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A Perturbation Method for Elastic Wave Propagation

2. Small Inhomogeneities[1]

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Abstract. A perturbation method to treat small inhomogeneities is developed. By means of the integral representation theorems of elastodynamics the solutions are expressed as quadratures of known quantities. The method is first presented for mediums which deviate slightly from homogeneity in a large region and then is modified to treat mediums with large deviations from homogeneity in a thin region. The method is suitable for treating dikes, lenses, and other geological formations. Numerical computations are carried out for the scattering of Love waves by a dike.

Introduction. *Knopoff* [1956] and *de Hoop* [1958] have recently discussed integral representation theorems for elastodynamics. These theorems are very important in connection with problems on elastic wave transmission because they reduce these problems to quadratures when the corresponding Green's functions are known and because they are very suitable for obtaining approximate solutions of any order using classical perturbation methods [*Herrera*, 1964*a*].

The basic ideas of the method are applicable to any problem formulated in a region (called the perturbed region) whose geometry and physical properties deviate only slightly from those of a second region (called the unperturbed region) for which (*a*) the corresponding Green's function is known and (*b*) the solution to the problem is known.

This type of problem is of great interest in geophysics [*Hudson and Knopoff*, 1964; *Knopoff and Hudson*, 1964*a*, *b*; *De Noyer*, 1961; *Mal*, 1962*a*, *b*; *Obukhov*, 1963] and in some other fields such as engineering. Because of this interest one of the authors intiated a program of research whose aim is to explore the possibilities of the integral representation theorems of elastodynamics when they are used together with classical perturbation methods.

In paper 1 of this series [*Herrera*, 1964*a*] the method was formulated for some problems in which the geometry of the region is perturbed. Specifically, it was formulated for problems involving a multilayered half-space with nonparallel boundaries.

In this paper the method is formulated for problems in which the physical properties of the materials are perturbed. Since in many cases the Green's function is known when the material is homogeneous, perturbation methods can be applied when the given medium deviates slightly from homogeneity. Here again the slight deviation from homogeneity may be interpreted in more than one way. On one hand, the physical properties (elastic tensor and density) of the perturbed region can differ by small amounts in a large region from the elastic properties of the unperturbed region. On the other hand, the elastic properties of the perturbed region may differ by a large amount in a small region from the elastic properties of the unperturbed medium. The first possibility leads to problems for which perturbation methods have already been used in

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the solution of some problems [*Knopoff and Hudson,* 1964*b*; *Karal and Keller,* 1964]. The second possibility leads to problems for which the formulation of perturbation methods is considerably complicated.

Both possibilities can be treated by methods which are being developed in the present series. The first type of perturbation, which leads to body force terms in the partial differential equations governing the motion, is well known, and therefore it is only discussed briefly. It is shown that in order to account for the boundary conditions at the interfaces some terms must be added to those previously mentioned in the literature. The second type of perturbation is developed in greater detail, because apparently it had not been discussed previously. It is presented for problems in which the elastic properties of the medium are perturbed in a thin region by amounts which are not necessarily small. The analysis in this case is considerably more complicated. In this paper it is developed only for *SH* waves. The geophysical interest of the present work lies in the fact that lenses, dikes, and many other geological formations may be included within the type of thin inhomogeneities treated here.

To illustrate the method, the scattering of Love waves by a dike is treated numerically.

Notation. Throughout this work boldface type symbols indicate vectors. The character of the integral (surface integral S or volume integral R) is indicated by a subindex below the integral sign. The symbols most often used are

$$a = \mu^1/\mu^0$$

$$A_m = A_{m1}$$

A_{mn}, amplitude of the perturbation in the nth mode for incident Love waves in the mth mode.

β_1, β_2, shear wave velocities in the layer and half-space, respectively.

C_{ijpq}, elasticity tensor (for isotropic material $C_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{jp}\delta_{iq})$).

C^0_{ijpq}, C^1_{ijpq}, superindices 0 and 1 refer to the unperturbed material and perturbation, respectively.

$\delta(\mathbf{x})$, Dirac's delta function.

δ_{ij}, Kronecker's delta.

D, depth of the inhomogeneity.

$\mathbf{e}$, unit normal vector to the skeleton.

ϵ, perturbation parameter.

$\mathbf{f}$, body forces.

$G(\mathbf{x}, \boldsymbol{\xi})$, Green's function (singularity at ξ) for SH waves.

$\mathbf{G}_k(\mathbf{x}, \boldsymbol{\xi}) = (G_{k1}, G_{k2}, G_{k3})$, Green's function with singularity at $\boldsymbol{\xi}$ and for which the concentrated force is in the k direction.

$\mathbf{G}_k^0$, G^0, Green's functions for the unperturbed medium.

g_i, coordinates of the points on the skeleton.

h_1, h_2, function describing the shape of the boundaries of the thin inhomogeneity.

H, layer thickness.

i, j, k, p, q, indices whose range is 1, 2, 3 and for which summation convention holds (i is also used for the square root of -1, but in every case the meaning is clear from the context).

k_m, wave number of the incident Love waves in the mth mode.

$L_i(\mathbf{u}) = \partial/\partial x_j(C_{ijpq}\, \partial u_p/\partial x_q) + \rho\omega^2 u_i$.

L_i^0, L_i^1, the superindices mean that the corresponding superindices must be added on C_{ijpq} and ρ in the expression for L_i.

λ, μ, Lamé's constants of the perturbed medium.

m, n, indices whose range is not necessarily 1, 2, 3 and for which the summation convention does not hold.

$\mathbf{n}$, unit normal vector.

$\partial/\partial n = n_i\, \partial/\partial x_i$.

ω, angular frequency of solutions of the form $\mathbf{u}e^{-i\omega t}$.

$r = (\mu^0\rho/\mu\rho^0) - 1$.

ρ, density of the perturbed medium.

ρ_1, μ_1; ρ_2, μ_2, density and rigidity in the layer and half-space, respectively.

R_1, region where the elastic properties of the medium have been perturbed.

s_1, s_2, parameters labeling the points of the skeleton.

s, length of the skeleton.

$\partial/\partial s$, derivative with respect to length.

S_0, skeleton of the thin inhomogeneity.

S_1, surface containing all the discontinuity surfaces of the perturbations of the elastic parameters (when SH waves are discussed it is used for the boundary of R_1).

$\tau_{ij}(\mathbf{u}) = C_{ijpq}\, \partial u_p/\partial x_q$.

$\mathbf{u}(\mathbf{x})$, $\mathbf{v}(\mathbf{x}) = (u_1, u_2, u_3),(v_1, v_2, v_3)$, displacement field corresponding to the exact solution outside and inside the inhomogeneity, respectively.

u, v, displacement field outside and inside the inhomogeneity, respectively, for Love waves.

u^0, u_1, $\cdots$ superindices 0 and 1 refer to unperturbed solution and perturbation, respectively.

$\mathbf{x}$, $\boldsymbol{\xi} = (x_1, x_2, x_3)$, $(\zeta_1, \zeta_2, \zeta_3)$, Cartesian coordinates of a point.

[ ], jump discontinuity across a surface. (The normal vector points toward the negative side of the surface, and the jump is defined as the value on the positive side minus the value on the negative side).

Representation theorems. We shall be concerned with the quasi-steady motion of elastic bodies which is governed by

$$\partial \tau_{ij}/\partial x_j + \rho\omega^2 u_i = -f_i \qquad (1a)$$

where

$$\tau_{ij} = C_{ijpq}\partial u_p/\partial x_q \qquad (1b)$$

and summation convention is understood.

The representation theorem that will be used in this work is [*Knopoff*, 1956; *de Hoop*, 1958; *Herrera*, 1964*a*, *b*]

$$u_k(\mathbf{x}) = \int_R f_i(\boldsymbol{\xi}) G_{ki}(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi}$$
$$+ \int_S \{G_{ki}(\boldsymbol{\xi}, \mathbf{x})\tau_{ij}(\mathbf{u}) - u_i\tau_{ij}(\mathbf{G}_k)\}n_j\, d\boldsymbol{\xi}$$
$$+ \int_{S_1} \{G_{ki}(\boldsymbol{\xi}, \mathbf{x})[\tau_{ij}(\mathbf{u})] - [u_i]\tau_{ij}(\mathbf{G}_k)\}n_j\, d\boldsymbol{\xi} \qquad (2a)$$

where the tensor Green's function $G_{ki}(\boldsymbol{\xi}, \mathbf{x})$ satisfies, for fixed $\mathbf{x}$ and k,

$$\frac{\partial}{\partial \xi_j}\left(C_{ijpq}\frac{\partial G_{kp}}{\partial \xi_q}\right) + \rho\omega^2 G_{ki}(\boldsymbol{\xi}, \mathbf{x})$$
$$= -\delta_{ik}\delta(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) \qquad (2b)$$

and yields displacements and normal stresses which are continuous across any interface, R is a bounded region, S its boundary, and S_1 are all the surfaces lying in the interior of R across which the normal stresses $\tau_{ij}n_j$ and the displacements have jump discontinuities. Under some conditions the representation theorems can be extended to unbounded regions. In the problems of scattering that we are going to consider, (2) is applicable in unbounded regions to the scattered field, i.e. to the perturbation of the displacement field [*Herrera*, 1964*c*]. (This is so only if the Green's function is the Green's function for outward radiation).

Small inhomogeneities. In this section we consider a medium whose physical properties differ slightly from the properties of the unperturbed medium in a region R_1 which is not necessarily small.

More precisely we assume that

$$C_{ijpq} = C^0_{ijpq} + \epsilon C^1_{ijpq} \qquad (3a)$$

$$\rho = \rho^0 + \epsilon\rho^1 \qquad (3b)$$

where ρ^0 and $C^0{}_{ijpq}$ are the density and elastic tensor when there is no perturbation, ϵ is a small number, and ρ^1 and $C^1{}_{ijpq}$ are continuously differentiable functions of position.

It is well known [*Knopoff and Hudson*, 1964*b*; *Karal and Keller*, 1964] that an approximation to the solution for the perturbed medium may be obtained by perturbing the differential equations

$$\partial\tau_{ij}/\partial x_j + \rho\omega^2 u_i = 0 \qquad (4)$$

Indeed, substituting (3) into (4), using (1*b*), we obtain

$$\frac{\partial}{\partial x_j}\left(C^0_{ijpq}\frac{\partial u_p}{\partial x_q}\right) + \rho^0\omega^2 u_i = -\epsilon L^1_i(u) \qquad (5)$$

where

$$L_i(\mathbf{u}) = \frac{\partial}{\partial x_j}\left(C_{ijpq}\frac{\partial u_p}{\partial x_q}\right) + \rho\omega^2 u_i \qquad (6)$$

and the superindex has obvious meaning.

Now

$$u_i = u_i^0 + u_i^1 \qquad (7)$$

where the unperturbed solution u_i^0 is such that

$$L_i^0(\mathbf{u}^0) = 0 \qquad (8)$$

and u_i^1 is the perturbation. Subtracting (8) from (5), using (6) and (7), we get

$$L_i^0(\mathbf{u}^1) = -\epsilon L_i^1(\mathbf{u}^0) - \epsilon L_i^1(\mathbf{u}^1) \qquad (9)$$

This equation may be interpreted as the equation of elasticity (1*a*) with body forces, $\epsilon L_i^1(\mathbf{u}^0) + \epsilon L_i^1(\mathbf{u}^1)$.

Using the representation theorem (2), it follows that

$$u_k^1(\mathbf{x}) = \epsilon \int_{R_1} L_i^1(\mathbf{u}^0) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} + \epsilon \int_{R_1} L_i^1(\mathbf{u}^1) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \tag{10}$$

where $\mathbf{G}_k^0$ is the tensor Green's function for the unperturbed medium. This is an integral equation for $\mathbf{u}^1$. Since $\mathbf{u}^1 = 0$ when $\epsilon = 0$, it is plausible to assume that the perturbation $\mathbf{u}^1$ together with its partial derivatives are $O(\epsilon)$. Then (10) may be written as

$$u_k^1(\mathbf{x}) = \epsilon \int_{R_1} L_i^1(\mathbf{u}^0) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} + O(\epsilon^2) \tag{11}$$

which gives $\mathbf{u}^1$ in terms of known quantities except by terms $O(\epsilon^2)$; i.e., (11) gives a first-order approximation for $\mathbf{u}^1$.

Contribution of interfaces. Equation 11 allows us to compute the perturbation $\mathbf{u}^1$ when C^0_{ijpq}, C^1_{ijpq}, ρ^0, and ρ^1 are continuously differentiable.

In this section we explain the modifications that must be made in the previous analysis to include cases in which C^1_{ijpq} and ρ^1 are only piecewise continuous differentiable functions of position, but C^0_{ijpq} and ρ^0 are still continuously differentiable. This is the case when the inhomogeneity is separated from the rest of the material by an interface, because across this interface the properties of the material jump abruptly and therefore C^1_{ijpq} and ρ^1 are discontinuous there. An important example of this situation is one for which C^1_{ijpq} and ρ^1 are constant in R_1 and vanish outside R_1.

At interfaces, continuity of the displacements and normal stresses is required

$$\left.\begin{aligned} [u_i] &= 0 \\ n_j[\tau_{ij}(\mathbf{u})] &= 0 \end{aligned}\right\} \text{ at interfaces} \tag{12}$$

Therefore, by (7) and (3a)

$$[u_i^1] = -[u_i^0] = 0 \tag{13a}$$

$$n_j C^0_{ijpq}\left[\frac{\partial u_p^1}{\partial x_q}\right] = -\epsilon n_j\left[C^1_{ijpq}\frac{\partial u_p}{\partial x_q}\right] = -\epsilon n_j[C^1_{ijpq}]\frac{\partial u_p^0}{\partial x_q} - \epsilon n_j\left[C^1_{ijpq}\frac{\partial u_p^1}{\partial x_q}\right] \tag{13b}$$

Therefore, the perturbation $\mathbf{u}^1$ satisfies (9) and at the interfaces has continuous displacements and a jump discontinuity in the normal stresses given by (13b). Applying the representation theorem (2), we obtain

$$\begin{aligned} u_k^1(\mathbf{x}) = {} & \epsilon \int_{R_1} L_i^1(\mathbf{u}^0) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \\ & - \epsilon \int_{S_1} G_{ki}^0(\boldsymbol{\xi}, \mathbf{x}) n_j [C^1_{ijpq}] \frac{\partial u_p^0}{\partial x_q}\, d\boldsymbol{\xi} \\ & + \epsilon \int_{R_1} L_i^1(\mathbf{u}^1) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \\ & - \epsilon \int_{S_1} G_{ki}^0(\boldsymbol{\xi}, \mathbf{x}) n_j \left[C^1_{ijpq}\frac{\partial u_p^1}{\partial x_q}\right] d\boldsymbol{\xi} \end{aligned} \tag{14}$$

where S_1 is a surface which contains all the discontinuity surfaces of the density and elastic tensor.

This is again an integral equation for $\mathbf{u}^1$. If $\mathbf{u}^1$ and its partial derivatives are $O(\epsilon)$,

$$\begin{aligned} u_k^1(x) = {} & \epsilon \int_{R_1} L_i^1(\mathbf{u}^0) G^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \\ & - \epsilon \int_{S_1} G_{ki}(\boldsymbol{\xi}, \mathbf{x}) n_j [C^1_{ijpq}] \frac{\partial u_p^0}{\partial x_q}\, d\boldsymbol{\xi} + O(\epsilon^2) \end{aligned} \tag{15}$$

which gives $\mathbf{u}^1$ in terms of known quantities except by terms $O(\epsilon^2)$.

Equation 15 shows that the interfaces give rise to contributions $O(\epsilon)$ which cannot be neglected in a first-order theory, a fact which apparently had not been pointed out previously.

Thin inhomogeneities. In this section we consider cases in which the perturbation of the elastic properties is not small; i.e., the density and elastic tensor will not be given by (3), but instead by

$$C_{ijpq} = C^0_{ijpq} + C^1_{ijpq} \tag{16a}$$

$$\rho = \rho^0 + \rho^1 \tag{16b}$$

and for simplicity we restrict attention to the case where all the elastic properties are constant.

The perturbed region, on the other hand, will be assumed to be thin; more precisely, we assume that ρ^1, C^1_{ijpq} are constant inside and zero outside a bounded region R_1 which is thin in the sense that its boundaries (Figure 1) admit of a parametric representation of the form

$$x_i = g_i(s_1, s_2) + \epsilon h_1(s_1, s_2) e_i(s_1, s_2) \tag{17a}$$

$$x_i = g_i(s_1, s_2) - \epsilon h_2(s_1, s_2) e_i(s_1, s_2) \tag{17b}$$

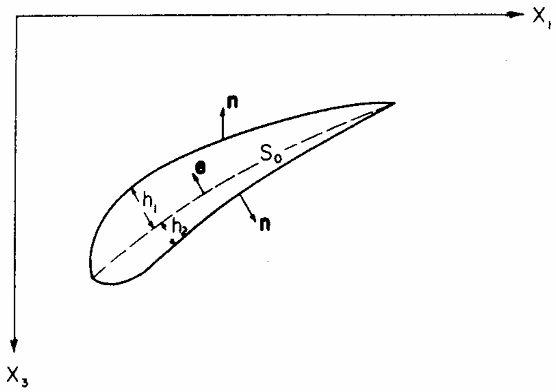

Fig. 1. Thin inhomogeneity.

where the equation

$$x_i = g_i(s_1, s_2) \tag{18}$$

represents a surface that will be called the 'skeleton' of the inhomogeneity, **e** is a unit normal vector to the skeleton, h_1 and h_2 are two functions describing the shape of the boundaries, and ϵ is the perturbation parameter.

The obvious procedure is to repeat the analysis which led to (14), to obtain

$$\begin{aligned} u_k^1(\mathbf{x}) &= \int_{R_1} L_i^1(\mathbf{u}^0) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \\ &- \int_{S_1} G_{ki}^0(\boldsymbol{\xi}, \mathbf{x}) n_j [C^1_{ijpq}] \frac{\partial u_p^0}{\partial x_q}\, d\boldsymbol{\xi} \\ &+ \int_{R_1} L_i^1(\mathbf{u}^1) G_{ki}^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} \\ &- \int_{S_1} G_{ki}^0(\boldsymbol{\xi}, \mathbf{x}) n_j \left[C^1_{ijpq} \frac{\partial u_p^1}{\partial \xi_q} \right] d\boldsymbol{\xi} \end{aligned} \tag{19}$$

Here the normal vector **n** to the interface is assumed to point outward with respect to the inhomogeneity.

We would like to be able to show that the third and fourth integrals in (19) are $O(\epsilon^2)$. However, this is not so. For the third integral the natural reasoning would be to say that since $L_i^1(\mathbf{u}^1)$ is $O(\epsilon)$ and the volume of integration is $O(\epsilon)$ (because the region is thin) it follows that the third integral is $O(\epsilon^2)$. But unfortunately $L_i^1(\mathbf{u}^1)$ is not $O(\epsilon)$ because the derivatives of $\mathbf{u}^1$ are not $O(\epsilon)$ in the interior of R_1. This may be most easily seen in the case of *SH* waves. In this case only one of the displacement components does not vanish (e.g., u_2) and all quantities involved are independent of x_2. In what follows it is convenient to represent by u this nonvanishing component of the displacement outside and by v the same component inside the region R_1. With this notation the continuity of normal stresses at the interfaces is

$$(\mu^0 + \mu^1)\partial v/\partial n = \mu^0 \partial u/\partial n$$

But for the unperturbed solution

$$\partial u^0/\partial n = \partial v^0/\partial n$$

so that

$$\frac{\partial v^1}{\partial n} = \frac{\mu^0}{\mu^0 + \mu^1}\frac{\partial u^1}{\partial n} - \frac{\mu^1}{\mu^0 + \mu^1}\frac{\partial u^0}{\partial n} \tag{20}$$

Therefore both

$$\partial v^1/\partial n; \qquad [\mu^0/(\mu^0 + \mu^1)]\partial u^1/\partial n$$

cannot be $O(\epsilon)$ because μ^1 is not small, and consequently

$$[\mu^1/(\mu^0 + \mu^1)]\partial u^0/\partial n$$

is $O(1)$.

All this shows that the method must be modified and these points carefully taken into account.

For simplicity we shall restrict our discussion to *SH* waves.

SH waves. With the notation introduced in the last section, equation 4 is

$$\mu^0 \partial^2 u/\partial x_i \partial x_i + \rho^0 \omega^2 u = 0 \tag{21a}$$

and

$$\mu \partial^2 v/\partial x_i \partial x_i + \rho \omega^2 v = 0 \tag{21b}$$

The boundary conditions at the interfaces are

$$v = u \tag{22a}$$

$$(\mu^0 + \mu^1)\partial v/\partial n = \mu^0 \partial u/\partial n \tag{22b}$$

It is convenient to introduce the parameter r, defined by

$$\rho/\mu = (1 + r)\rho^0/\mu^0$$

Since u^0 and v^0 satisfy (21a) it follows from (21) that

$$\mu^0 \partial^2 u^1/\partial x_i \partial x_i + \rho^0 \omega^2 u^1 = 0 \tag{23a}$$

and

$$\mu^0 \partial^2 v^1/\partial x_i \partial x_i + \rho^0 \omega^2 v^1 = -r\rho^0 \omega^2 v \tag{23b}$$

On the other hand, the unperturbed solution and its normal derivative are continuous across the interfaces, i.e.,

$$v^0 = u^0$$

$$\partial v^0/\partial n = \partial u^0/\partial n$$

Hence, by virtue of (22),

$$v^1 = u^1 \quad \text{at the interfaces} \qquad (24a)$$

$$\mu^0(\partial v^1/\partial n - \partial u^1/\partial n) = -\mu^1 \partial v/\partial n \quad \text{at the interfaces} \qquad (24b)$$

The representation theorem (2a) can now be used, taking into account (23) and (24). For points $\mathbf{x}$ outside R_1 we obtain

$$u^1(\mathbf{x}) = r\omega^2\rho^0 \int_{R_1} v(\boldsymbol{\xi})G^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} - \mu^1 \int_{S_1} G^0(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial v}{\partial n}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \qquad (25)$$

The last term in (25) is still not in the most convenient form. Using Green's second identity we can write

$$\int_{S_1} G^0 \frac{\partial v}{\partial n}\, d\boldsymbol{\xi} = \int_{R_1} \left\{ G^0 \frac{\partial^2 v}{\partial \xi_i\, \partial \xi_i} - v \frac{\partial^2 G^0}{\partial \xi_i\, \partial \xi_i} \right\} d\boldsymbol{\xi} + \int_{S_1} v \frac{\partial G^0}{\partial n}\, d\boldsymbol{\xi} \qquad (26)$$

Since $\mathbf{x}$ is outside R_1 by virtue of (2b) we have

$$\partial^2 G^0/\partial\xi_i\partial\xi_i + (\rho^0/\mu^0)\omega^2 G^0 = 0 \quad \text{in} \quad R_1$$

Using this fact and (21b), we can transform (26) into

$$\int_{S_1} G^0 \frac{\partial v}{\partial n}\, d\boldsymbol{\xi} = -\int_{R_1} \left(\frac{\rho}{\mu} - \frac{\rho^0}{\mu^0}\right)\omega^2 G^0 v\, d\boldsymbol{\xi} + \int_{S_1} v \frac{\partial G^0}{\partial n}\, d\boldsymbol{\xi}$$

When this result is substituted into (25), we get

$$u^1(\mathbf{x}) = (1 + a)r\omega^2\rho^2 \int_{R_1} v(\boldsymbol{\xi})G^0(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi} - \mu^1 \int_{S_1} v \frac{\partial G^0}{\partial n}\, d\boldsymbol{\xi} \qquad (27)$$

where we have written

$$a = \mu^1/\mu^0 \qquad (28)$$

The case we are treating is two dimensional. Therefore the boundaries of the inhomogeneities are not surfaces but only lines. Thus equations 17 reduce to

$$x_i = g_i(s) + \epsilon h_1(s)e_i(s) \qquad (29a)$$

$$x_i = g_i(s) - \epsilon h_2(s)e_i(s) \qquad (29b)$$

When the boundaries admit of this representation

$$v^1(\mathbf{g} + \epsilon h_1\mathbf{e}) = v^1(\mathbf{g} - \epsilon h_2\mathbf{e}) + \epsilon(h_1 + h_2)e_i \frac{\partial v^1}{\partial \xi_i}(\mathbf{g} - \epsilon h_2\mathbf{e}) + O(\epsilon^2) \qquad (30)$$

We have seen previously that both

$$\frac{\partial u^1}{\partial n} \quad \text{and} \quad \frac{\partial v^1}{\partial n}$$

cannot be $O(\epsilon)$. However, $u^1 = 0$ when $\epsilon = 0$ because $\epsilon = 0$ corresponds to the unperturbec medium. This in turn implies, provided that the limit with respect to ϵ commutes with the derivatives, that

$$\lim_{\epsilon \to 0} \frac{\partial u^1}{\partial n} = 0 \quad \text{at the interfaces}$$

It is therefore plausible to assume $\partial u^1/\partial n$ to b $O(\epsilon)$. If this is done, (20) becomes

$$\frac{\partial v^1}{\partial n} = -\frac{\mu^1}{\mu^0 + \mu^1} \frac{\partial u^0}{\partial n} + O(\epsilon) \quad \text{at the interface}$$

Therefore

$$e_i \frac{\partial v^1}{\partial \xi_i}(\mathbf{g} - \epsilon h_2\mathbf{e}) = -\frac{\mu^1}{\mu^0 + \mu^1} e_i \frac{\partial v^0}{\partial \xi_i}(\mathbf{g}) + O(\epsilon$$

And substituting in (30) yields

$$v^1(\mathbf{g} + h_1\mathbf{e}) = v^1(\mathbf{g} - h_2\mathbf{e}) - \epsilon(h_1 + h_2) \frac{\mu^1}{\mu^0 + \mu^1} e_i \frac{\partial v^0}{\partial \xi_i}(\mathbf{g}) + O(\epsilon^2) \quad (31$$

Since $v = v^0 + v^1$, (31) may be used to reduce (27). After some manipulations, in whicl the facts that v^1 is $O(\epsilon)$ and that the interface are assumed regular are used, we get

$$u^1(\mathbf{x}) = (1 + a)r\omega^2\rho^0 \int_{R_1} (v^0 + v^1)G^0(\boldsymbol{\xi}, \mathbf{x})\, d - \mu^1 \int_{S_1} v^0 \frac{\partial G^0}{\partial n}(\boldsymbol{\xi}, \mathbf{x})\, d\boldsymbol{\xi}$$

$$+ \frac{\epsilon(\mu^1)^2}{\mu^0 + \mu^1} \int_{S_0} (h_1 + h_2) e_i e_j \frac{\partial v^0}{\partial \xi_i} \frac{\partial G^0}{\partial \xi_j} (\xi, \mathbf{x}) \, d\xi + O(\epsilon^2) \qquad (32)$$

where S_0 stands for the skeleton.

The divergence theorem may be used to reduce the second integral in the right-hand member of (32) into a volume integral over R_1. The equation so obtained can be further reduced by means of (2b). Dividing the resulting equation by ϵ and letting $\epsilon \to 0$ yields

$$\left(\frac{\partial u}{\partial \epsilon}\right)_{\epsilon=0}(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{u^1(\mathbf{x})}{\epsilon}$$

$$= r(1 + a)\rho^0\omega^2 \int_{S_0} (h_1 + h_2) u^0 G^0 \, ds$$

$$+ \mu^1 \int_{S_0} (h_1' + h_2') u^0 \frac{\partial G^0}{\partial s} \, ds$$

$$- \frac{\mu^0 \mu^1}{\mu^0 + \mu^1} \int_{S_0} (h_1 + h_2) e_i e_j \frac{\partial u^0}{\partial \xi_i} \frac{\partial G^0}{\partial \xi_j} \, ds$$

$$- \mu^1 \int_{S_0} (h_1 + h_2) u^0 e_i e_j \frac{\partial^2 G^0}{\partial \xi_i \, \partial \xi_j} (\xi, \mathbf{x}) \, ds \qquad (33)$$

where primes stand for the derivatives with respect to the parameter s (which must be taken as the length of the skeleton) and we have written u^0 for the unperturbed solution everywhere. The first-order approximation is given by

$$u(\mathbf{x}) = u^0(\mathbf{x}) + \epsilon \left(\frac{\partial u}{\partial \epsilon}\right)_{\epsilon=0}(\mathbf{x}) + O(\epsilon^2)$$

Complementary remarks. So far it has been assumed that the boundary of the thin inhomogeneity is represented by means of (29) everywhere. If this is not the case, but the length of the part of the boundary where (29) does not hold is $O(\epsilon)$ (Figure 2), then the previous arguments may be modified slightly to obtain the linear approximation.

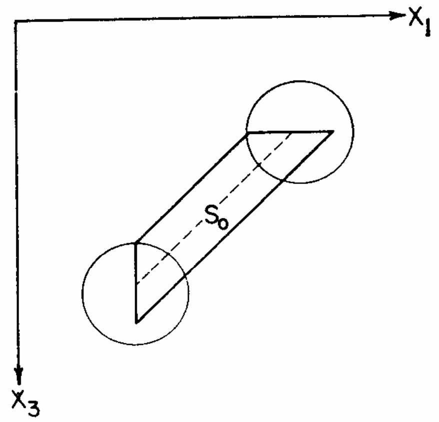

Fig. 2. Boundaries which do not admit of the representation (17) are encircled. Their lengths have to be small.

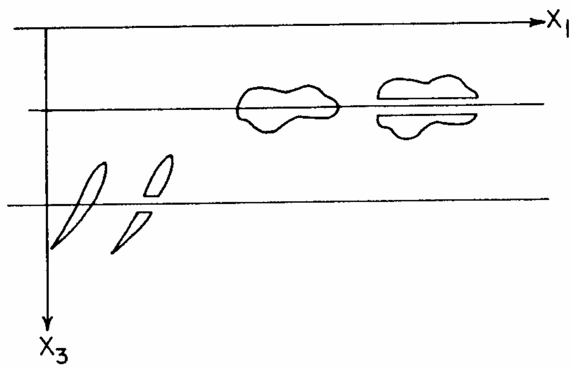

Fig. 3. Limit process by which a thin inhomogeneity crossing an interface may be treated as the superposition of two inhomogeneities both of which do not cross any interface.

Indeed, (27) was obtained regardless of the shape of R_1, and therefore it will still hold. That part of R_1 where (29) is applicable again gives the terms occurring in (33), and the subregion where (29) does not hold will give additional contributions according to (27). The contribution of the first integral in (27) is $O(\epsilon^2)$ because the area of the subregion where (29) does not hold is necessarily $O(\epsilon^2)$. Therefore, the only additional contribution is

$$\lim_{\epsilon \to 0} \left(-\frac{\mu^1}{\epsilon} \int_{S'} v^0 \frac{\partial G}{\partial n} \, d\xi \right) \qquad (34)$$

where S' is that part of the boundary where (29) does not hold. These contributions are concentrated doublets localized at corners or more generally at points where singularities of the boundary occur.

The discussion has been restricted to thin inhomogeneities contained in a homogeneous material. When the thin inhomogeneity crosses any interface (Figure 3) separating two materials of different kinds, it can be considered as the superposition of two different inhomogeneities, both of which are wholly contained in only one medium. The results obtained above hold for every one of these inhomogeneities separately. The net result is that there are additional contributions due to the interfaces.

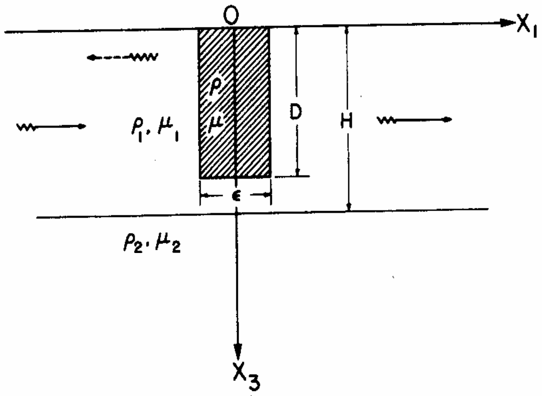

Fig. 4. Geometry of the dike problem.

Transmission of Love waves past a dike. As an illustration of the perturbation method we consider the transmission of Love waves in a single-layered half-space across a thin inhomogeneity whose skeleton is a vertical straight line, extending to a depth D, less than the layer thickness H (Figure 4). ρ and μ are the elastic constants inside the inhomogeneity, ρ_1, μ_1 and ρ_2, μ_2 are those in the layer and the half-space, respectively. Let the incident wave be represented by [cf. *Hererra,* 1964a]

$$e^{-i\omega t}u_m^0(\mathbf{x}) = g_m(\mathbf{x})e^{-i\omega t} = g_m(x)e^{-i\omega t} = f_m(x_3)e^{i(k_m x_1 - \omega t)}$$

where

$$f_m(x_3) = 2\{F'(k_m)\cos\sigma_m H\}^{-1/2}\cdot\cos\sigma_m x_3 \qquad 0 \le x_3 < H$$
$$= 2\{\cos\sigma_m H/F'(k_m)\}^{1/2}e^{-\sigma_m'(x_3 - H)} \qquad x_3 \ge H$$

$$\sigma_m = \left(\frac{\omega^2}{\beta_1^2} - k_m^2\right)^{1/2} \qquad \beta_1^2 = \mu_1/\rho_1$$

$$\sigma_m' = \left(k_m^2 - \frac{\omega^2}{\beta_2^2}\right)^{1/2} \qquad \beta_2^2 = \mu_2/\rho_2$$

and k_m is a root of the dispersion equation,

$$F(k_m) \equiv 2\mu_2\sigma_m'\cos\sigma_m H - 2\mu_1\sigma_m\sin\sigma_m H = 0 \qquad (36)$$

The Green's function which generates outgoing Love waves in the nth mode along the positive direction of the x_1 axis is [cf. *Herrera,* 1964*a*]

$$e^{-i\omega t}G(\mathbf{x}, \boldsymbol{\xi}) = e^{-i\omega t}\frac{i}{2}\sum_n g_n(\mathbf{x})g_n^*(\boldsymbol{\xi}) \qquad (37)$$

where ξ is the source point, and $g_n^*(\mathbf{x})$ is the complex conjugate of $g_n(\mathbf{x})$. If we substitute these expressions in (33) the perturbation $u_{mn}^1(\mathbf{x})$ in the nth mode of the transmitted wave can be written as

$$u_{mn}^1(\mathbf{x}) = \epsilon u_n^0(\mathbf{x})\left(\int_0^D \left\{r\rho^0\omega^2\frac{\mu}{\mu^0} + (\mu - \mu^0)k_n^2 - \frac{\mu^0}{\mu}(\mu - \mu^0)k_n k_m\right\}f_m(\xi_3)f_n(\xi_3)\,d\xi_3 + (\mu - \mu^0)f_m(D)\frac{df_n}{d\xi_3}(D)\right)$$

where ϵ is the thickness of the inhomogeneity and the last term in the right-hand member is a contribution due to the sharp ending of the inhomogeneity. Since the inhomogeneity does not cross the interface,

$$\mu^0 = \mu_1$$
$$\rho^0 = \rho_1$$

Thus the perturbation of the wave at an arbitrary point to the right of the inhomogeneity is

$$u_m^2(\mathbf{x}) = \sum_n u_{mn}^1(\mathbf{x})$$

and at the free surface

$$u_m^1(x_1, 0) = f_m(0)\sum_n A_{mn}ie^{ik_n x_1} \qquad (38)$$

where A_{mn} is conveniently defined and has the following explicit form:

$$A_{mn} = \frac{2\epsilon\sigma_n}{Hk_n}\left\{2\sigma_n H + \left(1 + \frac{\sigma_n^2}{\sigma_n'^2}\right)\sin 2\sigma_n H\right\}^{-1}\cdot\left[\frac{\left(\frac{\omega H}{\beta_1}\right)^2\left(\frac{\rho}{\rho_1} - \frac{\mu}{\mu_1}\right) + \left(\frac{\mu}{\mu_1} - 1\right)\left\{(k_n H)^2 - \frac{\mu_1}{\mu}(k_n H)(k_m H)\right\}}{(k_m H)^2 - (k_n H)^2}\right.$$
$$\left.\cdot(\sigma_n H\sin\sigma_n D\cos\sigma_m D - \sigma_m H\sin\sigma_m D\cos\sigma_n D)\left(\frac{\mu}{\mu_1} - 1\right)\sigma_n H\sin\sigma_n D\cos\sigma_m D\right] \qquad (39)$$

The total displacement at a point $(x_1, 0)$, $(x_1 \gg 0)$ on the surface can be written in the following form

$$u_m(x_1, 0) = [1 + i \sum_n A_{mn} e^{-i(k_m - k_n)x_1}] u_m^0(x_1, 0) \quad (40)$$

The linear approximation (neglecting body waves) to the transmission coefficient, which is defined as the ratio of the amplitude of the incident and transmitted wave at every point is, on the free surface,

$$T_m(x_1) = 1 + \sum_n A_{mn} \sin (k_m - k_n)x_1 \quad (41)$$

The phase shift is given by

$$\Theta(x_1) = \text{Im} \frac{u_m^1(x_1, 0)}{u_m^0(x_1, 0)} = \sum_n A_{mn} \cos (k_m - k_n)x_1 \quad (42)$$

The perturbation in the wave number,

$$k_m^1 = \frac{d\Theta}{dx_1} = -\sum_n (k_m - k_n) A_{mn} \cdot \sin (k_m - k_n)x_1 \quad (43)$$

and, consequently, the apparent phase velocity of waves of frequency ω is

$$c(x_1) = \omega / \{k_m + k_m^1\} \quad (44)$$

The apparent velocity as given by (44) is the phase velocity that would be determined if a sequence of seismographs were placed to the right of the dike and the records were Fourier analyzed. It is strongly dependent on the shape of the incoming motion, because it is the result of superposing surface waves in different modes.

The reflection coefficient can be calculated in a similar manner.

Numerical calculations and discussions. Equations 38 to 44 show that the transmitted wave in the same mode as the incident wave suffers no

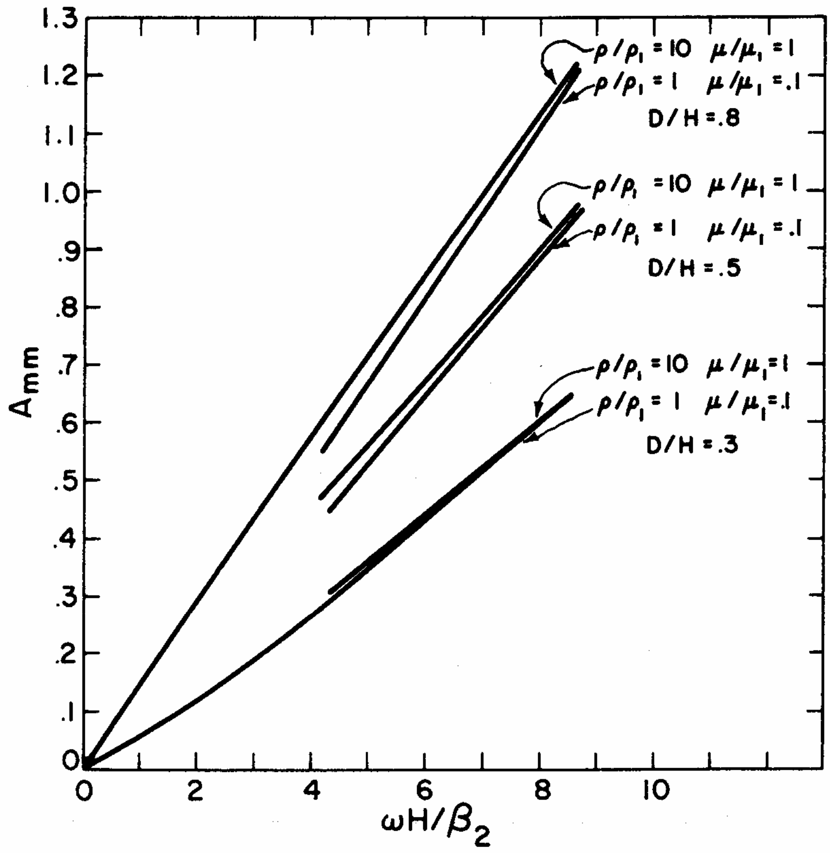

Fig. 5. Jump in the phase of the transmitted wave plotted against frequency for different values of μ/μ_1, ρ/ρ_1, and D/H.

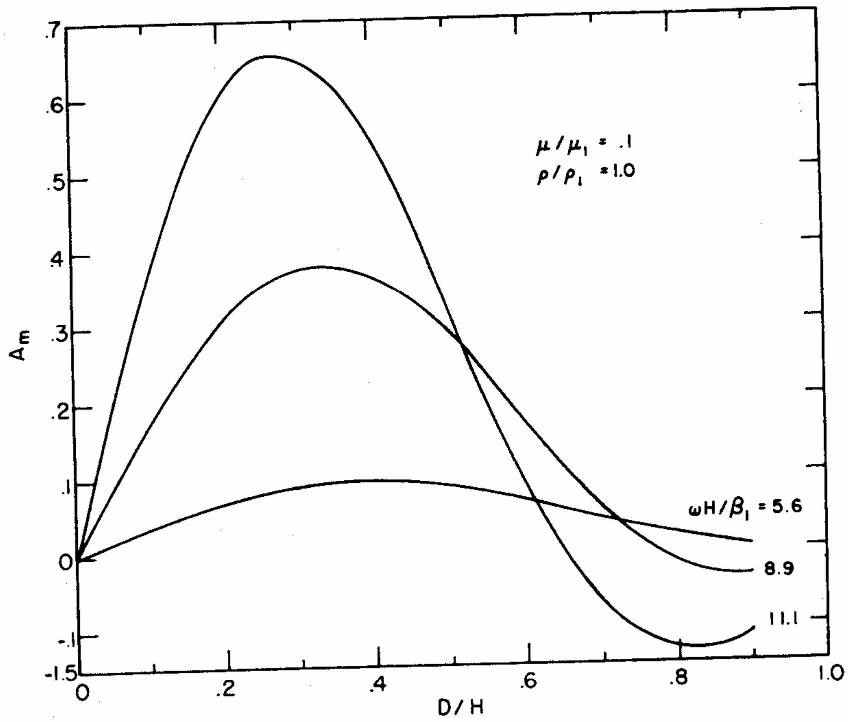

Fig. 6. Amplitude of the perturbations (A_m) plotted against the depth of the discontinuity for different frequencies, showing the effect of a discontinuity in the rigidity of the material (same density).

perturbation in the amplitude. However, the observed displacement of the transmitted wave consists of the superposition of all the perturbed modes. Since all these modes differ in phase velocity, the amplitude of the perturbed wave is naturally a function of x_1, as is evident from the expressions given above. The perturbation in amplitude is greatest at points where the perturbed modes are in the same phase. The extremum values are

$$\pm \sum_n |A_{mn}|$$

As a consequence of the dependence of T_m on x_1, the transmitted and reflected waves are augmented at some points and attenuated at others. The perturbation suffered by the incident mode itself can be written as

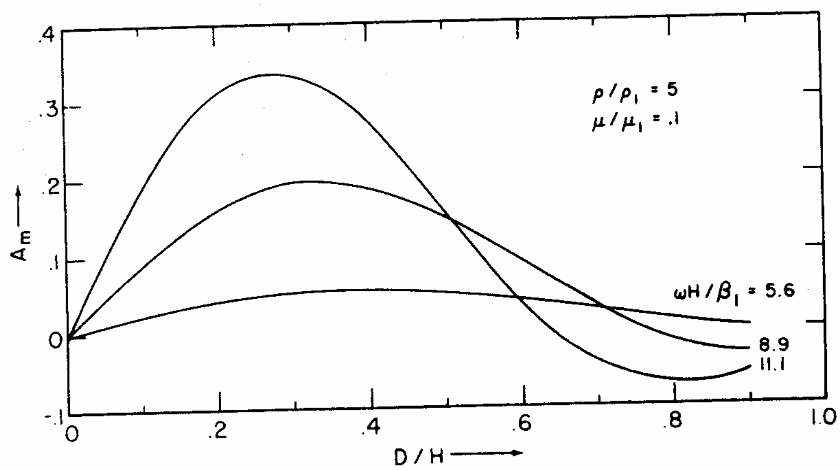

Fig. 7. Same as Figure 6, showing the effect of a density discontinuity (same rigidity).

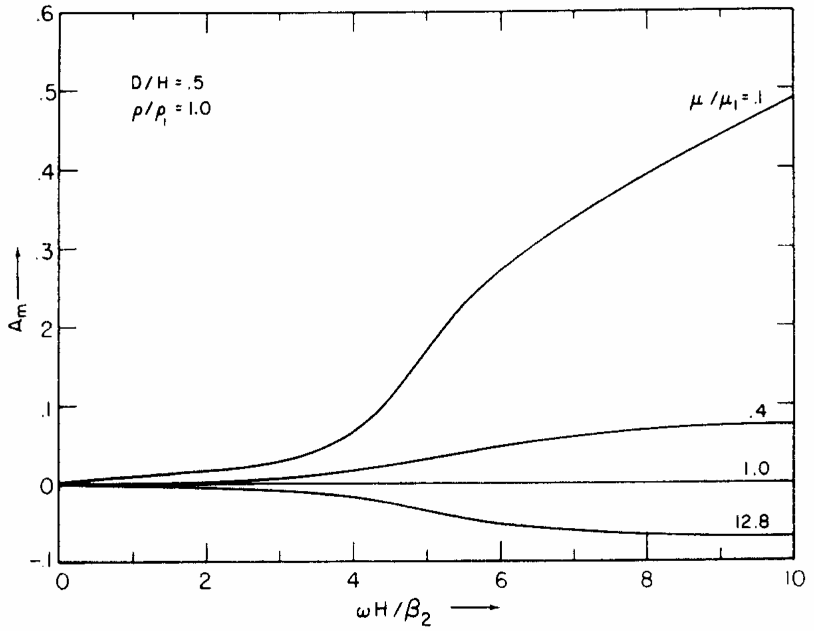

Fig. 8. A_m plotted against frequency for different values of the rigidity inside the discontinuity (same density).

$$u_{mm}{}^1(x_1, 0) = iA_{mm}u_m{}^0(x_1, 0)$$

where the real quantity A_{mm} has been obtained by proceeding to the limit of A_{mn} as $k_n \to k_m$. This shows that there is a sudden change in the phase of the transmitted waves just after crossing the dike. This jump in the phase has been plotted against frequency in Figure 5 for different values of the parameters involved. The total perturbation in phase consists of the superposition of this shift, which is independent of the point of observation and other terms (given by (42)) depending on x_1. The amplitudes of the position-dependent phase shift have been plotted in Figures 6, 7, and 8.

The number of real roots of the frequency equation (36) corresponding to frequency ω is given by the least integer exceeding

$$(\omega H/\pi)(1/\beta_1{}^2 - 1/\beta_2{}^2)^{1/2} \qquad (46)$$

Thus if

$$\omega H/\beta_2 \leq \pi/[(\beta_2/\beta_1)^2 - 1]^{1/2} \qquad (47)$$

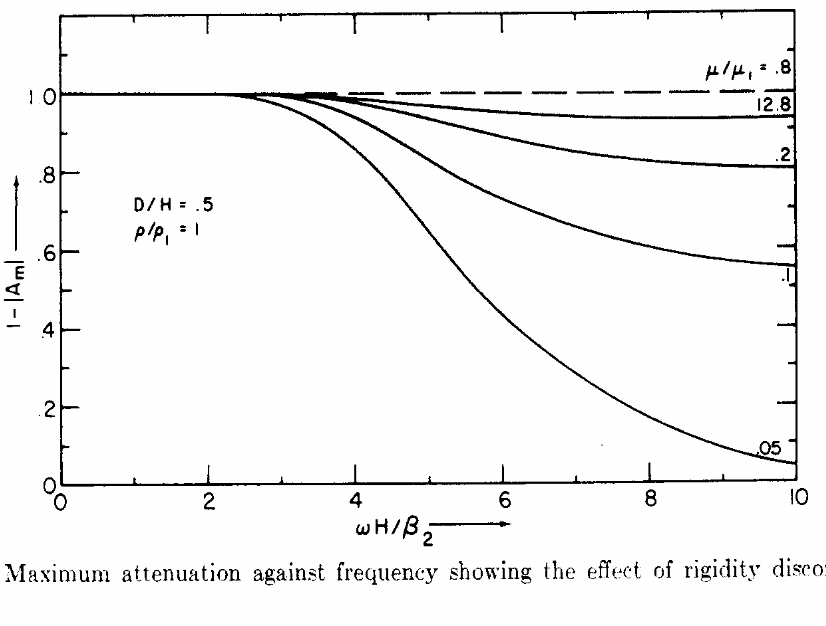

Fig. 9. Maximum attenuation against frequency showing the effect of rigidity discontinuity.

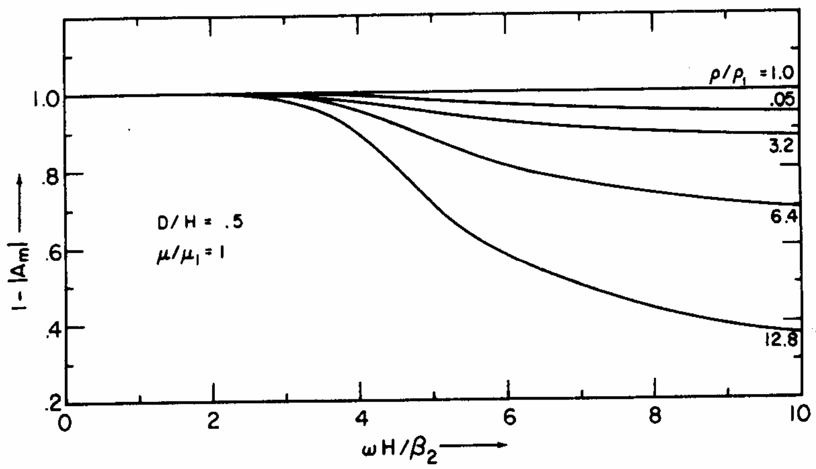

Fig. 10. Same as Figure 9, showing the effect of density discontinuity.

the Love wave equation has only one root. Equation 47 thus gives the critical frequency, so that the incident waves of lesser frequency will not be affected in amplitude by the discontinuity.

In the numerical example worked out we have restricted the computations to the case where the depth of the discontinuity is less than the layer thickness. If $D > H$, (39) has to be modified because of the contribution from the intersection of the interface with the discontinuity. The following physical properties have been assigned to the model:

$$\beta_2/\beta_1 = 1.278$$

$$\mu_2/\mu_1 = 1.8$$

$$2\epsilon/H = 0.05$$

The other parameters, D/H, μ/μ_1, and ρ/ρ_1, have been assigned different numerical values. With these values of the constants, the frequency equation (36) gives multiple roots only when $\omega H/\beta_2 \geq 4$. The incident wave is always assumed to be in the first mode which corresponds to $m = 0$. The computations have been made in the range of ω given by $4 \leq \omega H/\beta_2 \leq 12$. Equation 36 has three roots only when $\omega H/\beta_2 \geq 11$. In all other cases it has two roots. But the effect of the third root in this case is very small. Since A_{mm} is purely imaginary, in the range of ω we have considered that it is enough to analyze the behavior of the term $A_m = A_{m1}$ in (38) to (44) for the perturbation. The curves plotted in Figures 5 to 12 show the variation of the phase and amplitude perturbations with the different

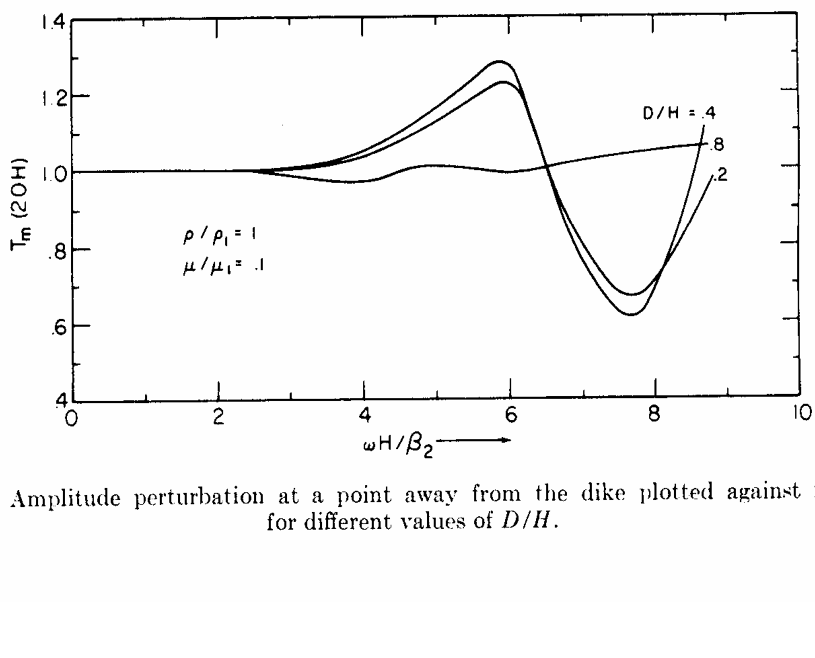

Fig. 11. Amplitude perturbation at a point away from the dike plotted against frequency for different values of D/H.

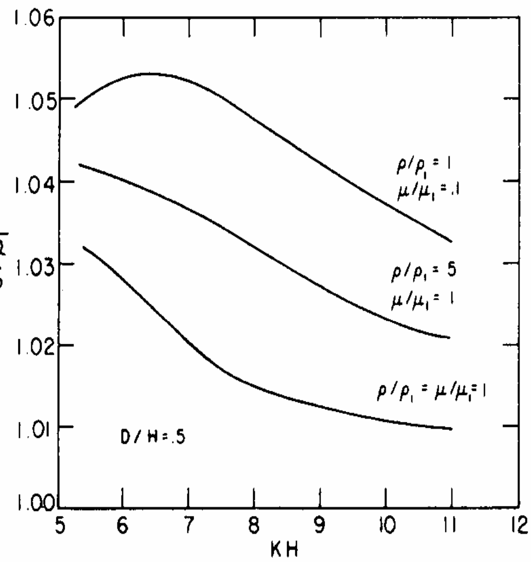

Fig. 12. Apparent velocities.

parameters as well as the frequency of the incident waves. In all the cases except that in Figure 11 the extreme behavior of the perturbation has been analyzed. The quantity A_m in Figures 6 to 8 will change sign at some points of observation. The quantity $1 - |A_m|$ in Figures 9 and 10 gives the maximum possible attenuation in the amplitude of the transmitted waves (it must be noted that they do not refer to the same point of observation). There are points to the right of the dike where the amplitude of the transmitted wave is exactly $1 + |A_m|$. Figure 11 is representative of the behavior of the transmission coefficient at a particular point, away from the dike. Figure 12 shows the maximum possible change in apparent velocity due to the presence of the dike.

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