A CORRESPONDENCE PRINCIPLE FOR VISCOELASTIC WAVE PROPAGATION

I. HERRERA
and
M. E. GURTIN

Reprinted from
QUARTERLY OF APPLIED MATHEMATICS
Vol. XXII, No. 4, January, 1965

A CORRESPONDENCE PRINCIPLE FOR VISCOELASTIC WAVE PROPAGATION*

By I. HERRERA, (*University of Mexico*) and M. E. GURTIN, (*Brown University*)

Introduction. It is a well known result of classical linear elasticity theory that the speed of propagation U of an acceleration wave, at a point x in an elastic solid of elasticity tensor $c_{ijkl}(x)$ and density $\rho(x)$, is a solution to the eigenvalue problem**

$$[c_{ijkl}(x)n_j n_l - \rho(x)U^2\delta_{ik}]a_k = 0. \tag{1.1}$$

Here the unit vector n is the direction of propagation at x. Since the elasticity tensor obeys the symmetry relations

$$c_{ijkl} = c_{klij} , \tag{1.2}$$

it is clear that (1.1) has three (not necessarily distinct) solutions U_1 , U_2 , U_3 . More-

*Received March 13, 1964.

**We use indicial notation and Cartesian tensors throughout. Thus subscripts have the range (1, 2, 3), x_i denote rectangular Cartesian coordinates, and $x = (x_1, x_2, x_3)$. Moreover summation over repeated subscripts is implied, δ_{ij} is Kronecker's delta, subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate.

over these solutions will be *real* and nonzero if and only if $c_{ijkl}(x)$ is strongly elliptic in the sense that

$$c_{ijkl}(x)n_jn_lv_iv_k > 0 \quad \text{whenever} \quad v \neq 0. \tag{1.3}$$

A recent result of Toupin and Gurtin [1], which is a generalization to anisotropic materials of a theorem due to Gurtin and Sternberg [2], demonstrates the connection between wave propagation and uniqueness. This result may be stated roughly as follows: the displacement problem of linear elastodynamics has at most one solution provided c_{ijkl} is constant and strongly elliptic. Thus, if the medium is homogeneous, a sufficient condition for uniqueness is that three linearly independent acceleration waves can propagate in every direction.

The foregoing results can be easily generalized to visco-elastic solids with the aid of a correspondence principle which we prove in this paper. This principle asserts that the speed of propagation U of an acceleration wave at a point x in a viscoelastic solid with relaxation tensor $G_{ijkl}(x, t)$ and density $\rho(x)$ satisfies

$$(G_{ijkl}(x, 0)n_jn_l - \rho(x)U^2\delta_{ik})a_k = 0. \tag{1.4}$$

Hence, by (1.1), this speed is identical to the speed of propagation through an elastic solid whose elasticity tensor is

$$c_{ijkl}(x) = G_{ijkl}(x, 0). \tag{1.5}$$

This correspondence principle brings out the remarkable (but intuitively obvious) fact that the speed of propagation of an acceleration wave through a viscoelastic solid depends only upon the *initial value* $G_{ijkl}(x, 0)$ of the relaxation tensor and is independent of the behavior of $G_{ijkl}(x, t)$ with t.

Assume for the remainder of this section that $G_{ijkl}(x, t)$ is *initially symmetric*, i.e.,*

$$G_{ijkl}(x, 0) = G_{klij}(x, 0). \tag{1.6}$$

Then the eigenvalue problem (1.4) has three solutions. Further these solutions will be *real* and non-zero if and only if $G_{ijkl}(x, t)$ is *initially strongly elliptic*, that is if and only if $G_{ijkl}(x, 0)$ is strongly elliptic.

Edelstein and Gurtin [3] have shown that the displacement problem of dynamic viscoelasticity theory has at most one solution provided $G_{ijkl}(x, t)$ is independent of x and initially strongly elliptic. Thus a sufficient condition for uniqueness to hold in a homogeneous viscoelastic medium is that three independent acceleration waves can propagate in every direction in the body.

The correspondence principle. The fundamental system of field equations in the linear theory of viscoelasticity consists of the following:**

$$2\epsilon_{ij}(x, t) = u_{i,j}(x, t) + u_{j,i}(x, t), \tag{2.1}$$

$$\sigma_{ij,j}(x, t) + f_i(x, t) = \rho(x)u_i^{(2)}(x, t), \qquad \sigma_{ij}(x, t) = \sigma_{ji}(x, t), \tag{2.2}$$

$$\sigma_{ij}(x, t) = \int_{-\infty}^{t} G_{ijkl}(x, t - s)\epsilon_{kl}^{(1)}(x, s)\, ds. \tag{2.3}$$

---

*Notice that $G_{ijkl}(x, 0)$ automatically has this property if it is isotropic.

**We use the notation $g^{(n)} = \partial^n g/\partial t^n$.

Here $u_i(x, t)$, $\epsilon_{ij}(x, t)$, $\sigma_{ij}(x, t)$, and $f_i(x, t)$ are the Cartesian components of the displacement vector, the strain tensor, the stress tensor, and the body force density vector defined for every pair (x, t) such that x is a point of the open region $\mathcal{R}$ occupied by the interior of the body and t $(-\infty < t < \infty)$ is the time. Further $\rho(x)$ is the mass density defined for every $x \in \mathcal{R}$, and $G_{ijkl}(x, t)$ are the components of the relaxation tensor which are defined for every $(x, t) \in \mathcal{R} \times [0, \infty)$ and satisfy

$$G_{ijkl} = G_{jikl} = G_{ijlk} . \tag{2.4}$$

The first symmetry relation follows from the symmetry of the stress tensor, while the second follows (without loss in generality) from the symmetry of the strain tensor.

We suppose once and for all that we are given a solution u_i, ϵ_{ij}, σ_{ij} of (2.1), (2.2), (2.3) on $\mathcal{R} \times (-\infty, \infty)$ which meets the initial condition

$$u_i(x, t) = 0, \qquad (x, t) \in \mathcal{R} \times (-\infty, 0) \tag{2.5}$$

and corresponds to data ρ, G_{ijkl}, f_i which satisfies:

(*i*) $\rho > 0$ is continuous;

(*ii*) G_{ijkl} is twice continuously differentiable and meets (2.4);

(*iii*) f_i is continuous.

Moreover we assume the existence of an *acceleration wave* propagating through $\mathcal{R}$—that is a one-parameter family of surfaces S_t $(0 \le t < \infty)$ which has the following properties:

(*iv*) the hypersurface

$$\textstyle\sum_S = \{(x, t) \mid x \in S_t (0 \le t < \infty)\}$$

is smooth and orientable;

(*v*) given an $x \in \mathcal{R}$, the set

$$D_x = \{t \mid x \in S_t\}$$

has measure zero;

(*vi*) u_i, $u_i^{(1)}$, $u_{i,j}$ are continuous on $\mathcal{R} \times (-\infty, \infty)$, $u_i^{(2)}$, $u_{i,j}^{(1)}$, $u_{i,jk}$ have *jump* discontinuities across $\sum_S$ but are continuous everywhere else on $\mathcal{R} \times (-\infty, \infty)$, and $u_{i,jk}$ is bounded on compact subsets of $\mathcal{R} \times (-\infty, \infty)$.

Let $(x, t) \in \sum_S$. Condition (iv) then implies the existence of a normal vector ν at (x, t). Indeed, such a vector is given by

$$\nu = (n_1, n_2, n_3, -U), \tag{2.6}$$

where n_i are the components of a unit vector which is normal to S_t at x and $U \ge 0$. The latter condition is imposed to insure the uniqueness of ν at points where $U \ne 0$. The number U is called the *speed of propagation* of S_t at x, while the unit vector n is called the *direction* of S_t at x. Condition (v) is merely the requirement that the surface S_t, as it progresses in time, does not pass through a given point x too many times. This condition together with condition (*vi*) implies that for any $x \in \mathcal{R}$, $u_{i,jk}(x, t)$, $u_{i,j}^{(1)}(x, t)$, and $u_i^{(2)}(x, t)$ are discontinuous only when $t \in D_x$. Finally we remark that all integrals are to be taken in the sense of Lebesgue.

We are now in a position to prove the

Correspondence Principle for Wave Propagation. *The speed of propagation U of the acceleration wave S_t at a point $x \in S_t$ is a solution of the eigenvalue problem*

$$(G_{ijkl} n_j n_l - \rho U^2 \delta_{ik}) a_k = 0, \tag{2.7}$$

where

$$c_{ijkl} = G_{ijkl}(x, 0) \tag{2.8}$$

and n is the direction of S_t at x. That is, the speed of propagation is identical to the speed of propagation of S_t through an elastic solid whose elasticity tensor is c_{ijkl}.

PROOF. Integrate (2.3) by parts and use (2.1), (2.5) to verify that

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} + G^{(1)}_{ijkl} * \epsilon_{kl} , \tag{2.9}$$

where c_{ijkl} is given by (2.8) and

$$G^{(1)}_{ijkl} * \epsilon_{kl}(x, t) = \int_{-\infty}^{t} G^{(1)}_{ijkl}(x, t - s)\epsilon_{kl}(x, s)\, ds. \tag{2.10}$$

Now substitute (2.1), (2.9) into (2.2) and use the symmetries (2.4) together with (2.8) to derive the displacement equations of equilibrium in the form

$$c_{ijkl}u_{k,lj} = \rho u^{(2)}_i - F_i , \tag{2.11}$$

where

$$F_i = c_{ijkl,j}u_{k,l} + G^{(1)}_{ijkl,j} * u_{k,l} + G^{(1)}_{ijkl} * u_{k,lj} + f_i . \tag{2.12}$$

Define the functions p_{kl} and q_i through

$$p_{kl} = u_{k,l} , \qquad q_i = u^{(1)}_i. \tag{2.13}$$

Then p_{kl} and q_i are continuous, have continuous derivatives except on $\sum_S$, and satisfy the compatibility conditions

$$p^{(1)}_{kl} = q_{k,l} . \tag{2.14}$$

Moreover, the jumps in the (four dimensional) gradients of p_{kl} and q_i are parallel to the normal ν of $\sum_S$ and hence, because of (2.6), take the form*

$$\begin{aligned} [p_{kl,j}] &= \lambda_{kl}n_j , \\ [p^{(1)}_{kl}] &= -\lambda_{kl}U, \\ [q_{k,l}] &= a_k n_l , \\ [q^{(1)}_k] &= -a_k U. \end{aligned} \tag{2.15}$$

Thus (2.13), (2.14), (2.15), together with the continuity of c_{ijkl} and ρ across $\sum_S$ imply

$$[c_{ijkl}u_{k,lj}] = c_{ijkl}\lambda_{kl}n_j , \qquad [\rho u^{(2)}_i] = -\rho a_i U, \qquad -\lambda_{kl}U = a_k n_l . \tag{2.16}$$

By (2.8) and the smoothness assumptions (ii), (iii), and (vi) it follows that

$$[c_{ijkl,j}u_{k,l}] = [G^{(1)}_{ijkl,j} * u_{k,l}] = [f_i] = 0. \tag{2.17}$$

We now prove that $[G^{(1)}_{ijkl} * u_{k,lj}] = 0$. Fix $(x, t) \in \sum_S$, let $A = G^{(1)}_{ijkl} * u_{k,lj}$, and notice that

---

*Here and in the sequel we use the usual notation for the jump $[g]$ of a function g across Σ_s. That is

$$[g](x, t) = \lim_{(x^+, t^+) \to (x, t)} g(x^+, t^+) - \lim_{(x^-, t^-) \to (x, t)} g(x^-, t^-),$$

where (x^+, t^+) and (x^-, t^-) lie on opposite sides of Σ_s.

$$A(x^+, t^+) = \int_0^t G^{(1)}_{ijkl}(x^+, t^+ - s)u_{k,lj}(x^+, s)\, ds + \int_t^{t^+} G^{(1)}_{ijkl}(x^+, t^+ - s)u_{k,lj}(x^+, s)\, ds. \tag{2.18}$$

Let Λ denote the *compact* set $\Omega \times [0, T]$, where Ω is a solid sphere centered at x and contained in $\mathfrak{R}$ while $T > t$. Then hypotheses (*ii*) and (*vi*) imply that $G^{(1)}_{ijkl}$ and $u_{k,lj}$ are bounded on Λ. Thus the second term in (2.18) tends to zero as $(x^+, t^+) \to (x, t)$. Moreover conditions (*ii*), (*v*), and (*vi*) yield

$$G^{(1)}_{ijkl}(x^+, t^+ - s)u_{k,lj}(x^+, s) \to G^{(1)}_{ijkl}(x, t - s)u_{k,lj}(x, s) \tag{2.19}$$

for almost every $s \leq t$, i.e., for $s \notin D_x$. Therefore the boundedness of $G^{(1)}_{ijkl}$ and $u_{k,lj}$ on Λ and Lebesgue's bounded convergence theorem imply

$$\lim_{(x^+, t^+) \to (x, t)} A(x^+, t^+) = \int_0^t G^{(1)}_{ijkl}(x, t - s)u_{k,lj}(x, s)\, ds. \tag{2.20}$$

Clearly the same result follows for $\lim_{(x^-, t^-) \to (x, t)} A(x^-, t^-)$ and thus

$$[G^{(1)}_{ijkl} * \dot{u}_{k,lj}] = 0. \tag{2.21}$$

Equations (2.12), (2.17), and (2.22) imply $[F_i] = 0$. It therefore follows from (2.11), (2.16) that

$$(c_{ijkl}n_j n_l - \rho U^2 \delta_{ik})a_k = 0, \tag{2.22}$$

and this completes the proof.

Acknowledgement. The authors are indebted to W. Edelstein who read the manuscript and offered many helpful criticisms. This research was supported in part (M. E. G.) by the Office of Naval Research under Contract Nonr 562(25) and in part (I. H) by the Institutes of Geophysics and Engineering at the University of Mexico.

References

1. R. A. Toupin and M. E. Gurtin, *A uniqueness theorem for the displacement boundary-value problem of linear elastodynamics*, to appear in Q. Appl. Math.
2. M. E. Gurtin and Eli Sternberg, *A note on uniqueness in classical elastodynamics*, Q. Appl. Math., 19 (1961) 169–171
3. W. S. Edelstein and M. E. Gurtin, *Uniqueness theorems in the linear theory of anisotropic viscoelastic solids*, Rept. No. 24, Contract Nonr 562 (25), Brown University, February, 1964.