ϵ of the perturbation parameter.

$\mathbf{u}^0(\mathbf{x})$ $= \mathbf{u}(\mathbf{x}, 0)$, unperturbed solution.

$\mathbf{u}^1(\mathbf{x})$ $= \partial \mathbf{u}/\partial \epsilon(\mathbf{x}, 0)$.

ω, angular frequency of solutions of the form $u_i e^{-i\omega t}$.

$\tau_{ij}(\mathbf{u})$ $= C_{ijpq}\ \partial u_p/\partial x_q$, stress associated with $\mathbf{u}$.

$\tau_{ij}{}^0$ $= \tau_{ij}(\mathbf{u}^0)$.

$\tau_{ij}{}^1$ $= \tau_{ij}(\mathbf{u}^1)$.

C_{ijpq}, elasticity tensor (for isotropic material $C_{ijpq} = \lambda\delta_{ij}\ \delta_{pq} + \mu(\delta_{ip}\ \delta_{jq} + \delta_{jp}\ \delta_{iq})$).

δ_{ij}, Kronecker's delta.

$\delta(x)$, Dirac's delta function.

μ, λ, Lamé constants (μ_1, λ_1 in R_1, μ_2, λ_2 in R_2).

ρ, density of the material (ρ_1 in R_1, ρ_2 in R_2).

S_1, free surface of the unperturbed region.

S_2, interface of the unperturbed region.

R_1, layer bounded by S_1 and S_2.

R_2, half-space bounded by S_2.

H, depth of the interface for the unperturbed region.

h_1, h_2, functions of x_1 and x_2 giving the shape of the boundaries.

ΔH $= (h_2 - h_1)\ \epsilon$.

[ ], jump discontinuity at the interface (value at R_2 minus value at R_1).

$h_{i,j}$ $= \partial h_i/\partial x_j (i, j = 1, 2)$.

n_i, unit vector normal to the boundaries.

$m_i{}^{(j)}$ = vectors normal to the free surface and interface ($j = 1, 2$).

$\mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi})$ $= (G_{k1}, G_{k2}, G_{k3})$, Green's function with singularity at $\boldsymbol{\xi}$ and for which the concentrated force is in the k direction.

$G(\mathbf{x}, \boldsymbol{\xi})$, = Green's function (singularity at $\boldsymbol{\xi}$) for SH waves.

$B(\mathbf{x}, \boldsymbol{\xi})$, body wave terms in $G(\mathbf{x}, \boldsymbol{\xi})$.

$v(\mathbf{x}, \epsilon)$, displacement field for SH waves in the perturbed region corresponding to the parameter ϵ.

$v^0(\mathbf{x})$ $= v(\mathbf{x}, 0)$, unperturbed SH waves.

$v^1(\mathbf{x})$ $= \partial v/\partial \epsilon(\mathbf{x}, 0)$.

$v_n{}^0(\mathbf{x})$ $= f_n(x_3)\ e^{ik_n x_1}$, Love wave in the nth mode.

$$f_n(x_3) = \begin{cases} \dfrac{2 \cos \sigma_n x_3}{(F'(k_n) \cos \sigma_n H)^{1/2}} & \text{if} \quad 0 \leq x_3 \leq H. \\ 2\left(\dfrac{\cos \sigma_n H}{F'(k_n)}\right)^{1/2} e^{-\sigma_n'(x_3 - H)} & \text{if} \quad H \leq x_3. \end{cases}$$

$\sigma_n{}^2$ $= (\rho_1\omega^2/\mu_1) - k_n{}^2; \quad \sigma_n \geq 0.$

$(\sigma_n')^2$ $= k_n{}^2 - (\rho_2\omega^2/\mu_2); \quad \sigma_n' \geq 0.$

$F(k_n)$ $= 2\mu_2\sigma_n' \cos \sigma_n H - 2\mu_1\sigma_n \sin \sigma_n H.$

k_n, nth zero of $F(k_n)$.

$\epsilon v_n{}^1(\mathbf{x})$, perturbation when the unperturbed displacement field is $v_n{}^0$.

$B_n{}^1(\mathbf{x})$, body wave terms in $v_n{}^1(\mathbf{x})$.

$C_{mn}{}^T$, transmission coefficient for the mth mode when the incoming wave is in the nth mode.

$C_{mn}{}^R$, reflection coefficient for the mth mode when the incoming wave is in the nth mode.

$$G_m(\boldsymbol{\xi}) = \frac{i}{2} f_m(\xi_3) \cdot \begin{cases} e^{-ik_m\xi_1} & \text{if} \quad x_1 > \xi_1. \\ e^{ik_m\xi_1} & \text{if} \quad x_1 < \xi_1. \end{cases}$$

v_{nn}, component in the nth mode of the perturbed SH wave.

N, number of parallel boundaries.

Perturbation method. The method to be described is applicable to any problem formulated in a region which deviates only slightly from another region, for which (a) the corresponding Green's function is known and (b) the solution to the problem is known.

However, for simplicity, the method is explained only in connection with problems of elastic wave transmission in a homogeneous half-space with an overlying layer of different material, as shown in Figure 1. The extension of the results to a multilayered medium is straightforward.

Assume that the displacements u_i^0 corresponding to some type of wave transmitted in that region are known. That is, u_i^0 satisfies the equations of elasticity in medium 1 and 2 together with the boundary conditions

$$\tau_{i3}^0 = 0$$

$$\left.\begin{aligned} [\tau_{i3}^0] &= 0 \\ [u_i^0] &= 0 \end{aligned}\right\} \quad x_3 = H$$

We are interested, however, in finding the solution to the same problem when the upper surface is given by

$$x_3 = \epsilon h_1(x_1, x_2)$$

and the interface is

$$x_3 = H + \epsilon h_2(x_1, x_2)$$

where h_1 and h_2 are bounded functions and ϵ is a small number. If we let ϵ be arbitrary we are led to a family of problems whose solutions will be represented by $u_i(\mathbf{x}, \epsilon)$ and $\tau_{ij}(\mathbf{x}, \epsilon)$. When these functions have continuous derivatives with respect to ϵ,

$$u_i(\mathbf{x}, \epsilon) \approx u_i(\mathbf{x}, 0) + \epsilon \frac{\partial u_i}{\partial \epsilon}(\mathbf{x}, 0)$$

$$= u_i^0(\mathbf{x}) + \epsilon u_i^1(\mathbf{x}) \qquad (1)$$

for sufficiently small ϵ.

Since $u_i^0(\mathbf{x})$ is known, our goal will be to determine $u_i^1(\mathbf{x})$. To this end we establish the boundary conditions which it satisfies.

Now

$$m_j^{(1)} \tau_{ij}(x_1, x_2, \epsilon h_1, \epsilon) = 0 \qquad (2)$$

$$\begin{aligned} m_j^{(2)}[\tau_{ij}] &= 0 \qquad x_3 = H + \epsilon h_2 \\ [u_i] &= 0 \qquad x_3 = H + \epsilon h_2 \end{aligned} \qquad (3)$$

where

$$\begin{aligned} m_j^{(i)} &= \epsilon h_{i,j} = \epsilon \frac{\partial h_i}{\partial x_j} \qquad i, j = 1, 2 \\ m_3^{(i)} &= -1 \end{aligned} \qquad (4)$$

are vectors normal to the boundaries and the summation convention has been adopted.

When using perturbation techniques to obtain the boundary conditions for the perturbation, one usually expands in power series with respect to ϵ the unperturbed boundary conditions, i.e. (2) and (3), and neglects terms of order ϵ^2 or higher in the equation so obtained. This is equivalent to taking the derivative with respect to ϵ of the unperturbed boundary conditions and setting $\epsilon = 0$ in the resulting equation. The latter procedure, however, must be preferred because, first, it avoids many cumbersome manipulations and, second, it is more sound theoretically since only continuous first-order derivatives with respect to ϵ of the quantities involved are assumed. Since this point of view is unconventional we shall discuss the method of obtaining the perturbed boundary conditions in detail.

Taking the derivative of (2) with respect to ϵ, we get for arbitrary ϵ

$$m_j^{(1)}\left\{\frac{\partial \tau_{ij}}{\partial \epsilon} + \frac{\partial \tau_{ij}}{\partial x_k}\frac{dx_k}{d\epsilon}\right\} + \frac{\partial m_j^{(1)}}{\partial \epsilon}\tau_{ij} = 0$$

$$x_3 = \epsilon h_1$$

Now, from (2) it follows that

$$\frac{dx_k}{d\epsilon} = h_1 \delta_{3k}$$

and from (4), since h_1 is independent of ϵ,

$$\frac{dm_j^{(1)}}{d\epsilon} = h_{1,j} \qquad j = 1, 2$$

$$\frac{dm_3^{(1)}}{d\epsilon} = 0$$

Therefore

$$m_j^{(1)}\left\{\frac{\partial \tau_{ij}}{\partial \epsilon} + h_1 \frac{\partial \tau_{ij}}{\partial x_3}\right\}$$

$$+ h_{1,1}\tau_{i1} + h_{1,2}\tau_{i2} = 0 \qquad x_3 = \epsilon h_1$$

Setting $\epsilon = 0$ in this equation yields

$$\tau_{i3}^1 = h_{1,1}\tau_{i1}^0 + h_{1,2}\tau_{i2}^0 - h_1 \frac{\partial \tau_{i3}^0}{\partial x_3} \qquad x_3 = 0 \qquad (5)$$

In a similar manner, from (3) it follows that

$$[\tau_{i3}^1] = \left[h_{2,1}\tau_{i1}^0 + h_{2,2}\tau_{i2}^0 - h_2 \frac{\partial \tau_{i3}^0}{\partial x_3}\right]$$

$$x_3 = H \qquad (6)$$

$$[u_i^1] = -h_2\left[\frac{\partial u_i^0}{\partial x_3}\right] \qquad x_3 = H$$

Since the equations of elasticity are linear, the u_i^1 also satisfy them.

Now the shape of the boundaries h_1 and h_2, the unperturbed solution u_i^0, and the corresponding stresses are known. Therefore (5) and (6) give the stresses at the free surface and the discontinuity jumps of the displacements and stresses at the interface in terms of known quantities. Thus the problem of finding u_i^1 has been reduced to finding a solution of the equations of elasticity meeting the boundary conditions given by (5) and (6) and satisfying appropriate conditions at infinity. This problem will be solved in the following section by means of the representation theorems mentioned previously.

Representation theorem. The representation theorems given by *de Hoop* [1958] are restricted to homogeneous regions. However, *Hudson and Knopoff* [1964] have shown how these theorems can be extended to regions which are made of several homogeneous subregions. Since, according to (6), the solution we are looking for has prescribed jump discontinuities at the interface, it is convenient to extend the representation theorem to cover this case. The representation theorem for quasi-steady motions of the form $u_k(\mathbf{x})e^{-i\omega t}$ is

$$u_k(\mathbf{x}) = \int_S \{G_{ki}(\boldsymbol{\xi}, \mathbf{x})\tau_{ij}(\mathbf{u}) - u_i(\boldsymbol{\xi})\tau_{ij}(\mathbf{G}_k)\}n_j\, dS \qquad (7)$$

where S is the boundary of the region considered, which is assumed bounded.

We choose $\mathbf{G}_k$, satisfying

$$\frac{\partial}{\partial x_i}\tau_{ij}(\mathbf{G}_k) + \rho\omega^2 G_{kj}(\mathbf{x}, \boldsymbol{\xi}) = -\delta_{jk}\delta(x_1 - \xi_1)\delta(x_2 - \xi_2)\delta(x_3 - \xi_3)$$

where conditions equivalent to continuity of the stress and displacement are applied at the interface, i.e.,

$$\left.\begin{aligned} n_j[\tau_{ij}(\mathbf{G}_k)] &= 0 \\ [\mathbf{G}_k] &= 0 \end{aligned}\right. \qquad x_3 = H \qquad (8)$$

We now apply (7) to region R_1 (Figure 1) to get

$$\int_{S_1+S_2} \{G_{ki}\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS = \begin{cases} u_k(\mathbf{x}) & \text{if} \quad \mathbf{x}\, \varepsilon\, R_1 \\ 0 & \text{if} \quad \mathbf{x}\, \varepsilon\, R_2 \end{cases} \qquad (9)$$

and then apply (7) to region R_2 to get

$$\int_{S_2} \{G_{ki}\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS = \begin{cases} 0 & \text{if} \quad \mathbf{x}\, \varepsilon\, R_1 \\ u_k(\mathbf{x}) & \text{if} \quad \mathbf{x}\, \varepsilon\, R_2 \end{cases} \qquad (10)$$

In these equations we have deleted the contribution coming from the surface integral over that part of a sphere necessary to close the boundary; this is valid only for a restricted class of functions and only when the Green's function is conveniently chosen. A detailed discussion of the conditions under which the representation theorems can be extended to unbounded regions has been given by *Herrera* [1964*a*] for two-dimensional problems.

Adding (9) and (10) we get

$$u_k(\mathbf{x}) = \int_{S_1} \{G_{ki}\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, dS + \int_{S_2} \{G_{ki}[\tau_{ij}(\mathbf{u})] - \tau_{ij}(\mathbf{G}_k)[u_i]\}n_j\, dS \qquad (11)$$

where the normal vector must be taken pointing upward in both integrals.

If we choose the Green's function G_{ki} in such a way that $n_j\, \tau_{ij}(\mathbf{G}_k) = 0$ on S_1, then

$$u_k(\mathbf{x}) = \int_{S_1} G_{ki}\tau_{ij}(\mathbf{u})n_j\, dS + \int_{S_2} \{G_{ki}[\tau_{ij}(\mathbf{u})] - \tau_{ij}(\mathbf{G}_k)[u_i]\}n_j\, dS \qquad (12)$$

Equation 12 now yields the desired solution when G_{ki} is known because $n_j\, \tau_{ij}(\mathbf{u}^1)$ at $x_3 = 0$ and $n_j[\tau_{ij}(\mathbf{u}^1)]$ and $[u_i^1]$ at $x_3 = H$ are given by (5) and (6), respectively.

Two-dimensional problems. It has been shown that (12) is valid for problems of surface wave transmission when the Green's function is properly chosen [*Herrera,* 1964*a*].

For *SH* waves, equation 12, when applied to the perturbation v^1, reduces to

$$v^1 = \int_{S_2} \left\{\mu \frac{\partial G}{\partial \xi_3}[v^1] - G\left[\mu \frac{\partial v^1}{\partial \xi_3}\right]\right\} dS - \int_{S_1} \mu \frac{\partial v^1}{\partial \xi_3} G\, dS \qquad (13)$$

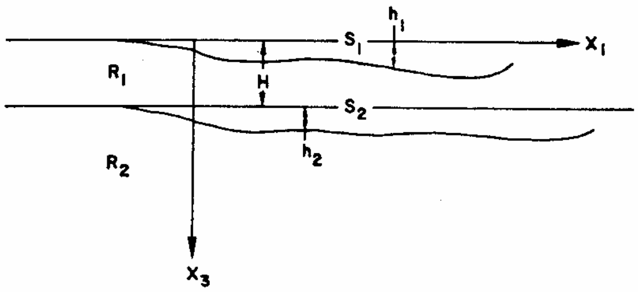

Fig. 2. Crustal layer of variable thickness.

The boundary values are given by (5) and (6), which in this case reduce to

$$\frac{\partial v^1}{\partial x_3} = h_1' \frac{\partial v^0}{\partial x_1} - h_1 \frac{\partial^2 v^0}{\partial x_3^2} \qquad x_3 = 0 \qquad (14)$$

$$\left[\mu \frac{\partial v^1}{\partial x_3}\right] = \left[\mu h_2' \frac{\partial v^0}{\partial x_1} - \mu h_2 \frac{\partial^2 v^0}{\partial x_3^2}\right]$$

$$[v^1] = -h_2\left[\frac{\partial v^0}{\partial x_3}\right]$$

$$x_3 = H \qquad (15)$$

where the primes denote first derivatives with respect to x_1.

Transmission of Love waves through a crustal layer of variable thickness. To illustrate the method explained above, we now study a problem of great geophysical interest, namely the effect of changes of the thickness of the crustal layer in mountainous regions on the transmissions of Love waves.

We idealize the configuration of the crustal layer in the manner shown in Figure 2.

If the incoming wave is a Love wave in the nth mode, then

$$v_n^0(\mathbf{x}) = f_n(x_3)e^{ik_n x_1} \qquad (16)$$

because in this manner

$$\int_0^\infty \mu f_n f_m \, dx_3 = \delta_{nm} \quad n, m = 1, 2. \cdots \qquad (17)$$

Equation 17 may be obtained by observing that the f_n are orthogonal [*Herrera*, 1964*b*], and then by direct computation of the norm of these functions. It must be recalled that $\mu = \mu_1$ if $0 \leq x_3 \leq H$ and $\mu = \mu_2$ if $x_3 > H$.

The Green's function to be used depends on the conditions at $-\infty$ and $+\infty$ [*Herrera*, 1964*a*]. Suppose that at $-\infty$ only the advancing Love waves were specified and they were given by (16) and at $+\infty$ it was stated that no receding Love waves were incoming; then the conditions for the perturbation v^1 would be that at $-\infty$ no advancing Love waves be incoming and that at $+\infty$ no receding Love waves be incoming. For such a problem (13) holds only if the Green's function G is defined by

$$G(\mathbf{x}, \boldsymbol{\xi}) = B(\mathbf{x}, \boldsymbol{\xi}) + \frac{i}{2} \sum_n f_n(\xi_3) f_n(x_3) \cdot \begin{cases} e^{ik_n(x_1 - \xi_1)} & \text{if} \quad x_1 > \xi_1 \\ e^{-ik_n(x_1 - \xi_1)} & \text{if} \quad x_1 < \xi_1 \end{cases} \qquad (18)$$

The Green's function given by (18) may be obtained by multiplying *Sezawa*'s [1935] source solution by the normalizing factor $\mu/4$. It has been used by Knopoff and Hudson in similar problems. The terms corresponding to body waves are represented by B.

The solution given by (13) may be written in the form

$$v_n^1(\mathbf{x}) = B_n^1 + \sum_m C_{mn}^T(x_1) f_m(x_3) e^{ik_m x_1} + \sum_m C_{mn}^R(x_1) f_m(x_3) e^{-ik_m x_1} \qquad (19)$$

where v_n^1 is the perturbation of an advancing wave in the nth mode, C_{mn}^T and C_{mn}^R are the transmission and reflexion coefficients, respectively, for Love waves (they depend on x_1), and B_n^1 are terms corresponding to body waves.

It must be observed that (19) does not define the coefficients C_{mn}^T and C_{mn}^R in a unique manner. To define them uniquely, we adopt here the definition given previously [*Herrera*, 1964*a*]. In the same paper it was shown that the derivatives dC_{mn}^T/dx_1 and dC_{mn}^R/dx_1 are independent of the conditions at infinity and that they can be obtained from the integral representation theorem. Indeed, when we substitute (18) into (13) we obtain

$$C_{mn}^T(x_1) = \int_{-\infty}^{x_1} \left\{\mu \frac{\partial G_m}{\partial \xi_3} [v^1] - G_m\left[\mu \frac{\partial v^1}{\partial \xi_3}\right]\right\} d\xi_1 - \int_{-\infty}^{x_1} \mu G_m \frac{\partial v^1}{\partial \xi_3} \, d\xi_1$$

$$(20)$$

$$C_{mn}^R(x_1) = \int_{x_1}^{\infty} \left\{\mu \frac{\partial G_m}{\partial \xi_3} [v^1] - G_m\left[\mu \frac{\partial v^1}{\partial \xi_3}\right]\right\} d\xi_1 - \int_{x_1}^{\infty} \mu G_m \frac{\partial v^1}{\partial \xi_3} \, d\xi_1$$

where

$$G_m(\xi) = \frac{i}{2} f_m(\xi_3) \cdot \begin{cases} e^{-ik_m\xi_1} & \text{if} \quad x_1 > \xi_1 \\ e^{ik_m\xi_1} & \text{if} \quad x_1 < \xi_1 \end{cases}$$

and it is understood that $\xi_3 = H$ and $\xi_3 = 0$ in the first and second terms, respectively, of the right-hand members.

From (20) it follows that

$$\frac{dC_{mn}{}^T}{dx_1} = \mu \frac{\partial G_m}{\partial \xi_3} [v^1] - G_m \left[\mu \frac{\partial v^1}{\partial \xi_3} \right] - \mu G_m \frac{\partial v^1}{\partial \xi_3} \tag{21}$$

$$\frac{dC_{mn}{}^R}{dx_1} = -\mu \frac{\partial G_m}{\partial \xi_3} [v^1] + G_m \left[\mu \frac{\partial v^1}{\partial \xi_3} \right] + \mu G_m \frac{\partial v^1}{\partial \xi_3}$$

Using (14) and (15) we get

$$\frac{dC_{mn}{}^T}{dx_1} = -\frac{ih_1}{2} \mu_1 \sigma_n{}^2 f_n(0) f_m(0) e^{i(k_n+k_m)x_1}$$
$$- \frac{ih_2}{2} \left\{ \mu_1 \sigma_n{}^2 + \mu_2 \sigma_n{}'^2 + \frac{\mu_2}{\mu_1} (\mu_2 - \mu_1) \sigma_n{}' \sigma_m{}' \right\} f_n(H) f_m(H) e^{i(k_n+k_m)x_1}$$
$$- \frac{h_1{}'}{2} \mu_1 k_n f_n(0) f_m(0) e^{i(k_n+k_m)x_1} - \frac{h_2{}'}{2} (\mu_2 - \mu_1) k_n f_n(H) f_m(H) e^{i(k_n+k_m)x_1} \tag{22}$$

$$\frac{dC_{mn}{}^R}{dx_1} = \frac{ih_1}{2} \mu_1 \sigma_n{}^2 f_n(0) f_m(0) e^{i(k_n-k_m)x_1}$$
$$+ \frac{ih_2}{2} \left\{ \mu_1 \sigma_n{}^2 + \mu_2 \sigma_n{}'^2 + \frac{\mu_2}{\mu_1} (\mu_2 - \mu_1) \sigma_n{}' \sigma_m{}' \right\} f_n(H) f_m(H) e^{i(k_n-k_m)x_1}$$
$$+ \frac{h_1{}'}{2} \mu_1 k_n f_n(0) f_m(0) e^{i(k_n-k_m)x_1} + \frac{h_2{}'}{2} (\mu_2 - \mu_1) k_n f_n(H) f_m(H) e^{i(k_n-k_m)x_1}$$

Observe that if $g(x_1) = re^{i\theta}$, where r and θ are real valued functions of x_1,

$$g(x_1) e^{i(k_m x_1 - \omega t)} = re^{i(k_m x_1 + \theta - \omega t)}$$

represents a wave whose wave number is $k_m + d\theta/dx_1$ and whose rate of increase of amplitude with distance is dr/dx_1. But

$$\frac{1}{g} \frac{dg}{dx_1} = \frac{1}{r} \frac{dr}{dx_1} + i \frac{d\theta}{dx_1}$$

Therefore, for $m \neq n$, the real part and imaginary part of $(dC_{mn}/dx_1)/C_{mn}$ have a direct physical interpretation, the real part giving the rate of increase of the log of the amplitude and the imaginary part plus k_m giving the wave number.

One of the most interesting things to look at is the perturbation suffered by the nth mode itself. For $m = n$ the form of the results is especially simple:

$$\frac{dC_{nn}{}^T}{dx_1} = \frac{i}{2} (h_2 - h_1) \mu_1 \sigma_n{}^2 f_n{}^2(0)$$
$$+ \frac{k_n}{2} f_n{}^2(0) \{ \mu_1 h_1{}' + (\mu_2 - \mu_1) h_2{}' \cos^2 \sigma_n H \} \tag{23}$$

Let v_{nn} be the component in the nth mode of perturbed wave, so that, by virtue of (16),

$$v_{nn}(\mathbf{x}) = \{1 + \epsilon C_{nn}{}^T(x_1)\} f_n(x_3) e^{ik_n x_1}$$

Therefore, if we neglect terms $O(\epsilon^2)$, we find that ϵ multiplied by the real and imaginary part of (23) gives, respectively, the rate of increase of the log of the amplitude and the increase in the wave number. Thus at any point the wave number is

$$k_n + \frac{2(h_2 - h_1)\mu_1 \sigma_n{}^2}{F'(k_n) \cos \sigma_n H} \epsilon$$

and the amplitude is

$$\left(1 + \frac{2k_n \epsilon}{F'(k_n) \cos \sigma_n H} \cdot \{ \mu_1 h_1 + (\mu_2 - \mu_1) h_2 \cos^2 \sigma_n H \} \right) f_n(x_3)$$

where we have integrated the real part in (23) from $-\infty$ to x_1, assuming that h_1 and $h_2 \to 0$ as $x_1 \to -\infty$. The factor $f_n(x_3)$ is introduced in order to obtain the amplitude at any level x_3.

The perturbation in the wave number is precisely the same as that which is obtained by taking the derivative with respect to H of k_n, using the condition

$$\mu_2\sigma_n' \cos \sigma_n H - \mu_1\sigma_n \sin \sigma_n H \equiv F(k_n)/2 = 0$$

and multiplying this derivative by $\triangle H$.

Thus to this order of approximation the predicted velocity is the same as the velocity of a wave traveling in its nth mode from $-\infty$ to $+\infty$ in a medium with an upper layer whose thickness is $H + \triangle H$, an approximation which has been extensively used. However, we are now in a position to predict the amplitude as well. If we normalize the incoming wave so that it has unit amplitude on the free surface, the amplitude on the free surface of the perturbed wave will be given by

$$1 + \frac{2k_n\epsilon}{F'(k_n) \cos \sigma_n H} \{\mu_1 h_1 + (\mu_2 - \mu_1) h_2 \cos^2 \sigma_n H\}$$

so that it depends only on the change in height of the free surface and interface at the point of observation, a fact that may be useful when determining the thickness of the crustal layer.

Conclusion. The method presented here has been for a medium having only one interface, but its extension to more than one interface is obvious. As a matter of fact, the only change in the results will be that every interface will give rise to additional jump contributions, so that instead of only one integral over the interface we shall have a sum of integrals over the interfaces. Thus (12) must be replaced by

$$u_k(\mathbf{x}) = \int_{S_1} G_{ki}\tau_{ij}(\mathbf{u})n_j \, dS + \sum_{n=2}^{N} \int_{S_n} \{G_{ki}[\tau_{ij}(\mathbf{u})]_n - \tau_{ij}(\mathbf{G}_k)[u_i]_n\} n_j \, dS$$

and the jumps are given by

$$[\tau_{i3}]_n = [h_{n,1}\tau_{i1}{}^0 + h_{n,2}\tau_{i2}{}^0 - h_n \, \partial\tau_{i3}{}^0/\partial x_3]_n$$

$$[u_i{}^1] = -h_n[\partial u_i/\partial x_3]_n$$

where the subscript n refers to the interface number and it is assumed that $n = 2, \cdots, N$; i.e., there are $N - 1$ interfaces.

Thus u_i is given in terms of integrals of known quantities by the appropriate Green's functions, and methods are available to obtain the latter [*Harkrider,* 1964; *Herrera,* 1964*b*].

By repeated derivation of the boundary conditions with respect to ϵ the method may be extended to obtain higher-order perturbations.

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