FUNCTIONAL PARTIAL DIFFERENTIAL

EQUATIONS OF HYPERBOLIC TYPE

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HYPERBOLIC SYSTEMS OF FUNCTIONAL

PARTIAL DIFFERENTIAL EQUATIONS.

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1.- Introduction. In the general theory of materials with memory the stress is assumed to be a functional of the whole history of the strain [1, 2, 3, 4, 5, 6], so that the equations of motion which for purely elastic materials constitute a system of partial differential equations, in the theory of materials with memory they are partial differential equations whose coefficients are functionals of the strain. It seems natural to call an equation of this type "Functional Partial Differential Equation".

Functional partial differential equations occur in applications, not only in the theory of materials with memory. They occur in some other fields such as the theory of nuclear reactors.

In spite of their interest from the practical point of view, apparently they have not been so far the subject of a systematic treatment from a purely mathematical point of view.

This is not the case for functional differential equations for functions of a single variable, which have been extensively studied [7, 8, 9, 10, 11, 12, 13]. It seems therefore that a systematic treatment of functional partial differential equations is wanting . The present work is a first attempt to fill this gap.

For some time during the development of the theory of viscoelastic materials, it was thought that surfaces of discontinuity were unable to exist in viscoelastic media [14]. The possibility of existence of surfaces of discontinuity is related to the hyperbolic character of the governing equations.

Apparently both were overlooked for some time. Glanz and Lee [15] considered a model which led to a hyperbolic system of partial differential equations (with memory) and this was integrated by the method of characteristics. A systematic study of discontinuous solutions in materials with memory has been done only recently [14]. Herrera and Gurtin [16] calculated the acoustical tensor in full generality and later Coleman and Gurtin partly in collaboration with Herrera [17, 18, 19] developed a general theory of wave propagation in materials with memory. Varley [14 , 26] independently, neglecting thermal effects, calculated a general partial differential equation for the change of amplitude as the wave progresses and integrated it explicitly for plane, cylindrical, or spherical waves propagating into an infinite body at rest. Although the above work is not concerned with the existence of solutions, discontinuous solutions can be easily constructed [20, 27]. As a consequence of all these results, it has become clear that many properties corresponding to hyperbolic differential equations are preserved by the equations of motion for materials with memory if the functionals defining the memory are assumed to satisfy appropriate hypothesis.

It is therefore natural to try to develop a theory parallel to the theory of hyperbolic partial differential equations for functional differential equations.

In this paper attention will be restricted to systems of functional differential for functions of two independent variables. In a manner similar to what has been done for partial differential equations [21] we first introduce the notion of conservation law and then define what we mean by a quasi-linear system of functional differential equations. Then some necessary conditions for the existence of weak discontinuties are discussed and a definition of hyperbolic system is given.

The main results of the paper are existence and uniqueness theorems in a determinacy domain for a Cauchy problem of the hyperbolic system of functional partial differential equations.

The paper is based on a previous paper by Friedrichs [22]. Some of the ideas due to Douglis [23] have also been helpful in the development of the work.

The hypothesis of the main theorems refer to some sort of generalized Lifshitz conditions. This type of conditions can be deduced from the existence of the Frechet derivatives of sufficiently high order when they are assumed to exist and appropriate smoothness hypothesis are imposed on them. This was perhaps desirable to do, but the author preferred to assume directly conditions of the Lifshitz type because they are intuitively appealing and in applications can be directly checked. Besides, the paper is already considerably long and it was thought convenient to avoid enlarging it even more with the discussion of a point which is not essential for the theory.

2.- Notation. The functional differential equations which we shall investigate refer to a system $u=\{u_i\}$ of ν functions $u_1, \dots, u_\nu$, simply called "a function", of the two variables x and t. The coefficients of the equations, given as square matrices

$$A=\{A_{ij}\} \quad ; \quad B=\{B_{ij}\} \quad ; \quad i,j=1,\dots,\nu$$

and the inhomogeneous term $C=\{C_n\}$ are functionals of u. More c for these two functions and we shall do so.

In a similar manner, we associate with every point $(x, t, \tau) \in \hat{R}(\beta, \delta)$ a functional $B(x, t, \tau, \cdot)$ which is defined in the same set of functions as A and C. Using the convention adopted before, we define the function ${}^u b$ in $\hat{R}(\beta, \delta)$ by

$${}^u b(x,t,\tau) = B(x,t,\tau,u)$$

and we shall drop the superindice u whenever this does not lead to confussion. It must be emphasized that the functional $B(x, t, \tau, \cdot)$ is defined on functions defined in the interval $[0, t]$.

In section 8 we define some functionals in terms of the functionals A, B, C, and we shall use for them conventions similar to those introduced here.

For vectors $f = \{f_i\}$ and matrices $r = \{r_{ij}\}$ we introduce the absolute value

$$|f| = \max_{i=1,\dots,\nu} |f_i|$$

$$|r| = \max_{i=1,\dots,\nu} \sum_j |r_{ij}|$$

so that
$$|rf| \leq |r|\,|f|$$

If f and r depend on x, t in $R(\beta, \delta)$, we introduce the notation

$$|f|_t = \max_{0\leq\tau\leq t} |f(x,\tau)| \quad ; \text{ fixed } x$$

$$|r|_t = \max_{0\leq\tau\leq t} |r(x,\tau)| \quad ; \text{ fixed } x$$

Observe that
$$|r(x,t)| \leq |r|_t$$

We define further

$$\|f\| = \max_{(x,t)\in \mathcal{R}} |f(x,t)|$$

$$\|r\| = \max_{(x,t)\in \mathcal{R}} |r(x,t)|$$

When f and r depend on x, t, τ in $\hat{\mathcal{R}}(\beta,\delta)$,

$$\|f\| = \max_{(x,t,\tau)\in \hat{\mathcal{R}}} |r(x,t,\tau)|$$

$$\|r\| = \max_{(x,t,\tau)\in \hat{\mathcal{R}}} |r(x,t,\tau)|$$

With these definitions

$$\|rf\| \leq \|r\| \, \|f\|$$

The partial derivatives will be denoted most often by means of subindices.

The restriction to the set $\mathcal{J}$ of a function u defined in $\mathcal{R}(\beta,\delta)$, will be denoted by $\bar{u}$, i.e.

$$\bar{u}(x) = u(x,0) \; ; \quad X_1 \leq x \leq X_2$$

The set of continuous functions defined on $\mathcal{R}(\beta,\delta)$ will be denoted by $\mathfrak{F}(\beta,\delta)$. The set of those with first order continuous derivatives by $\mathfrak{F}^1(\beta,\delta)$ and the set of those with second order continuous derivatives by $\mathfrak{F}^2(\beta,\delta)$. We shall write $u \in \mathfrak{F}_\Omega(\beta,\delta)$ if and only if $u \in \mathfrak{F}(\beta,\delta)$ and

$$\|u\| < \Omega \tag{2.1.a}$$

Also, $u \in \mathfrak{F}^1_\Omega(\beta,\delta)$ if and only if $u \in \mathfrak{F}^1_\Omega(\beta,\delta)$ and

$$\|u\| + \|u_x\| + \|u_t\| < \Omega \tag{2.1.b}$$

Finally, $u \in \mathfrak{F}^2_\Omega(\beta,\delta)$ if and only if $u \in \mathfrak{F}^2(\beta,\delta)$ and

$$\|u\| + \|u_x\| + \|u_t\| + \|u_{xx}\| + \|u_{xt}\| + \|u_{tt}\| < \Omega \qquad (2.1.c)$$

This notation will be used for vectors as well as for matrices

Analogous definitions are given for $\mathcal{J}(\beta, \delta)$, $\hat{\mathcal{J}}^1(\beta, \delta)$ and $\hat{\mathcal{J}}^2(\beta, \delta)$ replacing $\mathcal{R}(\beta, \delta)$ by $\hat{\mathcal{R}}(\beta, \delta)$ and including in the left hand of the inequalities(2.1.b) and(2.1.c) the partial derivatives with respect to τ

3. Conservation Laws. For fixed x and t, let $F(x, t, \cdot)$ and $C(x, t, \cdot)$ be vector valued functionals defined on some subset of the set of functions

$$f : (-\infty, t] \longrightarrow R^\nu \qquad (3.1)$$

A system of conservation laws is an equation of the form

$$u_t + \{F(x,t,u)\}_x + C = 0 \qquad (3.2)$$

Following Coleman, [Coleman, Gurtin and Herrera 17] we consider the set of functions $f_{(r)}:(-\infty, t) \longrightarrow R^\nu$ obtained by taking for every f its restriction $f_{(r)}$ to the interval $(-\infty, t)$. Without specifying further, assume defined a norm $\|f_r\|_t$ in this set of functions. We define the norm of a function f by

$$\|f(\cdot)\|_t^{\oplus} = \|f_{(r)}(\cdot)\|_t + |f(t)| \qquad (3.3)$$

We suppose that for fixed x and t, $F(x, t, \cdot)$ has a functional derivative in the sense of Frechet with respect to this norm,

$$F(x,t,f+g) = F(x,t,f) + DF(x,t,f)\, g(t) \qquad (3.4)$$

$$+ \delta F(x,t,f \,|\, g_r(\cdot)) + o(\|g(\cdot)\|^{\oplus})$$

where the operator introduced by Coleman [24, 25] is such that

$DF(x, t, f)$ is a matrix valued functional, so that $DF(x, t, f)g(t)$ is linear in $g(t)$, on the other hand $\delta F(x, t, f|\cdot)$ is a linear functional of the restriction g_r of g.

We assume even more, that for fixed f, the function $F(x, t, f)$ has a partial derivative with respect to x. In this case a functional $F_x(x, t, \cdot)$ may be defined in the same set of functions as F. It is given by

$$F_x(x,t,f) = \frac{\partial}{\partial x}\{F(x,t,f)\}$$

Recall that here f is a fixed function independent of x.

In this notation

$$\{F(x,t,u)\}_x = DF(x,t,u)\,u_x(t) + \delta F\left(x,t,u|(u_x)_r\right) + F_x(x,t,u)$$

and (3.2) becomes

$$u_t + Au_x + \delta F\left(x,t,u|(u_x)_r\right) + C = 0 \qquad (3.5)$$

where $A(x, t, \cdot)$ is a matrix valued functional defined by

$$A(x,t,u) = DF(x,t,u)$$

and we have modified the definition of C adding F_x.

In what follows we restrict attention to those cases for which the linear functional of $(u_x)_r$, $\delta F(x, t, u\,(u_x)_r)$ can be written as an integral i.e.

$$\delta F\left(x,t,u|(u_x)_{(r)}\right) = \int_{-\infty}^{t} B(x,t,\tau,u)\,u_x(\tau)\,d\tau$$

where $B(x, t, \tau, \cdot)$ for fixed x, t, τ, is a matrix valued functional defined in the same set of functions as F.

Then

$$u_t + A(x,t,u)u_x + \int_{-\infty}^{t} B(x,t,\tau,u)u_x(\tau)d\tau + C(x,t,u) = 0 \quad (3.6)$$

In what follows any system of the form (3.6) will be called quasi-linear system of functional partial differential equations, independently of whether it can be derived from a conservation law of the form (3.2) or not.

4.- Propagation of discontinuities. Characteristics. Assume that a solution of (3.6) exists in a region R, which is continuous in R, and whose first order derivatives are continuous except across some line where they have jump discontinuities.

Let $\phi = 0$ be the equation of the line of discontinuity and define α by

$$\frac{\partial \phi}{\partial t} + \alpha \frac{\partial \phi}{\partial x} = 0 \qquad (4.1)$$

Here we assume that grad $\phi \neq 0$.

Then α is a function defined on the line of discontinuity which has the interpretation of the speed of propagation of the discontinuity. If $\frac{\partial \phi}{\partial x} = 0$, such an α does not exist and we say that the speed of propagation is infinite.

Since u is continuous we have the Rankin-Hungoniot relations

$$[u_t] + \alpha [u_x] = 0 \qquad (4.2)$$

for the jumps in the partial derivatives of u.

On the other hand, assume that for the given continuous function $u(x, t)$, the functions $B(x, t, \tau, u)$ and $u_x(x, \tau)$ are absolutely integrable in the interval $(-\infty < \tau \leq t)$ for every $(x, t) \in R$ and that

$$\int_{-\infty}^{t} |B(x,t,\tau,u)|\, d\tau$$

and

$$\int_{-\infty}^{t} |u_x(x,\tau)|\, d\tau$$

are continuous as functions of (x, t). Assume further that

$$^{u}c(x,t) = C(x,t,u)$$

and

$$^{u}a(x,t) = A(x,t,u)$$

are continuous. Then, it follows from (3.6) that

$$[u_t] + A(x,t,u)[u_x] = 0 \tag{4.3}$$

Equations (4.2) and (4.3) together imply

$$A[u_x] = \alpha[u_x] \tag{4.4}$$

Thus, the well known results for systems of partial differential equations remain valid. Any line whose speed of propagation α satisfies (4.4) will be called characteristic. *Observe that (4.4) is in agreement with results obtained by Coleman, Gurtin and Herrera [16, 17].*

We shall say that the system (3.6) is hyperbolic at a point (x, t) with respect to the function $f:(-\infty, t] \to R^{\nu}$ if all the eigen values of $A(x, t, f)$ in (4.4) are real and either they are all different to zero or $B \equiv 0$

This definition is identical to the one used for systems of partial differential equations, except by the requirement that all the speeds of propagation of weak discontinuitiesbe non-zero. A requirement of this sort is necessary because otherwise the character of the equation is changed. It will be shown that syste' s o. 'u' :tional differential equations of this type behave in many res; e(s . ke 1; erbolic systems of differentia' equ :ions. However, if the above requirement is not introduced in the defin..uon of hyperbolic systems the single equation

$$u_t(t) - \int_{-\infty}^{t} u_x(\tau)\, d\tau = 0 \tag{4.5}$$

would be hyperbolic. But if we define

$$v(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$$

Th n '4.! becor es

$$v_{tt} = v_x$$

and thi s is a parabolic equation.

By the definition of hyperbolicity, the eigen values of the matrix A have to be real and finite. We shall assume in what follows without any further e<plic mention that all the eigen values of A are real and are uniformly bounded by a number $\beta > 0$. This implies that there exists a matrix q a'd it: irverse q^{-1} such that

$$q^{-1} a\, q = d$$

where d is a diagonal matrix. Even more, the regions $R(\beta,\delta)$ will allways be determinacy domains [22] of the partial differential equations to be considered because β is a bound for the speeds of propagation of characteristics.

An equation of the form

$$u_t + A(x,t,u)u_x + C(x,t,u) \equiv 0 \tag{4.6}$$

obtained by dropping the term containing the integral in equation (3.6) will be called "reduced equation". Sometimes to be more explicit we call equation (3. 6) the "full equation".

The reduced and full equations will be discussed separately. When dealing with the full equation we shall assume that all the eigenvalues are real and different to zero and uniformly bounded away from zero by some number $\rho > 0$. This implies the existence of an inverse matrix A^{-1}.

Our main goal will be to discuss uniqueness and existence of solution of a Cauchy problem for equation (3.6). In this problem we will prescribe $u(x, t)$ in $\mathcal{J} \times (-\infty, 0]$. When we restrict attention to functions which are equal to the prescribed function in $\mathcal{J} \times (-\infty, 0]$, the functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$, $B(x, t, \tau, \cdot)$ may be thought as functionals defined in a set of functions $f: [0, t] \to R^{\nu}$. Consequently

$$\int_{-\infty}^{0} B(x,t,\tau,u)\, u_x(\tau)\, d\tau$$

is also a functional defined for functions $f: [0, t] \to R^{\nu}$, which can be incorporated in C. All these considerations lead us to restrict attention to the system of equations

$$u_t + A(x,t,u)\,u_x + \int_0^t B(x,t,\tau,u)\,u_x(\tau)\,d\tau + C(x,t,u) = 0 \qquad (4.6)$$

where $A(x, t, \cdot)$, $B(x, t, \ , \cdot)$, $C(x, t, \cdot)$ for every x, t, and τ, are functionals defined in a set of functions $f: [0, t] \to R^{\nu}$

5.- Preliminary Results. In this section we prove some results for linear hyperbolic systems of partial differential equations which will be used in the sequel. Most of these results are easy consequences of results originally obtained by Friedrichs and this section uses nomenclature introduced by him [22].

The system to be considered is

$$u_t + a u_x + r u + c = 0 \tag{5.1}$$

where a, r and c are functions of x and t only. It is assumed that the system is hyperbolic, i.e. that there exists a matrix q and its inverse q^{-1} such that

$$q^{-1} a q = d \tag{5.2}$$

where d is a diagonal matrix.

The eigen values of a are assumed to be bounded in absolute value by $\beta > 0$. Under these conditions we prove

THEOREM 5.1. Assume that in a region $\mathcal{R}(\beta, \delta)$, a, q, q^{-1} are bounded functions, with bounded continuous first derivatives, while r and c are bounded continuous functions. Then, if u is a solution in the wider sense of (5.1) and u is continuous on $\mathcal{J}$, there exists a number $N > 0$ such that given any $\varepsilon > 0$, a number $\lambda_0 > 0$ can be chosen so that

$$\| u e^{-\lambda t} \| \leq N \| \bar{u} \| + \varepsilon \| c e^{-\lambda t} \| \tag{5.3}$$

whenever $\lambda > \lambda_0$. Even more, the choice of N and λ_0 depends only on the bounds for a, q, q^{-1}, r and the first derivatives of q.

Proof. This theorem follows immediately from Frierich's results [22]. According to section 5 of that paper u is said to be a solution of (5.1) in the wider sense, if and only if

$$v = \underset{\sim}{q}^{-1} u$$

satisfies in the wider sense, the equation

$$v_t + d v_x + \underset{\sim}{q}^{-1}(\underset{\sim}{q}_t + a\underset{\sim}{q}_x + r\underset{\sim}{q})v + \underset{\sim}{q}^{-1} c = 0 \tag{5.4}$$

Since $R'(\rho, \delta)$ is a determinacy domain, by Lemma 4.1 of Friedrich's paper

$$(\lambda - M)\|v e^{-\lambda t}\| \leq \lambda \|\bar{v}\| + \|\underset{\sim}{q}^{-1} c\, e^{-\lambda t}\| \tag{5.5}$$

where

$$M = \|\underset{\sim}{q}^{-1}(\underset{\sim}{q}_t + a\underset{\sim}{q}_x + r\underset{\sim}{q})\| \leq \|\underset{\sim}{q}^{-1}\|\|\underset{\sim}{q}_t\| + \ldots + \|\underset{\sim}{q}^{-1}\|\|r\|\|\underset{\sim}{q}\|$$

Let M* be a bound for

$$\|\underset{\sim}{q}^{-1}\|\|\underset{\sim}{q}_t\| + \cdots + \|\underset{\sim}{q}^{-1}\|\|r\|\|\underset{\sim}{q}\|$$

and use the fact that qv = u so that

$$\|u e^{-\lambda t}\| = \|\underset{\sim}{q} v e^{-\lambda t}\| \leq \|\underset{\sim}{q}\|\,\|v e^{-\lambda t}\|$$

Therefore,

$$(\lambda - M^*)\|u e^{-\lambda t}\| \leq \lambda\|\underset{\sim}{q}\|\|\underset{\sim}{q}^{-1}\|\|\bar{u}\| + \|\underset{\sim}{q}\|\|\underset{\sim}{q}^{-1}\|\|c e^{-\lambda t}\|$$

by (5.5). Hence

$$\|u e^{-\lambda t}\| \leq 2\|\underset{\sim}{q}\|\|\underset{\sim}{q}^{-1}\|\|\bar{u}\| + \frac{\|\underset{\sim}{q}\|\|\underset{\sim}{q}^{-1}\|}{\lambda - M^*}\|c e^{-\lambda t}\|$$

whenever $\lambda > 2M^*$.

Take

$$N > 2\|\underset{\sim}{q}\|\|\underset{\sim}{q}^{-1}\|$$

and given $\varepsilon > 0$, choose λ_0 such that

$$\lambda_0 > 2M^* \quad \text{and} \quad \lambda_0 > \frac{\|q\|\,\|q^{-1}\|}{\varepsilon} + M^*$$

With this choice, $\lambda > \lambda_0$ implies

$$\|u e^{-\lambda t}\| \leq N\|\bar{u}\| + \varepsilon\|c e^{-\lambda t}\|$$

<u>THEOREM 5.2</u>. Under the hypothesis of Theorem 5.1, there exists a number M such that

$$\|u\| < M$$

If the function $\bar{u}$ is kept fixed, the choice of M depends only on the bounds for a, q, q^{-1}, r, c and the first derivatives of q.

<u>Proof</u>. Using (5.3)

$$\begin{aligned} \|u\| = \|u e^{-\lambda t} e^{\lambda t}\| &\leq \|e^{\lambda t}\|\,\|u e^{-\lambda t}\| \\ &\leq \|e^{\lambda t}\|\left\{N\|\bar{u}\| + \varepsilon\|c\|\right\} \end{aligned} \qquad (5.6)$$

Observe that the bound given by (5.6) has the properties asserted in the theorem.

<u>COROLLARY 5.1</u>. Assume that a, q, q^{-1} and c possess continuous first derivatives in $\mathcal{R}(\beta, \delta)$. Let u be a solution in the wider sense of

$$u_t + a u_x + c = 0 \qquad (5.7)$$

in $\mathcal{R}(\beta, \delta$, uc.. that $\bar{u}$ has first continuous derivatives on $\mathcal{D}$. Then there exists $N > 0$ such that given $\varepsilon > 0$ we can choose $\lambda_0 > 0$ for which $\lambda > \lambda_c$ implies

$$\|u e^{-\lambda t}\| \leq N\|\bar{u}\| + \varepsilon\|c e^{-\lambda t}\| \qquad (5.8)$$

$$\|u_x e^{-\lambda t}\| \leq N\|\bar{u}_x\| + \varepsilon\|c_x e^{-\lambda t}\| \qquad (5.8.a)$$

$$\|u_t e^{-\lambda t}\| \leq N\{\|\bar{u}_x\| + \|\bar{u}_t\|\} + \varepsilon\{\|c_x e^{-\lambda t}\| + \|c_t e^{-\lambda t}\|\} \qquad (5.9.b)$$

Even more the choice of N and λ_0 depends only on the bounds for a, q, q^{-1} and the first derivatives of a and q.

<u>Proof.</u> Observe that $\bar{u}_t = -a\bar{u}_x - c$ on $\mathcal{J}$. By theorem 5.3 of Friedrick's paper, u_x and u_t exist, are continuous and satisfy in the wider sense

$$(u_x)_t + a(u_x)_x + a_x u_x + c_x = 0 \qquad (5.10.a)$$

$$(u_t)_t + a(u_t)_x + a_t u_x + c_t = 0 \qquad (5.10.b)$$

By theorem 5.1, there is a number N such that given any $\varepsilon' > 0$, we can chose λ_0 with the property that $\lambda > \lambda_0$ implies

$$\|u e^{-\lambda t}\| \leq N\|\bar{u}\| + \varepsilon'\|c e^{-\lambda t}\| \qquad (5.11.a)$$

$$\|u_x e^{-\lambda t}\| \leq N\|\bar{u}_x\| + \varepsilon'\|c_x e^{-\lambda t}\| \qquad (5.11.b)$$

$$\|u_t e^{-\lambda t}\| \leq N\|\bar{u}_t\| + \varepsilon'\|(c_t + a_t a_x) e^{-\lambda t}\| \qquad (5.11.c)$$

Observe that although the numbers N and λ_0 predicted by theorem 5.1 may be different for every one of the equations (5.11), they can be taken equal if we choose N and λ_0 as the greatest of each triplet. Now, from theorem 5.1 it follows that the choice of N and λ_0 depends only on the bounds for a, q, q^{-1} and the first derivatives of a and q

On the other hand

$$\therefore \|(c_t + a_t u_x)e^{-\lambda t}\| \leq \|c_t e^{-\lambda t}\| + \|a_t\| \; \|u_x e^{-\lambda t}\| .$$
$$\leq \|c_t e^{-\lambda t}\| + \|a_t\| \{N \|\bar{u}_x\| + \varepsilon' \|c_x e^{-\lambda t}\|\}$$

If ε' si such that

$$\varepsilon' \|a_t\| < 1 \tag{5.12}$$

then

$$\|u_t e^{-\lambda t}\| \leq N\{\|\bar{u}_x\| + \|\bar{u}_t\|\} + \varepsilon'\{\|c_x e^{-\lambda t}\| + \|c_t e^{-\lambda t}\|\} \tag{5.13}$$

Thus, given $\varepsilon > 0$ we take $\varepsilon' < \varepsilon$ such that (5.12) holds. Then from (5.11) and (5.13), the relations (5.8) and (5.9) follow.

<u>COROLLARY 5.2</u>. Under hypothesis of corollary 5.1 there exists a number M such that

$$\|u\| + \|u_x\| + \|u_t\| < M \tag{5.14}$$

When $\bar{u}$ is given, the choice of M depends only on the bounds for a, q, q^{-1}, c and their first derivatives.

Even more, if in addition a, q, q^{-1} and c possess continuous second derivatives in $\mathcal{R}(\beta, \delta)$ and $\bar{u}$ continuous second derivative in $\mathcal{J}$, then there exists a number M such that

$$\|u\| + \|u_x\| + \|u_t\| + \|u_{xx}\| + \|u_{xt}\| + \|u_{tt}\| < M \tag{5.15}$$

When $\bar{u}$ is kept fixed, the choice of M depends only on the bounds for a, q, q^{-1}, c and their first and second derivatives.

<u>Proof</u>. By the help of the argument used to prove theorem (5.2) the first part of the corollary follows from (5.8), (5.9) and the fact that

$$\bar{u}_t = -a(\bar{u})_x - c \tag{5.16}$$

To prove the second part observe that the second derivatives of u by theorem 5.4 of Friedrichs' paper, satisfy in the wider sense the equations

$$(u_{xx})_t + a(u_{xx})_x + 2a_x u_{xx} + a_{xx} u_x + c_{xx} = 0 \tag{5.17.a}$$

$$(u_{xt})_t + a(u_{xt})_x + a_x u_{xt} + a_t u_{xx} + a_{xt} u_x + c_{xt} = 0 \tag{5.17.b}$$

$$(u_{tt})_t + a(u_{tt})_x + 2a_t u_{xt} + a_{tt} u_x + c_{tt} = 0 \tag{5.17.c}$$

From (5.17.a) and theorem 5.2 it follows that there exists M_1 such that

$$\| u_{xx} \| < M_1 \tag{5.18}$$

For given $\bar{u}_{xx}$ (i. e. for given $\bar{u}$) the choice of M_1 depends only on the bounds for a, q, q^{-1}, a_x, $a_{xx} u_x + c_{xx}$ and the first derivatives of q. By the part already proved of the corollary for given $\bar{u}$ the bound for u_x depends only on the bounds for a, q, q^{-1}, c and their first derivatives. Therefore, given $\bar{u}$, the choice of M_1 depends only on the bounds for a, q, q^{-1}, c and their first and second derivatives.

Using (5.17.b, c) and the fact that

$$\bar{u}_{xt} = -a(\bar{u})_{xx} - a_x(\bar{u})_x - c \tag{5.19.a}$$

$$\bar{u}_{tt} = -a\bar{u}_{xt} - a_t \bar{u}_x - c_t \tag{5.19.b}$$

the corresponding relations can be shown to hold for u_{xt} and u_{tt}, from which the corollary follows.

<u>THEOREM 5.3.</u> Let u satisfy in the wider sense

$$u_t + a u_x + c = 0. \tag{5.20}$$

in $Q(\beta, \delta)$ and assume that a, q, q^{-1} and c have continuous second derivatives there. Then, if $\bar{u}$ has continuous second order derivatives on $\mathcal{J}$, given $\varepsilon > 0$ there exists $\delta_o > 0$ such that

$$\|u\| < \|\bar{u}\| + \varepsilon \tag{5.21.a}$$

$$\|u_x\| < \|\bar{u}_x\| + \varepsilon \tag{5.21.b}$$

$$\|u_t\| < \|\bar{u}_t\| + \varepsilon \tag{5.21.c.}$$

$$\|u_{xx}\| < \|\bar{u}_{xx}\| + \varepsilon \tag{5.21.d}$$

$$\|u_{xt}\| < \|\bar{u}_{xt}\| + \varepsilon \tag{5.21.e}$$

$$\|u_{tt}\| < \|\bar{u}_{tt}\| + \varepsilon \tag{5.21.f}$$

when the norm is taken with respect to $Q(\beta, \delta_o)$ and the initial values $\bar{u}_t$, $\bar{u}_{xt}$, $\bar{u}_{tt}$ are given by (5.16) and (5.19). Even more, if $\bar{u}$, $\bar{u}_x$, $\bar{u}_t$, $\bar{u}_{xx}$, $\bar{u}_{xt}$ and $\bar{u}_{tt}$ are kept fixed, then for a given $\varepsilon > 0$ the choice of δ_o depends only on the bounds for a, q, q^{-1}, c and their first and second partial derivatives.

Proof. We first prove the following

LEMMA. Let u satisfy in the wider sense the equation (5.20) in $Q(\beta, \delta)$, where c and $\bar{u}$ are continuous and a, q, q^{-1} have first continuous derivatives in Q. Then given any $\varepsilon > 0$ there exists $\delta_o > 0$ such that

$$\|u\| < \|\bar{u}\| + \varepsilon$$

when the norm is taken with respect to $Q(\rho, \delta_o)$. Even more, if $\bar{u}$, $\bar{q}$ and $\overline{q^{-1}}$ are kept fixed, then for a given $\varepsilon > 0$ the choice of δ_o depends only on the bounds for a, q, q^{-1}, c and the first derivatives of q.

Proof. To prove this Lemma we use arguments closely related to those presented by A. Douglis [23]. Let us denote by $\frac{d}{dt_i}$ the derivatives along characteristics. Define $v = q^{-1} u$, then from (5.4) it follows that

$$\frac{dv_i}{dt_i} = f_i \tag{5.22}$$

where f_i contains a, q, q^{-1}, c and the first derivatives of q. Equation (5.22) and the definition of v imply

$$\begin{aligned} u_j(x,t) &= \sum_i q_{ji}(x_i(x,t),0)v_i(x_i(x,t),0) \\ &\quad + \sum_i \int_0^{t} \left(q_{ji} \frac{dv_i}{dt_i} + \frac{dq_{ji}}{dt_i} \right) dt_i = \\ &= \sum_i q_{ji}(x_i(x,t),0)v_i(x_i(x,t),0) + \sum_i \int_0^t \left(q_{ji} f_i + \frac{dq_{ji}}{dt_i} v_i\right) dt_i \end{aligned} \tag{5.23}$$

where $x_i(x, t)$ represents the point where the i-th characteristic passing through the point (x, t) intersects the x-axis, q_{ji} are the elements of the matrix q and the integration must be carried out along every one of the characteristics. Given $\varepsilon > 0$, we can choose δ_1 in such a way that

$$\sum_i \left| \int_0^t \left(q_{ji} f_i + \frac{dq_{ji}}{dt_i} v_i\right) dt_i \right| < \frac{\varepsilon}{2} \tag{5.24}$$

whenever $0 < t < \delta_1$. The choice of δ_1 depends only on the bounds for a, q, q^{-1}, c, u and the first derivatives of q. But a bound for u, when $\bar{u}$ is kept fixed, by theorem 5.2, depends only on the bounds for a, q, q^{-1}, c, and the first derivatives of q. Thus, the choice of δ_1 depends only on the bounds for a, q, q^{-1}, c and the first derivatives of q.

On the other hand

$$\sum_i q_{ji}(x_i(x,t),0)v_i(x_i(x,t),0) = \sum_i q_{ji}(x_1(x,t),0)v_i(x_1(x,t),0)$$
$$+ \sum_i \left\{ q_{ji}(x_i(x,t),0)v_i(x_i(x,t),0) - q_{ji}(x_1(x,t),0)v_i(x,t),0) \right\} =$$
$$= U_j(x_1(x,t),0) + \sum_i \left\{ q_{ji}(x_i(x,t),0)v_i(x_i(x,t),0) - q_{ji}(x_1(x,t),0)v_i(x_1(x,t),0) \right\} \tag{5.25}$$

The functions $q(x, 0)$, $q^{-1}(x, 0)$ and $u(x, 0)$ are uniformly continuous on $\mathcal{J}$ (because $\mathcal{J}$ is closed and bounded), v also is uniformly continuous on $\mathcal{J}$ and therefore we can chose $\delta_2 > 0$ such that

$$\left| \sum_i \left\{ q_{ji}(x_i,0)v_i(x_i,0) - q_{ji}(x_1,0)v_i(x_1\ 0) \right\} \right| < \frac{\varepsilon}{2} \tag{5.26}$$

whenever $|x_i - x_1| < \delta_2$ for every i. Observe that the choice of δ_2 (for a given ε) depends only on the functions $\bar{q}$, $\overline{q^{-1}}$ and $\bar{u}$. On the other hand β is a bound for the speed of propagation of the characteristics, so that

$$|x_i(x,t) - x_1(x,t)| < \delta_2 \qquad \text{whenever} \qquad t < \delta_3 = \frac{\delta_2}{2\beta}$$

Therefore, from (5.23), (5.24), (5.25), (5.26) it follows that

$$\|u\| < \|\bar{u}\| + \varepsilon$$

where the norm is taken on $Q(\beta, \delta_0)$ and $\delta_0 = \min(\delta_3, \delta_1)$.
It may be checked in the above arguments that the choice of δ_0 depends only on the bounds for a, q, q^{-1}, c, and the first derivatives of q, when $\bar{u}$, $\bar{q}$, and $\overline{q^{-1}}$ are kept fixed. This completes the proof of the Lemma.

The theorem follows now from (5.10) and (5.17), using Corollary 5.2 (relation (5.15)).

6.- Uniqueness of solution. The uniqueness of solution of a Cauchy problem for hyperbolic systems of functional differential equations can be shown in the large with assumptions weaker than those that will be used when proving existence.

In this section we treat the reduced equation. The full equation will be discussed in section 8.

In what follows, given two functions u, v defined in $Q(\beta, \delta)$, we introduce the notation

$$\Delta u = v - u \qquad (6.1.a)$$

$$\Delta a = {}^{v}a - {}^{u}a \qquad (6.1.b)$$

$$\Delta b = {}^{v}b - {}^{u}b \qquad (6.1.c)$$

$$\Delta c = {}^{v}c - {}^{u}c \qquad (6.1.d)$$

and similar notation for the partial derivatives.

<u>THEOREM 6.1.</u> Let the functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$ be such that the functions a, q, q^{-1} and c possess continuous first order derivatives in $Q(\beta, \delta)$ whenever $u \in \mathcal{F}_{\Omega}^{1}(\beta, \delta)$ and assume there is a $k > 0$ such that for every $(x, t) \in Q(\beta, \delta)$

$$|\Delta a| \leq k|\Delta u|_t \quad ; \quad |\Delta c| \leq k|\Delta u|_t \tag{6.2}$$

whenever $u, v \in \mathcal{F}_{\Omega}^{1}(\beta, \delta)$. Then if two functions $u, v \in \mathcal{F}_{\Omega}^{1}(\beta, \delta)$ satisfy the equation

$$u_t + Au_x + C = 0 \tag{6.3}$$

and assume the same initial values on $\mathcal{J}$ we necessarily have

$$u = v \qquad \text{in} \quad Q(\beta, \delta)$$

<u>Proof.</u> Observe that Δu satisfies the equation

$$(\Delta u)_t + a(\Delta u)_x + (\Delta a)v_x + \Delta c = 0 .$$

where we have written a for ${}^u a$. This is an equation for Δu of the type (5.1), with r identically zero. Even more $-a$, $-q$, $-q^{-1}$ are bounded with bounded first continuous derivatives, because $Q(\beta, \delta)$ is closed and similarly $(\Delta a)v_x + \Delta c$ is a bounded continuous function.

Therefore, theorem 5.1 can be applied. Since $\overline{\Delta u} = 0$ on $\mathcal{J}$, given any $\varepsilon > 0$ we can chose $\lambda_0 > 0$ such that

$$\|\Delta u\, e^{-\lambda t}\| \leq \varepsilon \|\{(\Delta a)v_x + \Delta c\} e^{-\lambda t}\| \tag{6.4}$$

whenever $\lambda > \lambda_0$

In view of (6.2) and the fact that v_x is bounded, it follows that $\|(\Delta u) e^{-\lambda t}\| \leq \mu \| |\Delta u|_t\, e^{-\lambda t}\|$

for some $0 < \mu < 1$, taking ε sufficiently small.

Finally observe that for every $\lambda > 0$

$$\| e^{-\lambda t} \Delta u \| = \| |\Delta u|_t \, e^{-\lambda t} \| \tag{6.6}$$

because for every $t \geq 0$,

$$| e^{-\lambda t} \Delta u | \leq \| e^{-\lambda \tau} \Delta u \|$$

i.e.

$$| \Delta u | \leq \| e^{-\lambda \tau} \Delta u \| e^{\lambda t}$$

and therefore

$$| \Delta u |_t \leq \| e^{-\lambda \tau} \Delta u \| e^{\lambda t}$$

Hence

$$\| |\Delta u|_t \, e^{-\lambda t} \| \leq \| e^{-\lambda t} \Delta u \|$$

On the other hand the inequality

$$\| |\Delta u|_t e^{-\lambda t} \| \geq \| e^{-\lambda t} \Delta u \| = \| e^{-\lambda t} |\Delta u| \|$$

is obvious becuase

$$|\Delta u|_t \geq |\Delta u|$$

Substituting (6.6) into (6.5) we obtain

$$\| |\Delta u|_t e^{-\lambda t} \| \leq \mu \| |\Delta u|_t e^{-\lambda t} \|$$

which can hold with $\mu < 1$ only if $\Delta u = 0$ for every $(x, t) \in Q(\rho, \delta)$. This completes the proof of the theorem.

When discussing functional differential equations, the Cauchy problem considered in theorem 6.1 is not a natural one. It is natural to consider instead functionals $\mathcal{A}(x, t, \cdot)$, $\mathcal{C}(x, t, \cdot)$ defined for every $(x, t) \in \mathcal{R}(\beta, \delta)$ in a set of functions $f: (-\infty, t] \to R^{\nu}$. The initial value problem would be one for which a function $u(x, t)$ is prescribed in the semi-infinite strip $\mathcal{J} x (-\infty, 0]$ and we look for an extension of $u(x, t)$ to $\{\mathcal{J} x (-\infty, 0]\} \cup \mathcal{R}(\beta, \delta)$ in such a way that the restriction of u to $\mathcal{R}(\beta, \delta)$ belongs to $\mathcal{F}^1(\beta, \delta)$ and

$$u_t + \mathcal{A} u_x + \mathcal{C} = 0 \qquad (6.7)$$

holds for the restriction of u to $\mathcal{R}(\beta, \delta)$. The unuqueness of solution for this problem is established in the following

<u>COROLLARY 6.1</u>. Let $u(x, t)$ be defined in $\{\mathcal{J} x (-\infty, 0]\} \cup \mathcal{R}(\beta, \delta)$ and such that its restriction to $\mathcal{R}(\beta, \delta)$ belongs to $\mathcal{F}^1_{\Omega}(\beta, \delta)$, and satisfies 6.7. For every $(x, t) \in \mathcal{R}(\beta, \delta)$ define the functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$, for functions $f: [0, t] \to R^{\nu}$ by

$$A(x,t,f) = \mathcal{A}(x,t,g) \quad ; \quad C(x,t,f) = \mathcal{C}(x,t,g)$$

where

$$g(\tau) = u(x, \tau) \quad ; \text{ if } \quad -\infty < \tau < 0$$
$$g(\tau) = f(\tau) \quad ; \text{ if } \quad 0 \le \tau \le t$$

Assume that for the given u the functionals A and C so defined satisfy the hypothesis of theorem 6.1. Then if any function v defined in $\{\mathcal{J} x (-\infty, 0]\} \cup \mathcal{R}(\beta, \delta)$ is such that $u = v$ in $\mathcal{J} x (-\infty, 0]$, its restriction to $\mathcal{R}(\beta, \delta)$ belongs to $\mathcal{F}^1_{\Omega}(\beta, \delta)$ and satisfies (6.7), we necessarily have $u \equiv v$, *in* $\mathcal{R}(\beta, \delta)$.

<u>Proof.</u> The proof is obvious by theorem 6.1 because u and v satisfy

$$u_t + A u_x + C = 0 \qquad \text{on} \quad \mathcal{R}(\beta, \delta)$$

and assume the same initial values on $\mathcal{J}$.

Similar considerations apply to the existence theorems to be given in later sections, but we shall refrain from going into further discussion of this matter. It is worthwhile mentioning that the hypothesis 6.2 in theorem 6.1 may be replaced by a hypothesis of the form

$$|\Delta a| \leq k \|\Delta u\|_t \quad ; \quad |\Delta c| \leq k \|\Delta u\|_t \tag{6.8.a}$$

where

$$\|\Delta u\|_t = \int_0^t |\Delta u(\tau)| \, d\tau \tag{6.8.b}$$

This is proved in the following

<u>LEMMA 6.1</u>. Let u and v be continuous in $Q(\beta, \delta)$. Then

$$\|\Delta u\|_t \leq t\,|\Delta u|_t \leq \delta\,|\Delta u|_t \text{ in } Q(\beta, \delta) \tag{6.9}$$

<u>Proof.</u> This result trivially follows from the inequality

$$|\Delta u(\tau)| \leq |\Delta u|_t \quad ; \quad 0 \leq \tau \leq t$$

for fixed x

7. <u>Existence of Solution</u>. In this section we prove the existence of solution for the reduced system. We use again the notation introduced in section 6.

<u>THEOREM 7.1</u>. Let the family of functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$ and the number $\delta_0 > 0$ be such that

I) The functions a, q, q^{-1} and c have continuous second derivatives in $\mathcal{R}(\beta,\delta)$ for every $u\in\mathcal{F}_{\Omega}^{2}(\beta,\delta)$ and continuous first derivatives for every $u\in\mathcal{F}_{\Omega}^{1}(\beta,\delta)$ whenever $0<\delta\leq\delta_0$

II) For every $0<\delta\leq\delta_0$ there is a number M which bounds in $\mathcal{R}(\beta,\delta)$ the functions a, q, q^{-1}, c and the first and second derivatives of these functions for every $u\in\mathcal{F}_{\Omega}^{2}(\beta,\delta)$ whenever $0<\delta\leq\delta_0$. This bound is independent of δ and of the particular function u chosen.

III) There exists a k such that for every $u, v\in\mathcal{F}_{\Omega}^{1}(\beta,\delta)$ we have at every $(x,t)\in\mathcal{R}(\beta,\delta)$

$$|\Delta a|\leq k|\Delta u|_t \quad ; \quad |\Delta c|\leq k|\Delta u|_t \tag{7.1.a.,b}$$

$$|\Delta a_x|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\} \tag{7.2.a}$$

$$|\Delta c_x|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\} \tag{7.2.b}$$

$$|\Delta a_t|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\} \tag{7.3.a}$$

$$|\Delta c_t|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\} \tag{7.3.b}$$

whenever $0<\delta\leq\delta_0$.

IV) Let the function $\hat{u}(x)$ defined on $\mathcal{J}$, possess continuous second derivatives on $\mathcal{J}$, and be such that the "initial data" $\hat{u}_x$, $\hat{u}_t$, $\hat{u}_{xx}$, $\hat{u}_{xt}$, $\hat{u}_{tt}$ implied by the equation

$$u_t+au_x+c=0 \tag{7.4}$$

and the derived relation

$$(u_t)_t + a(u_t)_x + a_t u_x + c_t = 0 \tag{7.5}$$

exist and satisfy

$$|\hat{u}| + |\hat{u}_x| + |\hat{u}_t| + |\hat{u}_{xx}| + |\hat{u}_{xt}| + |\hat{u}_{tt}| \leq \Omega^* < \Omega \tag{7.6}$$

on $\mathcal{J}$.

Then there is a function $u \in \mathcal{F}^1_\Omega(\beta, \delta')$ which satisfies (7.4) in $R(\beta, \delta')$ for some $0 < \delta' \leq \delta_0$, and

$$\bar{u}(x) = u(x,0) = \hat{u}(x) \quad \text{on} \quad \mathcal{J}.$$

Proof. We define first the "initial data" $\hat{u}_x$, $\hat{u}_t$, $\hat{u}_{xx}$, $\hat{u}_{xt}$, $\hat{u}_{tt}$ and show that our definition is such that when they exist they are uniquely determined on $\mathcal{J}$ by the given function $\hat{u}$.

We say that a function $w \in \mathcal{F}^2_\Omega(\beta, \delta)$, $\delta > 0$, is a "solution of (7.4) initially" if and only if

$$w(x,0) = \hat{u}(x) \quad \text{on} \quad \mathcal{J}$$

and w satisfies (7.4) and (7.5) on $\mathcal{J}$, with

$$a = {}^w a, \qquad c = {}^w c.$$

In that case we write

$$\hat{u}_x(x) = w_x(x,0) = (\hat{u})_x \quad ; \quad \hat{u}_t(x,0) = w_t(x,0)$$

$$\hat{u}_{xx}(x) = w_{xx}(x,0) = (\hat{u})_{xx}(x,0) \quad ; \quad \hat{u}_{xt}(x,0) = w_{xt}(x,0) \quad ,$$

$$\hat{u}_{tt}(x,0) = w_{tt}(x,0)$$

The existence of the initial data

must be understood as implying the existence of an initial solution. It is important to recall that given $\hat{u}$ the definition of the initial data is independent of the particular initial solution w chosen. That this is so for u_x and u_{xx} is obvious because

$$\hat{u}_x(x) = (\hat{u})_x(x)$$

$$\hat{u}_{xx}(x) = (\hat{u})_{xx}(x)$$

On the other hand, let v and w be two initial solutions, then $v, w \in \mathcal{F}_\Omega^2(\beta, \delta)$ for some $\delta > 0$. Therefore by (7.1)

$$ {}^{w}a(x,0) = {}^{v}a(x,0) \;;\; {}^{w}c(x,0) = {}^{v}c(x,0), \text{ on } \mathcal{J} \qquad (7.7.a)$$

and in view of (7.4),

$$w_t = -{}^{w}a\, w_x - {}^{w}c = -{}^{v}a\, v_x - {}^{v}c, \text{ on } \mathcal{J}. \qquad (7.7.b)$$

Thus $\hat{u}_t$ is uniquely defined.

Hence, $\hat{u}_{xt}$ is also independent of the particular w chosen because

$$\hat{u}_{xt}(x) = w_{xt}(x,0) = (w_t(x,0))_x = (\hat{u}_t(x))_x$$

Therefore by (7.5) and (7.3)

$$\hat{u}_{tt} = w_{tt} = -{}^{w}a\, w_{tx} - {}^{w}a_t\, w_t - {}^{w}c_t, \text{ on } \mathcal{J}$$

is also independent of the choice of w.

From the above arguments it can be seen even more, that if two functions (not necessarily initial solutions) $v, w \in \mathcal{F}_\Omega^2(\beta, \delta)$ for some $\delta > 0$, are such that

$$w(x,0) \equiv v(x,0) \equiv \hat{u}(x,0), \text{ on } \mathcal{J} \qquad (7:8:a)$$

then

$$v_x(x,0) = w_x(x,0), \text{ on } \partial \tag{7.9.a}$$

$$v_{xx}(x,0) = w_{xx}(x,0), \text{ on } \partial \tag{7.9.b}$$

and

$$^{v}a\, v_x + {}^{v}c = {}^{w}a\, w_x + {}^{w}c, \text{ on } \partial. \tag{7.9.c}$$

If in addition

$$v_t = w_t, \text{ on } \partial \tag{7.8.b}$$

then

$$^{v}a\, v_{xt} + {}^{v}a_t\, v_x + {}^{v}c_t = {}^{w}a\, w_{xt} + {}^{w}a_t\, w_x + {}^{w}c_t \quad \text{on } \partial. \tag{7.9.d}$$

Using equations (7.8) and (7.9) it can be seen that the assumption made in theorem 7.1about the existence of at least an initial solution, may be replaced by the assumption that certain inequalities are satisfied. Indeed, assume

$$|\hat{u}| + |(\hat{u})_x| + |(\hat{u})_{xx}| \leq \Omega^* < \Omega \tag{7.10.a}$$

and define

$$v(x,t) = \bar{u}(x), \quad (x,t) \in Q(\beta, \delta_0)$$

Then $v \in \mathcal{F}^{1,2}_{\Omega}(\beta, \delta_0)$. Construct now $V(x, t) \in \mathcal{F}^{1,2}(\beta, \delta_0)$ so that

$$V_{tt} = 0 \quad \text{on } \partial$$

$$V_t = -{}^{v}a\, v_x - {}^{v}c \quad \text{on } \partial$$

$$V = \bar{u} \quad \text{on } \partial$$

If we assume that

$$|V| + |V_x| + |V_t| + |V_{xx}| + |V_{xt}| < \Omega \,, \text{ on } \mathcal{J} \qquad (7.10.b)$$

then $v \in \mathcal{F}^2_\Omega(\beta, \delta'_0)$ *for some* $\delta'_0 > 0$. *Define finally* $w \in \mathcal{F}^2(\beta, \delta_0)$, *so that*

$$w_{tt} = -{}^v a V_{xt} - {}^v a_t V_x - {}^v c_t \,, \text{ on } \mathcal{J}$$

$$W_t = -{}^v a V_x - {}^v c \qquad , \text{ on } \mathcal{J}$$

$$W = \hat{u} \qquad , \text{ on } \mathcal{J} \,.$$

If we assume that

$$|W| + |W_x| + |W_t| + |W_{xx}| + |W_{xt}| + |W_{xx}| < \Omega \quad \text{on } \mathcal{J} \qquad (7.10.c)$$

then $w \in \mathcal{F}^2_\Omega(\beta, \delta''_0)$ taking $\delta''_0 > 0$ sufficiently small. From equations (7.8) and (7.9) and the way w was constructed, it can be easily seen that w is an initial solution.

Conversely if an initial solution exists and (7.6) is satisfied, then *inequalities* (7.10) are necessarily satisfied when carrying the above construction.

We prove now the theorem by iteration. For every $u \in \mathcal{F}^2_\Omega(\beta, \delta'_0)$ such that

$$u(x,0) = \hat{u}(x) \quad ; \quad u_t(x,0) = \hat{u}_t(x) \qquad (7.11)$$

we define the transformation $\mathcal{T}$ by

$$U = \mathcal{T}(u) \qquad (7.12.a)$$

where

$$U_t + {}^u a U_x + {}^u c = 0 \qquad (7.12.b)$$

and

$$U(x,0) = \hat{u}(x) \tag{7.12.c}$$

The existence of initial solutions of (7.4) implies that the set $\mathcal{S}$ of functions satisfying (7.11) and belonging to $\mathcal{F}_\Omega^2(\beta, \delta_0')$ is not void. On the other hand the functions ${}^u a(x, 0)$, ${}^u a_x(x, 0)$ and ${}^u a_t(x, 0)$ depend only on the initial values of u, u_x and u_t (inequalities (7.1.a), (7.2.a) and (7.3.a)), so that if we define

$$\hat{a}(x) = {}^u a(x,0)\,;\ \hat{a}_x(x) = {}^u a_x(x,0)\,;\ \hat{a}_t(x) = {}^u a_t(x,0)$$

on $\mathcal{J}$, the definition of these functions does not depend on the particular $u \in \mathcal{S}$ chosen, by virtue of (7.8) and (7.9).

By theorems 5.3 and 5.4 of Friedrichs' paper and equations (7.11) and (7.12) it follows that $U \in \mathcal{F}^2(\beta, \delta_0')$ and U is an initial solution, i.e.

$$\bar{U} = \hat{u} \tag{7.13.a}$$

$$\bar{U} = \hat{u}_x \tag{7.13.b}$$

$$\bar{U}_t = \hat{u}_t \tag{7.13.c}$$

$$\bar{U}_{xx} = \hat{u}_{xx} \tag{7.13.d}$$

$$\bar{U}_{xt} = \hat{u}_{xt} \tag{7.13.e}$$

$$\bar{U}_{tt} = \hat{u}_{tt} \tag{7.13.f}$$

on $\mathcal{J}$.

By hypothesis a, q, q^{-1}, c and their first two partial derivatives have bounds which hold independently of $u \in \mathcal{F}_\Omega^2(\beta, \delta)$ as long as $0 < \delta \leq \delta_0' \leq \delta_0$. Therefore, by theorem 5.3 given $\varepsilon > 0$, we can chose $\delta' > 0$ independent of $u \in \mathcal{S}$, such that

$$\| U \| < \| \hat{u} \| + \varepsilon$$

$$\| U_x \| < \| \hat{u}_x \| + \varepsilon$$

$$\| U_t \| < \| \hat{u}_t \| + \varepsilon$$

$$\| U_{xx} \| < \| \hat{u}_{xx} \| + \varepsilon$$

$$\| U_{xt} \| < \| \hat{u}_{xt} \| + \varepsilon$$

$$\| U_{tt} \| < \| \hat{u}_{tt} \| + \varepsilon$$

when the norm is taken on $\mathcal{R}(\beta, \delta')$. Observe that theorem 5.3 can be applied because $\overline{U}$, $\overline{U}_x$, $\overline{U}_t$, $\overline{U}_{xx}$, $\overline{U}_{xt}$, $\overline{U}_{xx}$ are kept fixed by (7.13). Taking ε sufficiently small, we have

$$\|U\| + \|U_x\| + \|U_t\| + \|U_{xx}\| + \|U_{xt}\| + \|U_{tt}\| < \Omega \tag{7.14.a}$$

therefore, the transformation $\mathcal{T}$ maps the set of functions $\mathcal{S}'$ into itself, if $\mathcal{S}'$ is taken as the set of functions $u \in \mathcal{F}^2_\Omega(\beta, \delta')$ satisfying (7.11).

We prove next that $\mathcal{T}$ is a contraction with respect to the norm

$$\| |u|_t e^{-\lambda t} \| + \| |u_x|_t e^{-\lambda t} \| + \| |u_t|_t e^{-\lambda t} \| \tag{7.14.b}$$

for some $\lambda > 0$.

Let $v \in \mathcal{S}'$ and $V = \mathcal{T}(v)$

Then

$$(\Delta U)_t + a(\Delta U)_x + (\Delta a) V_x + \Delta c = 0 \tag{7.15.a}$$

and

$$\overline{\Delta U} = 0 \; ; \; \overline{(\Delta U)_x} = 0 \; ; \; \overline{(\Delta U)_t} = 0 \qquad (7.15.b)$$

where we have written a for and Δa and Δc stand for ${}^{v}a - {}^{u}a$, ${}^{v}c - {}^{u}c$ respectively. By (7.14.a) Ω is a uniform bound for V_x, V_{xx} and V_{xt} whenever $v \in \mathcal{S}'$. On the other hand a, q and q^{-1} have first order continuous derivatives which by hypothesis II are uniformly bounded. All this implies by corollary 5.1 that given any $\varepsilon > 0$ there exists $\lambda > 0$ such that on $\mathcal{R}(\beta, \delta')$

$$\|e^{-\lambda t}\Delta U\| \le \varepsilon\Omega\|e^{-\lambda t}\Delta a\| + \varepsilon\|e^{-\lambda t}\Delta c\|$$

$$\|e^{-\lambda t}\Delta U_x\| \le \varepsilon\Omega\|e^{-\lambda t}\Delta a\| + \varepsilon\Omega\|e^{-\lambda t}\Delta a_x\| + \varepsilon\|e^{-\lambda t}\Delta c,$$

$$\|e^{-\lambda t}\Delta U_t\| \le \varepsilon\Omega\|e^{-\lambda t}\Delta a\| + \varepsilon\Omega\|e^{-\lambda t}\Delta a_x\| + \varepsilon\|e^{-\lambda t}\Delta c_x\| + \varepsilon\Omega\|e^{-\lambda t}\Delta a\| + \varepsilon\Omega\|e^{-\lambda t}\Delta a_t\| + \varepsilon\|e^{-\lambda t}\Delta c_t\|$$

whenever $u, v \in \mathcal{S}'$

For any $\lambda > 0$, relations of the form (6.6) hold. Therefore, using (7.1), (7.2) and (7.3) and by an appropriate choice of ε, it follows that for any number $0 < \mu < 1$ we can choose $\lambda > 0$ such that

$$\|e^{-\lambda t}|\Delta U|_t\| + \|e^{-\lambda t}|\Delta U_x|_t\| + \|e^{-\lambda t}|\Delta u_t|_t\| \le \mu\left\{\|e^{-\lambda t}|\Delta u|_t\| + \|e^{-\lambda t}|\Delta u_x|_t\| + \|e^{-\lambda t}|\Delta u_t|_t\|\right\} \qquad (7.16)$$

on $\mathcal{R}(\beta, \delta')$.

The fixed point u of the contraction necessarily belongs to $\mathcal{J}^{1,1}_{\Omega}(\beta, \delta')$ because the partial derivatives are uniformly bounded on

δ'. In addition $u(x, 0) = \hat{u}(x)$ (and even more $\bar{u}_t = \hat{u}_t$).

We consider now a particular approximating sequence of functions $u^{(\sigma)}$; $\sigma = 1, \ldots, n, \ldots$ which goes to the fixed point of the contraction Let it be given by

$$u_t^{(\sigma+1)} + a^{(\sigma)} u_x^{(\sigma+1)} + c^{(\sigma)} = 0 \qquad (7.17.a)$$

where $a^{(\sigma)}$ and $c^{(\sigma)}$ have obvious meanings. Then

$$u_t^{\sigma+1} + a u_x^{\sigma+1} + (a^{\sigma} - a) u_x^{\sigma+1} + c + (c^{\sigma} - c) = 0 \qquad (7.17.b)$$

where $a = {}^u a$. Since $u_x^{\sigma+1}$ is uniformly bounded, and $a^{\sigma} - a$ and $c^{\sigma} - c$ go to zero uniformly as $\sigma \to \infty$ by (7.1), it follows that u is a solution of (7.4) in the strict sense.

T existence theorem 7.1 is restricted to the small because $\delta' \leq \delta_0$. When the functional differential equation (7.4) is linear this restriction can be removed, and convergence can be shown in the large (i.e. $\delta = \delta_0$).

Using the arguments explained at the end of section 6, the case for which the coefficients in the functional differential equation are functionals defined on functions whose domain is $(-\infty, t]$ may be transformed into one for which the domain of these functions is $[0, t]$. In the linear case this transformation gives rise to an additional inhomogeneous term, which does not introduce any further difficulty because the proof of the existence in the large is given for the linear inhomo... , in the following

<u>THEOREM 7.2.</u> Let the families of functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$ and the number $\delta_0 > 0$ be such that for every $(x, t) \in \mathcal{R}(\beta, \delta_0)$

$$A(x,t,\cdot) = a(x,t) \; ; \; C(x,t,\cdot) = \mathcal{L}'(x,t,\cdot) + d(x,t) \qquad (7.18)$$

where $\mathcal{L}$ is a linear functional and a, d are two fixed functions (the same independently of u) defined in $\mathcal{R}(\beta, \delta_0)$ and $a, q, q^{-1}, d \in \mathcal{F}^2(\beta, \delta_0)$.

Assume in addition:

I) The functional $\mathcal{L}(x, t, \cdot)$ is such that the function $l(x, t) = \mathcal{L}(x, t, u) \in \mathcal{F}^2(\beta, \delta_0)$ whenever $u \in \mathcal{F}^2(\beta, \delta_0)$ and $l \in \mathcal{F}^1(\beta, \delta_0)$ whenever $u \in \mathcal{F}^1(\beta, \delta_0)$.

II) There exists a k such that for every $(x, t) \in \mathcal{R}(\beta, \delta_0)$ we have

$$|l| \leq k|u|_t \qquad (7.19.a)$$

$$|l_x| \leq k\{|u|_t + |u_x|_t + |u_t|_t\} \qquad (7.19.b)$$

$$|l_t| \leq k\{|u|_t + |u_x|_t + |u_t|_t\} \qquad (7.19.c)$$

whenever $u \in \mathcal{F}^1(\beta, \delta_0)$.

Then given any function $\hat{u}(x)$ defined on $\mathcal{J}$ and possessing second order continuous derivatives there, there is a function $u \in \mathcal{F}^1(\beta, \delta_0)$ which satisfies (7.4) in $\mathcal{R}(\beta, \delta_0)$ and such that

$$u(x,0) = \hat{u}(x)$$

<u>Proof.</u> The proof of theorem 7.2 is similar to the proof of thoerem 7.1.

Observe that a, q, q^{-1}, d together with their first and second derivatived are uniformly bounded in $\hat{\mathcal{R}}(\beta, \delta)$ because they are continuous in $\mathcal{R}(\beta, \delta_0)$.

Now the construction of an initial solution w is always possible because there are not inequalities to be satisfied. This fact permits us to define them on the whole $\mathcal{R}(\beta, \delta_0)$, i.e. $w \in \mathcal{F}^2(\beta, \delta_0)$.

We define the set $\mathcal{S}$ as the collection of functions $u \in \mathcal{F}^2(\beta, \delta_0)$ satisfying (7.11). The transformation $\mathcal{T}$ defined by (7.12) maps $\mathcal{S}$ into itself by theorems 5.3 and 5.4 of Friedrichs' paper (because the image under $\mathcal{T}$ of any element of $\mathcal{S}$ is an initial solution).

We prove next that there is a $\lambda > 0$ for which $\mathcal{T}$ is a contraction with respect to the norm

$$\| |u|_t e^{-\lambda t} \| + \| |u_x|_t e^{-\lambda t} \| + \| |u_t|_t e^{-\lambda t} \|$$

where the norm $\| \quad \|$ is taken on $\mathcal{R}(\beta, \delta_0)$.

Equation (7.15.a) reduces now to

$$(\Delta U)_t + a(\Delta U)_x + \Delta l = 0$$

because a and d are two fixed functions independent of u and v. By corollary 5.1, it follows that given any $\varepsilon > 0$ we can chose $\lambda > 0$ (whose choice is independent of $u, v \in \mathcal{S}$) such that

$$\|e^{-\lambda t} \Delta U\| \leq \varepsilon \|e^{-\lambda t} \Delta l\|$$
$$\|e^{-\lambda t} \Delta U_x\| \leq \varepsilon \|e^{-\lambda t} \Delta l_x\|$$
$$\|e^{-\lambda t} \Delta U_t\| \leq \varepsilon \|e^{-\lambda t} \Delta l_x\| + \varepsilon \|e^{-\lambda t} \Delta l_t\|$$

From (7.19), using relations of the form (5.6), it follows that for any number $0 < \mu < 1$ we can chose $\varepsilon > 0$ in such a way that (7.16) holds.

To prove that the fixed point of the contraction is a solution of (7.4) we proceed as in the proof of theorem 7.1 but observe that equation (7.17.b) satisfied by the approximating sequence u^{σ}, reduces now to

$$u_t^{\sigma+1} + a u_x^{\sigma+1} + \ell + (\ell^{\sigma} - \ell) = 0$$

Since $\ell^{\sigma} - \ell$ goes to zero uniformly it follows that u is a solution of (7.4) in the strict sense on $\mathcal{R}(\beta, \delta_0)$.

8.- The Full Equation. Under some conditions the full equation may be shown to be equivalent to a reduced equation. In this section we establish this equivalence and formulate un ueness and existence theorems for the full equation.

Given the functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$ and $B(x, t, \tau, \cdot)$, $(x, t) \in \mathcal{R}(\beta, \delta)$; $(x, t, \tau) \in \tilde{\mathcal{R}}(\beta, \delta)$ we define the functional $A^{-1}(x, t, \cdot)$ (the existence of this functional is a consequence of the exclusion of the zero speed of propagation for characteristics) in an obvious manner. Then for any given function $u \in \mathcal{F}^1(\beta, \delta)$ and a fixed x such that $(x, 0) \in \mathcal{J}$ we define a linear operator ${}^uL(x)$ which transforms any continuous function $f: [0, t^*(x)] \longrightarrow R^{\nu}$ into another function of the same set, according to the rule

$$[{}^uL(x)f](t) = -\int_0^t {}^ub(x,t,\tau)\, {}^ua^{-1}(x,\tau) f(\tau)\, d\tau \tag{8.1}$$

Here we have written $t^*(x)$ for the least upper bound of the set $\{\tau \mid (x, \tau) \in \mathcal{R}(\beta, \delta)\}$. For brevity we shall frequently write L instead of ${}^{u}L$. The powers L^n of L are defined in the usual manner, for the powers of an operator. When f has first order continuous derivative in $[0, t^*(x)]$, we define the operator ${}^{u}P(x)$ by

$$[{}^{u}P(x)f](t) = \sum_{=1}^{\infty} \left[L^n \left(\frac{df}{dt} + c \right) \right](t) \tag{8.2}$$

when this series converges point-wise in the interval $[0, t^*(x)]$. When $v \in \mathfrak{F}^1(\beta, \delta)$, for fixed x, $v(x, t)$ defines a function in the interval $[0, t^*(x)]$, to which we can apply the operator ${}^{u}P(x)$, if we now permit x to vary we obtain a function defined on $\mathcal{R}(\beta, \delta)$ which we represent by ${}^{u}Pv$. We define now

$$p(x,t) = ({}^{u}P_u)(x,t) \quad ; (x,t) \in \mathcal{R}(\beta, \delta)$$

Given two functions $u, v \in \mathfrak{F}^1(\beta, \delta)$ the linear operators ΔL^n are defined for every n by

$$\Delta L^n = {}^{v}L^n - {}^{u}L^n \tag{8.3}$$

<u>THEOREM 8.1</u>. Assume that the functional s $A(x, t, \cdot)$, $A^{-1}(x, t, \cdot)$, $C(x, t, \cdot)$ and $B(x, t, \tau, \cdot)$ are such that the functions a, a^{-1}, c and b are continuous in $\mathcal{R}(\beta, \delta)$ the first three and in $\tilde{\mathcal{R}}(\beta, \delta)$ the last one, whenever $u \in \mathfrak{F}^1_{\Omega}(\beta, \delta)$. Then a function $u \in \mathfrak{F}^1_{\Omega}(\beta, \delta)$ satisfies

$$u_t - Au_x + C + \int_0^t B(x, t, \tau, u)\, u_x(\tau)\, d\tau = 0 \tag{8.4.a}$$

in $\mathcal{R}(\beta, \delta)$ if and only if

$$u_t + Au_x + c + {}^uPu = 0 \qquad (8.4.b)$$

holds in $\mathcal{R}(\beta, \delta)$.

Proof. We observe first $P(x, t, \cdot)$ is well defined for any function $u \in \mathcal{F}_{\alpha}^{1}(\beta, \delta)$ and that even more, the function $p(x, t)$ defined in this way on $\mathcal{R}(\beta, \delta)$ is continuous. This can be shown observing that ba^{-1} and $u_t + c$ are bounded in $\mathcal{R}(\beta, \delta)$ because b, a^{-1}, u_t and c are continuous in the closed region $\mathcal{R}(\beta, \delta)$.

Let M be such that

$$\|u_t + c\| < M \quad \text{and} \quad \|ba^{-1}\| < M$$

Then from (8.1) it follows that

$$\left| L_0^n (u_t + c) \right| \leq \frac{M^{n+1} t^n}{n!}$$

This shows that the series in (8.2) defining ${}^uPu$ is absolutely and uniformly convergent and therefore $p(x, t)$ is continuous. Here the obvious fact that every term $L^n(u_t + c)$ is continuous in $\mathcal{R}(\beta, \delta)$ has been used.

We show first that (8.4.a) implies (8.4.b). Equation (8.4.a) implies

$$u_x = -a^{-1}\{u_t + c\} - a^{-1}\int_0^t b(x, t, \tau)\, u_x(x, \tau)\, d\tau \qquad (8.5)$$

Write, for fixed x

$$f(t) = \int_0^t b(x, t, \tau)\, u_x(\tau)\, d\tau \qquad (8.6)$$

multiply (8.5) by $b(x, t_0, t)$ and integrate the resulting equation with respect to t from 0 to t_0, to obtain

$$f(t_0)=\left[{}^{u}L(x)(u_t+c)\right](t_0)-\int_0^{t_0} b(x,t_0,t)\,a^{-1}(x,t)\,f(t)\,dt \tag{8.7}$$

this is an integral equation for f whose only solution is

$$f = p \tag{8.8}$$

When we substitute this result in (8.4.a) using (8.6), (8.4.b) follows.

Conversely, assume (8.4.b) and apply L to (8.4.b) to obtain using (8.6)

$$f(t)=\left[L(x)(u_t+c)\right](t)-\int_0^{t} b(x,t,\tau)\,a^{-1}(x,\tau)\,p(\tau)\,d\tau \tag{8.9.a}$$

But by its definition, p is a solution of the integral equation (8.7), therefore

$$p(t)=\left[L(x)(u_t+c)\right](t)-\int_0^{t} b(x,t,\tau)\,a^{-1}(x,\tau)\,p(\tau)\,d\tau \tag{8.9.b}$$

Equation (8.9.a) and (8.9.b) together imp:

$$p(t) = f(t)$$

When this result is substituted in (8.4.b) u:'n (8.6), equation (8.4.a) follows. This completes the proof of theorem 8.·.

Having established the equivalence of equations (8.4), our aim will be to prove that the theorems of sections 6 and 7 are applicable to equation (8.4.b) when the functional $B(x, t, \tau, \cdot)$ satisfies smoothness assumptions analogous to those required of A and · in those theorems. To make easier the understanding of the discussion to follow it is helpful to explain briefly its motivation. Suppose we had defined the function

$$g(x,t)=\int_0^t B(x,t,\tau,u)\,u_x(\tau)\,d\tau\ ;\quad (x,t)\in R(\beta,\delta)$$

for every $u\in\mathcal{F}^1_\Omega(\beta,\delta)$. In order to be able to apply the theorems of sections 6 and 7 to equation (8.4.a) we would have had to prove inequalities of the form

$$|\Delta g|\leq k|\Delta u|_t$$

and

$$|\Delta g_x|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\}$$

$$|\Delta g_t|\leq k\{|\Delta u|_t+|\Delta u_x|_t+|\Delta u_t|_t\}$$

However such bounds seem to be rather unnatural because the definition of g contains u_x, and therefore Δg can be bounded only by terms containing the first derivatives of Δu. The definition of p on the other hand is

$$p(x,t)=\sum_{n=1}^{\infty}\left\{L^n\left(\frac{\partial u}{\partial t}+c\right)\right\}(x,t)$$

At first glance it may be thought that this function presents the same type of difficulty because it contains u_t. However, u_t is contained in an integral with respect to t. The main point of the discussion to follow will be to avoid the introduction of higher order derivatives of u, using integration by parts.

Given u define the operator ${}^uL_x$ by

$${}^uL_x v=-\int_0^t\{b(x,t,\tau)a^{-1}(x,\tau)\}_x\,v(\tau)\,d\tau \qquad (8.10.a)$$

and ${}^uL_{xx}$ by

$${}^uL_{xx} v=-\int_0^t\{b(x,t,\tau)a^{-1}(x,\tau)\}_{xx}\,v(\tau)\,d\tau \qquad (8.10.b)$$

when the right-hand members in (8.10) exist.

With this notation we prove some lemmas

LEMMA 8.1. Assume the functional $A^{-1}(x, t, \cdot)$ to exist for every $(x, t) \in \mathcal{R}(\beta, \delta)$. Let the functionals $A(x, t, \cdot)$, $A^{-1}(x, t, \cdot)$ $C(x, t, \cdot)$ and $B(x, t, \tau, \cdot)$ be such that the functions a, a^{-1}, c and b possess continuous derivatives in $\mathcal{R}(\beta, \delta)$ the first three and in $\tilde{\mathcal{R}}(\beta, \delta)$ the last one, whenever $u \in \mathcal{F}^1_\Omega(\beta, \delta)$. Assume $u \in \mathcal{F}^1_\Omega(\beta, \delta)$ and write L for ${}^uL$. Then

I) If $v \in \mathcal{F}(\beta, \delta)$ and $n = 1, 2, \ldots$ we have:

a) $\{L^n(v)\}_t \in \mathcal{F}(\beta, \delta)$

$$\{L(v)\}_t = -b(x,t,t)\, a^{-1}(x,t)\, v(x,t) - \int_0^t b_t(x,t,\tau)\, a^{-1}(x,\tau)\, v(x,\tau)\, d\tau \tag{8.11.a}$$

b). For a given u, there is $M > 0$ (which depends only on the bounds for a^{-1}, b and their first order derivatives) such that

$$|L^n v|_t \leq \frac{M^n t^n}{n!} |v|_t \tag{8.11.b}$$

$$|\{L^n v\}_t|_t \leq \frac{M^n t^{n-1}}{(n-1)!} |v|_t \tag{8.11.c}$$

II) If $v \in \mathcal{F}^1(\beta, \delta)$ and $n = 1, 2, \ldots$ we have:

a) $L^n(v) \in \mathcal{F}^1(\beta, \delta)$

b) $$\{L^n(v)\}_x = \sum_{i=1}^{n} L^{i-1} L_x L^{n-1}(v) + L^n(v_x) \tag{8.12.a}$$

c) For a given u, there is $M > 0$ (which depends only on the bounds for a^{-1}, b and their first order derivatives) such that

$$\left| \left\{ L^n v \right\}_x \right|_t \leq \frac{M^n t^n}{n!} \left\{ n |v|_t + |v_x|_t \right\} \tag{8.12.b}$$

d) . $L^n v_t \in \mathcal{F}^1(\beta,\delta)$

III)

a) $\varphi \in \mathcal{F}^1(\beta,\delta)$ and the derivatives of φ are obtained differentiating term by term the series defining φ.

b) If for some $\Omega > 0$ the functions a^{-1}, b, c and their first order partial derivatives have a uniform bound in $\mathcal{R}(\beta,\delta)$ which is independent of $u \in \mathcal{F}^1_\Omega(\beta,\delta)$, then also φ and its first order partial derivatives have a uniform bound in $\mathcal{R}(\beta,\delta)$ which is independent of $u \in \mathcal{F}^1_\Omega(\beta,\delta)$

Proof. Ia) Follows, taking derivative with respect to t in the definition (8.1) of L, to obtain (8.11.a). This shows that $\{L^n(v)\}_t \in \mathcal{F}^1(\beta,\delta)$ for $n = 1$. For general n the same is easily shown by induction using (8.11.a).

To prove Ib) observe that ba^{-1} and $b_t a^{-1}$ are bounded in $\mathcal{R}(\beta,\delta)$ because these functions are continuous and the region is closed. Let M be a common bound for these functions. Then from (8.1) it follows

$$|Lv|_t \leq M \int_0^t |v|_\tau \, d\tau \tag{8.13}$$

Hence

$$|Lv|_t \leq M t |v|_t$$

Thus, we have shown (8.11.b) for $n = 1$.

Assume (8.11.b) valid for arbitrary n, i.e.

$$|L^n v|_\tau \leq \frac{M^n \tau^n}{n!} |v|_\tau .$$

then by (8.13)

$$|L^{n+1} v|_t \leq M \int_0^t |L^n v|_\tau \, d\tau \leq \frac{M^{n+1} t^{n+1}}{n!} |v|_t$$

which completes the proof of (8.11.b) by induction. To prove (8.11.c) take the bound M sufficiently large so that it bounds also to $|ba^{-1}| + |b_t a^{-1}| \delta$. Then by (8.11.a) we have

$$|\{L(v)\}_t|_t \leq M |v|_t$$

which is (8.11.c) for $n = 1$. Using (8.13) the inequality (8.11.c) follows by induction.

By Ia), to prove IIa) it is only necessary to prove that $\{L^n(v)\}_x \in \mathcal{F}^1(\beta, \delta)$. This will follow from formula (8.12.a), which in turn can be proved by induction. The formula

$$(Lv)_x = L_x v + L v_x \tag{8.14}$$

follows directly from (8.1). This shows (8.12.a) for $n = 1$. Equation (8.12.a) for general n follows by induction using (8.14).

To prove IIc) take M sufficiently large so that it bounds also $(ba^{-1})_{xx}$. Then from (8.12.a), (8.11.b) and repeated use of (8.13), the inequality (8.12. b) follows.

By IIa), to prove IId) it will be enough to show that

$Lv_t \in \mathcal{F}^1(\beta,\delta)$. By hypothesis $v_t \in \mathcal{F}(\beta,\delta)$, Therefore by Ia) we have $\{L(v_t)\}_t \in \mathcal{F}(\beta,\delta)$. It remains only to show that $\{L(v_t)\}_x \in \mathcal{F}(\beta,\delta)$. By (8.1) for any $h \neq 0$, we have

$$\frac{L(x+h)\,v_t(x+h) - L(x)\,v_t(x)}{h} = -\frac{1}{h}\int_0^t \{b(x+h,t,\tau)a^{-1}(x+h,\tau)$$

$$-\,b(x,t,\tau)a^{-1}(x,\tau)\}v_t(x+h,\tau)\,d\tau - \frac{1}{h}\int_0^t b(x,t,\tau)a^{-1}(x,\tau)\{v_t(x+h,\tau)-v_t(x,\tau)\}d\tau$$

Integrating by parts the second integral of the right hand member, we obtain after taking the limit when h goes to zero:

$$\frac{\partial L v_t}{\partial x} = L_x v_t - b(x,t,t)\,a^{-1}(x,t)v_x(x,t) + b(x,t,0)a^{-1}(x,0)v_x(x,0)$$
$$+\int_0^t \{b(x,t,\tau)a^{-1}(x,\tau)\}_\tau\, v_x(x,\tau)\,d\tau \qquad (8.15)$$

This shows that $(Lv_t)_x \in \mathcal{F}(\beta,\delta)$.

By (8.2) we have

$$p = {}^u P_u = \sum_{n=0}^{\infty} L^n w + \sum_{n=1}^{\infty} L^n c \qquad (8.16.a)$$

where

$$w = L u_t \qquad (8.16.b)$$

Observe that $w \in \mathcal{F}^1(\beta,\delta)$ by IId) and by hypothesis $c \in \mathcal{F}^1(\beta,\delta)$ also. Then, by IIa) every term in the series of the right hand member of (8.16.a) belongs to $\mathcal{F}^1(\beta,\delta)$. By (8.11.c) and (8.12.b) the series obtained taking the partial derivatives term by term in the series of the right hand member of (8.16.a) are uniformly convergent in $\mathcal{R}(\beta,\delta)$. In addition, the series themselves are uniformly convergent in $\mathcal{R}(\beta,\delta)$; by

(8.11.b). Therefore $\Omega \in \mathfrak{F}^1(\beta, \delta)$, and its first partial derivatives may be obtained differentiating term by term (8.16.a).

Observe that under hypothesis of IIIb) the bounds M occurring in (8.11.b), (8.11.c) and (8.12.b) are independent of $u \in \mathfrak{F}^1(\beta, \delta)$. But $t \leq \delta$, and by (8.11.b and c)

$$|w|_t \leq M t |u_t|_t \leq M \delta \Omega$$
$$|w_t|_t \leq M |u_t|_t \leq M \Omega$$

Also using (8.15) bounds for $|w_x|$ in $\mathcal{R}(\beta, \delta)$ can be constructed which are independent of the particular $u \in \mathfrak{F}^1_\Omega(\beta, \delta)$ considered. All *these facts together with inequalities (8.11) and (8.12.b) imply that the function*

$$\sum_{n=0}^{\infty} L^n w$$

and its first order partial derivatives possess bounds which hold in $\mathcal{R}(\beta, \delta)$ independently of the particular $u \in \mathfrak{F}^1_\Omega(\beta, \delta)$ considered. A similar argument shows that the same is true for

$$\sum_{n=1}^{\infty} L^n c$$

<u>LEMMA 8.2</u>. If in addition to the assumptions about the functionals A, A^{-1}, B, C in lemma 8.1 the functions a, a^{-1}, b, c and their first order partial derivatives have a bound in $\mathcal{R}(\beta, \delta)$ which holds for all $u \in \mathfrak{F}^1_\Omega(\beta, \delta)$ and there is a $k > 0$ such that

$$|\Delta a^{-1}| \leq k |\Delta u|_t \; ; |\Delta c| \leq k |\Delta u|_t \text{ in } \mathcal{R}(\beta, \delta) \qquad (8.17.a)$$

and

$$|\Delta b| \leq k |\Delta u|_t \quad \text{in } \tilde{R}(\beta,\delta) \tag{8.17.b}$$

whenever $u, v \in \mathcal{F}^1_\Omega(\beta,\delta)$. Then there is a $k' > 0$ such that

$$|\Delta p| = |{}^v p_v - {}^u p_u| \leq k' |\Delta u|_t, \text{ in } R(\beta,\delta) \tag{8.18}$$

whenever $u, v \in \mathcal{F}^1_\Omega(\beta,\delta)$.

Proof. To prove this lemma observe that for any $n \geqslant 1$

$${}^v L^n\, {}^v c - {}^u L^n\, {}^u c = \sum_{i=1}^{n} {}^v L^{i-1} (\Delta L) L^{n-i}\, {}^u c + {}^v L^n \Delta c \tag{8.19.a}$$

$${}^v L^n v_t - {}^u L^n u_t = \sum_{i=1}^{n} {}^v L^{i-1} (\Delta L) L^{n-i} u_t + {}^v L^n \Delta u_t \tag{8.19.b}$$

On the other hand, by definition

$$(\Delta L) w = - \int_0^t \{[\Delta b(x,t,\tau)] a^{-1}(x,\tau) + b(x,t,\tau)\Delta a^{-1}(x,\tau)\} w(x,\tau)\, d\tau \tag{8.20}$$

for any $w \in \mathcal{F}_1(\beta,\delta)$. By the assumptions of Lemma 8.2, it follows that there exists a number $k'' > 0$ independent of $u, v \in \mathcal{F}^1_\Omega(\beta,\delta)$ such that

$$|(\Delta L) w| \leq k'' |\Delta u|_t \int_0^t |w|_\tau \, d\tau \tag{8.21}$$

By means of this relation and repeated use of (8.13) it follows that

$$|{}^v L^{i-1} (\Delta L) L^{n-i} w| \leq \frac{k'' M^{n-1} t^n}{n!} |w|_t |\Delta u|_t \tag{8.22.a}$$

where M is independent of the particular couple of functions $u, v \in \mathcal{F}^1_\Omega(\beta,\delta)$ considered.

The inequalities (8.11.b) and (8.17.a) together imply

$$|{}^{v}L^{n}\Delta c| \leq \frac{kM^{n}t^{n}}{n!}|\Delta u|_t \qquad (8.22.b)$$

A bound for ${}^{v}L^{n}\Delta u_t$ in terms of $|\Delta u|_t$ can be obtained integrating by parts the definition (8.1)of ${}^{v}L$:

$${}^{v}L\Delta u_t = {}^{v}b(x,t,0)\,{}^{v}a^{-1}(x,0)\Delta u(x,0) - {}^{v}b(x,t,t)\,{}^{v}a^{-1}(x,t)\Delta u(x,t)$$

$$+\int_0^t \{{}^{v}b(x,t,\tau)\,{}^{v}a^{-1}(x,\tau)\}_{\tau}\,\Delta u(x,\tau)\,d\tau \qquad (8.22.c)$$

Since ${}^{v}b\,{}^{v}a^{-1}$ and its partial derivatives admit bounds which hold regardless of the particular $v \in \mathcal{F}^{1}_{\Omega}(\beta,\delta)$ considered, it follows that we can choose $M>0$ sufficiently large so that

$$|{}^{v}L\Delta u_t| \leq M|\Delta u|_t \qquad (8.23.a)$$

and (8.22.a, b) holds simultaneously. From (8.11.b) it now follows that

$$|{}^{v}L^{n}\Delta u_t| \leq \frac{M^{n}t^{n-1}}{(n-1)!}|\Delta u|_t \qquad (8.23.b)$$

Relations (8.19), (8.22.a,b) and (8.23.b) imply that

$$|{}^{v}L^{n}\,{}^{v}c - {}^{u}L^{n}\,{}^{u}c| \leq \frac{(n+1)M^{n+1}t^{n-1}}{(n-1)!}|\Delta u|_t \qquad (8.24.a)$$

$$|{}^{v}L^{n}v_t - {}^{u}L^{n}u_t| \leq \frac{(n+1)M^{n+1}t^{n-1}}{(n-1)!}|\Delta u|_t \qquad (8.24.b)$$

if M is conveniently chosen Recall however that the choice of M is independent of $u, v \in \mathcal{F}^{1}_{\Omega}(\beta,\delta)$. By the definition (3.2) of P, we have

$${}^{v}P_v - {}^{u}P_u = \sum_{n=1}^{\infty}\left({}^{v}L^{n}v_t - {}^{u}L^{n}u_t\right) + \sum_{n=1}^{\infty}\left({}^{v}L^{n}\,{}^{v}c - {}^{u}L^{n}\,{}^{u}c\right)$$

From this equation and (8.24) it follows that if

$$k' = 2\sum_{n=1}^{\infty} \frac{(n+1)\,M^{n+1}\,\delta^{n-1}}{(n-1)!}$$

then

$$|\Delta p| = |{}^{v}P_{v} - {}^{u}P_{u}| \leq k'\,|\Delta u|_t \quad ; \text{ in } \mathcal{R}(\beta,\delta)$$

where k' is independent of $u, v \in \mathfrak{F}_{\Omega}^{1}(\beta,\delta)$.

LEMMA 8.3. Assume the functional $A^{-1}(x, t, \cdot)$ to exist and let the functionals $A(x, t, \cdot)$, $A^{-1}(x, t, \cdot)$ $C(x, t, \cdot)$, $B(x, t, \sigma, \cdot)$ be such that

I) The functions a, a^{-1}, c and b have continuous second derivatives in $\mathcal{R}(\beta,\delta)$ the first three and in $\widetilde{\mathcal{R}}(\beta,\delta)$ the last one, whenever $u \in \mathfrak{F}_{\Omega}^{2}(\beta,\delta)$ and continuous first order derivatives in $\mathcal{R}(\beta,\delta)$ the first three and in $\widetilde{\mathcal{R}}(\beta,\delta)$ the last one, whenever $u \in \mathfrak{F}_{\Omega}^{1}(\beta,\delta)$.

II) There is a number $M > 0$ which bounds a, a^{-1}, c and their first and second derivatives in $\mathcal{R}(\beta,\delta)$ and b and their first and second derivatives in $\widetilde{\mathcal{R}}(\beta,\delta)$ whenever $u \in \mathfrak{F}_{\Omega}^{2}(\beta,\delta)$ (M is independent of u).

III) There exists $k > 0$ such that in $\mathcal{R}(\beta,\delta)$

$$|\Delta a| \leq k\,|\Delta u|_t\,;\ |\Delta c| \leq k\,|\Delta u|_t\,;\ |\Delta a^{-1}| \leq k\,|\Delta u|_t \tag{8.25.a}$$

$$|\Delta a_x| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}\,;\ |\Delta a^{-1}_x| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.25.b}$$

$$|\Delta a_t| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}\,;\ |\Delta a^{-1}_t| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.25.c}$$

$$|\Delta c_x| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.25.d}$$

$$|\Delta c_t| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.25.e}$$

and in $\mathcal{R}(\beta,\delta)$

$$|\Delta b| \leq k|\Delta u|_t \tag{8.26.a}$$

$$|\Delta b_x| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.26.b}$$

$$|\Delta b_t| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.26.c}$$

$$|\Delta b_\tau| \leq k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.26.d}$$

Then:

i) The function

$$p = {}^u P_u$$

has continuous second derivatives in $\mathcal{R}(\beta,\delta)$ whenever $u \in \mathcal{J}^2_\Omega(\beta,\delta)$, and continuous first derivatives in $\mathcal{R}(\beta,\delta)$ whenever $u \in \mathcal{J}^1_\Omega(\beta,\delta)$.

ii) There is a positive number which bounds p and its first and second derivatives in $\mathcal{R}(\beta,\delta)$ for every $u \in \mathcal{J}^2_\Omega(\beta,\delta)$.

iii) There is a $k' > 0$ such that in $\mathcal{R}(\beta,\delta)$

$$|\Delta p| \leq k'|\Delta u|_t \tag{8.27.a}$$

$$|\Delta p_x| \leq k'\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.27.b}$$

$$|\Delta p_t| \leq k'\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\} \tag{8.27.c}$$

whenever u $v \in \mathcal{J}^1_\Omega(\beta,\delta)$.

Proof. To prove this lemma we use arguments similar to those used to prove lemmas 8.1 and 8.2. Therefore we only sketch the proof, emphasizing the points which seem to be less obvious.

By Lemma 8.1 p possess continuous first derivatives in $\mathcal{R}(\beta,\delta)$ whenever $u \in \mathcal{J}^1_\Omega(\beta,\delta)$, (therefore, whenever $u \in \mathcal{J}^2_\Omega(\beta,\delta)$) and, p. and its derivatives possess bounds which are independent of the particular u chosen.

To prove that $\underset{\smile}{\eta}$ possess continuous second order derivatives when $u \in \mathcal{F}^{1\,2}_{\Omega}(\beta, \delta)$ we show that the series (8.16.a) can be differentiated term by term to obtain a uniformly convergent series in $\mathcal{Q}(\beta, \delta)$. O... program is first to show that $w \in \mathcal{F}^{1\,2}(\beta, \delta)$ (w given by (8.16.b)). ... is immediate by (8.11.a) and (8.15). Second we show that w and its first and second order partial derivatives are uniformly bounded in $\mathcal{Q}(\beta, \delta)$ the bound being independent of $u \in \mathcal{F}^{1\,2}_{\Omega}(\beta, \delta)$. Part of this assertion has already been shown in lemma 8.2, the rest follows taking the partial derivatives in (8.11.a) and (8.15)

Using the above facts it is not difficult to show that the series

$$\sum_{n=0}^{\infty} \{L^n w\}_{xx} \quad ; \sum_{n=0}^{\infty} \{L^n v\}_{xt} \quad ; \sum_{n=0}^{\infty} \{L^n v\}_{tt} \tag{8.28}$$

are absolutely and uniformly convergent. When proving this assertion, the formula

$$\{L^n w\}_{xx} = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{i-1} L^{i-1} L^{j-1} L_x L^{i-1-j} L_x L^{n-1} w +$$
$$\sum_{j=1}^{n-i} L^{i-1} L_x L^{j-1} L_x L^{n-i-j} w + L^{i-1} L_{xx} L^{n-i} w_x \Big\} + L^n w_{xx}$$

has to be used. Also by (8.11.a) we have

$$\{L^n w\}_{tt} = -(ba^{-1})_t L^{n-1} w - ba^{-1}(L^{n-1} w)_t - b_t a^{-1} L^{n-1} w - \int_0^t b_{tt} a^{-1} (L^{n-1} w)\, d\tau$$

Simmilar formulas may be obtained for

$$\{L^n w\}_{xt} = \{L^n w\}_{tx}$$

by simultaneous use of (8.11.a) and (8.12.b). The bounds for the series (8.28) ... [illegible]

Similar arguments apply to

$$\sum_{n=1}^{\infty}\{L^n c\}_{xx}\quad ;\ \sum_{n=1}^{\infty}\{L^n c\}_{xt}\ ;\sum_{n=1}^{\infty}\{L^n c\}_{tt}$$

By (8.16), this completes the proof of i) and ii), because the series obtained by taking term by term the first order partial derivatives in (8.16.a) has already been shown to be uniformly and absolutely convergent in Lemma 8.2.

The inequality (8.27.a) was also shown in Lemma 8.2. Therefore, to prove iii), it remains to prove (8.27.b, c). By (8.11.a), we have,

$$\{L^n u_t\}_t = -b(x,t,t)a^{-1}(x,t)L^{n-1}u_t - \int_0^t b_t(x,t,\tau)a^{-1}(x,\tau)L^{n-1}u_t\,d\tau$$

Therefore

$$\{\Delta L^n u_t\}_t = (\Delta b a^{-1})L^{n-1}u_t - {}^v b\, {}^v a^{-1}(\Delta L^{n-1}u_t) -$$

$$-\int_0^t(\Delta b_t a^{-1})L^{n-1}u_t\,d\tau - \int_0^t {}^v b_t\, {}^v a^{-1}(\Delta L^{n-1}u_t)\,d\tau$$

Using (8.11.b), (8.24.b), the assumptions (8.25) and arguments completely analogous to prove lemma 8.2 allow to prove that

$$\sum_{n=1}^{\infty}\left({}^vL^n v_t - {}^uL^n u_t\right)_t < k''\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

for some $k'' > 0$ independent of $u, v \in \mathcal{C}^{n+1}_{\Omega}(\beta,\delta)$. A similar argument applies to

$$\sum_{n=1}^{\infty}\left({}^vL^n\, {}^vc - {}^uL^n\, {}^uc\right)_t$$

and therefore (8.27.c) follows.

In the following argument [illegible] the notation

$$\Delta w = {}^{v}L v_t - {}^{u}L u_t$$

and we write Δw_x and Δw_t for the partial derivatives of Δw. Observe that these partial derivatives exist and are uniformly bounded by lemma 8.1 in spite of the fact that according to the assumption in iii), $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$.

By (8.19), we have for $n \geq 1$,

$$^{v}L^{n+1} v_t - {}^{u}L^{n+1} u_t = \sum_{i=1}^{n} {}^{v}L^{i-1} (\Delta L) L^{n-i} w + {}^{v}L^{n} \Delta w$$

and therefore using (8.12.a)

$$\begin{aligned}
({}^{v}L^{n+1} v_t - {}^{u}L^{n+1} u_t)_x &= \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{i-1} {}^{v}L^{j-1}\, {}^{v}L_x\, {}^{v}L^{i-1-j} (\Delta L) L^{n-i} w \\
&+ {}^{v}L^{i-1} (\Delta L_x) L^{n-i} + \sum_{j=1}^{n-i} {}^{v}L^{i-1} (\Delta L) L^{j-1} L_x L^{n-i-j} w \\
&+ {}^{v}L^{i-1} (\Delta L) L^{n-i} w_x + {}^{v}L^{i-1}\, {}^{v}L_x L^{n-i} \Delta w \Big\} \\
&+ L^{n} \Delta w_x \qquad\qquad (8.29)
\end{aligned}$$

By means of this formula the process of proving that there exists a k'' independent of $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$ such that

$$\Big| \sum_{n=0}^{\infty} ({}^{v}L^{n+1} v_t - {}^{u}L^{n+1} u_t)_x \Big| \leq k'' \big\{ |\Delta u| + |\Delta u_x|_t + |\Delta u_t|_t \big\}$$

is completely analogous to that of proving lemma 8.2. The only thing peculiar is the proof of the existence of a number k''' independent of $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$ such that

$$|\Delta w_x| \leq k''' \big\{ |\Delta u| + |\Delta u_x|_t + |\Delta u_t|_t \big\}$$

This is achieved using (8.15) and the assumptions III) of lemma 8.3. The proof that similarly

$$\left|\sum_{n=1}^{\infty}\left({}^{v}L^{n}v_{c} - {}^{u}L^{n}u_{c}\right)_x\right| \leq k''\{|\Delta u| + |\Delta u_x|_t + |\Delta u_t|_t\}$$

does not offer difficulty because a relation similar to (8.29) holds. All this shows that

$$|\Delta p_x| = \left|\sum_{n=1}^{\infty}\left({}^{v}L^{n}v_t - {}^{u}L^{n}u_t\right)_x + \sum_{n=1}^{\infty}\left({}^{v}L^{n}v_c - {}^{u}L^{n}u_c\right)_x\right|$$

$$\leq k'\{|\Delta u| + |\Delta u_x|_t + |\Delta u_t|_t\}$$

for some $k' > 0$ whenever $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$.

By means of theorem 8.1 and lemmas 8.1 and 8.2 the uniqueness of solution for a Cauchy problem of the full equation can be shown.

<u>THEOREM 8.2</u>. Let the functionals $A(x, t, \cdot)$, $A^{-1}(x, t, \cdot)$, $C(x, t, \cdot)$, $B(x, t, \tau, \cdot)$ be such that the functions a, a^{-1}, q, q^{-1}, c and b and their first order derivatives exist and are bounded in $\mathcal{R}(\beta, \delta)$ the first five and in $\tilde{\mathcal{R}}(\beta, \delta)$ the last one, when $u \in \mathfrak{F}^1_\Omega(\beta, \delta)$, the bound being independent of u. Assume even more that there is a $k > 0$ such that in $\mathcal{R}(\beta, \delta)$

$$|\Delta a| \leq k|\Delta u|_t \quad ; \quad |\Delta a^{-1}| \leq k|\Delta u|_t \, ; |\Delta c| \leq k|\Delta u|_t$$

and in $\tilde{\mathcal{R}}(\beta, \delta)$

$$|\Delta b| \leq k|\Delta u|_t$$

whenever $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$. Then if two functions $u, v \in \mathfrak{F}^1_\Omega(\beta, \delta)$ satisfy the equation

$$u_t + Au_x + C + \int_0^t B(x, t, \tau, u)\, u_x(\tau)\, d\tau = 0$$

and assume the same initial values on $\mathcal{J}$, we necessarily have

$$u(x,t) = v(x,t)$$

whenever $(x, t) \in \mathcal{R}(\beta, \delta)$.

Proof. By theorem 8.1 u and v satisfy

$$u_t + A u_x + C + {}^uP_u = 0$$

in $\mathcal{R}(\beta, \delta)$. This is an equation of the form (6.3). By lemmas 8.1 and 8.2 the hypothesis of theorem 6.1 are satisfied. Therefore

$$u = v$$

in $\mathcal{R}(\beta, \delta)$. This completes the proof of theorem 8.2.

The existence of solution for a Cauchy problem of the full equation is given in the following:

THEOREM 8.3. Let the functionals $A(x, t, \cdot)$, $A^{-1}(x, t, \cdot)$ (which is assumed to exist), $C(x, t, \cdot)$, $B(x, t, \tau, \cdot)$ and the number $\delta_0 > 0$, be such that for any $0 < \delta \leq \delta_0$, we have

I) The functions a, a^{-1}, q, q^{-1}, c and b have continuous second derivatives in $\mathcal{R}(\beta, \delta_0)$ the first five and in $\tilde{\mathcal{R}}(\beta, \delta_0)$ the last one, whenever $u \in \mathcal{F}^2_\Omega(\beta, \delta)$, and continuous first order derivatives in $\mathcal{R}(\beta, \delta)$ the first five and in $\tilde{\mathcal{R}}(\beta, \delta_0)$ the last one, whenever $u \in \mathcal{F}^1_\Omega(\beta, \delta)$.

II) There is a common bound M for a, a^{-1}, q, q^{-1}, c and their first and second derivatives in $\mathcal{R}(\beta, \delta)$ and for b and their first and second derivatives in $\tilde{\mathcal{R}}(\beta, \delta)$ whenever $u \in \mathcal{F}^2_\Omega(\beta, \delta)$. This bound is independent of u and of δ.

III) There exists $k > 0$ (independent of δ) such that for every $u, v \in \mathcal{F}^1_\Omega(\beta, \delta)$, we have

$$|\Delta a| \leq k|\Delta u|_t \; ; |\Delta c| \leq k|\Delta u|_t \; ; |\Delta a^{-1}| \leq k|\Delta u|_t$$

$$|\Delta a_x| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\};\ |\Delta a_x^{-1}| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

$$|\Delta a_t| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\};\ |\Delta a_t^{-1}| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

$$|\Delta c_x| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

$$|\Delta c_t| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

in $R(\beta, \delta)$

$$|\Delta b| \le k|\Delta u|_t$$

$$|\Delta b_x| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

$$|\Delta b_t| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

$$|\Delta b_\tau| \le k\{|\Delta u|_t + |\Delta u_x|_t + |\Delta u_t|_t\}$$

in $\tilde{R}(\beta, \delta)$.

IV). Let the function $\bar{u}(x)$ possess continuous second derivatives on $\mathcal{J}$, and be such that the initial data u_x, u_t, u_{xx}, u_{xt}, u_{tt} implied by the equation

$$u_t + Au_x + C + Pu = 0 \qquad (8.30.a)$$

and the derived relation

$$(u_t)_t + a(u_t)_x + a_t u_x + c_t + p_t = 0 \qquad (8.30.b)$$

satisfy

$$\|\hat{u}\| + \|\hat{u}_x\| + \|\hat{u}_t\| + \|\hat{u}_{xx}\| + \|\hat{u}_{xt}\| + \|\hat{u}_{tt}\| \le \Omega^* < \Omega \qquad (8.31)$$

Then there is a function $u \in \mathcal{F}^1_\Omega(\beta, \delta')$ which satisfies

$$u_t + au_x + c + \int_0^t b(x,t,\tau)u_x(x,\tau)\,d\tau = 0 \qquad (8.32)$$

in $R(\beta, \delta')$ for some $0 < \delta' \le \delta_0$, and $u(x, 0) = \hat{u}(x)$, on $\mathcal{J}$.

Proof. Consider equation (8.30.a). By lemmas 8.1, 8.2 and 8.3, this equation satisfies the hypothesis of theorem 7.1. Therefore for some $\delta' > 0$ there is a solution $u \in \mathcal{F}^1_\Omega(\beta, \delta')$ of (8.30.a) such that

$$u(x,0) = \hat{u}(x) \quad \text{on } \mathcal{D}$$

By theorem 8.1, u satisfies (8.32) in $Q(\beta, \delta')$. This completes the proof of theorem 8.3.

9. The Linear Equation. In the linear case the existenc of solution again can be shown in the large.

THEOREM 9.1. Let the family of functionals $A(x, t, \cdot)$, $C(x, t, \cdot)$, $B(x, t, \tau, \cdot)$ and the number $\delta_0 > 0$ satisfy the hypothesis of theorem 7.2 and be such that:

i) $a^{-1} \in \mathcal{F}^2(\beta, \delta_0)$

ii) $b \in \tilde{\mathcal{F}}^2(\beta, \delta_0)$

Then given any function $\hat{u}(x)$ defined on $\mathcal{D}$ and possessing second order continuous derivatives there, there is a function $u \in \mathcal{F}^1(\beta, \delta_0)$ which satisfies (8.32) in $Q(\beta, \delta_0)$ and such that

$$u(x,0) = \hat{u}(x)$$

on $\mathcal{D}$.

Proof. Observe that in this case

$$\beta = {}^uP_u = \sum_{n=1}^{\infty} L^n u_t + \sum_{n=1}^{\infty} L^n \mathcal{L} u + \sum_{n=1}^{\infty} L^n d \qquad (9.1)$$

Our goal will be to show that equation (8.30.a) satisfies the hypothesis of theorem 7.2, with

$${}^uC = \mathcal{L}^1 u + d^1 \qquad (9.2)$$

where

$$\mathcal{L}^1 u = \sum_{n=1}^{\infty} L^n u_t + \sum_{n=1}^{\infty} L^n \mathcal{L} u + \mathcal{L} u \tag{9.3.a}$$

and

$$d^1 = d + \sum_{n=1}^{\infty} L^n d \qquad)9.3.b)$$

Observe that $\mathcal{L}^1$ as given by (9.3.a) is a linear functional of u. Comparing the hypothesis of theorem 9.1 with those of theorem 7.2 it is seen that all what remains to prove is that c as given by (9.2) satisfies the hypothesis of theorem 7.2. Because of the uniform and absolute convergence of the series in (9.3.b) as well as those obtained by taking the first two partial derivatives, it follows that d^1 satisfies the hypothesis satisfied by a in theorem 7.2. A similar argument holds for

$$\mathcal{L} u + \sum_{n=1}^{\infty} L^n \mathcal{L} u$$

because we assume that $\ell \in \mathfrak{F}^1(\beta, \delta_0)$ whenever $u \in \mathfrak{F}^1(\beta, \delta_0)$ and $\ell \in \mathfrak{F}^2(\beta, \delta_0)$ whenever $u \in \mathfrak{F}^2(\beta, \delta_0)$, where $\ell(x, t) = \mathcal{L} u$. On the other hand for some M

$$\left| \sum_{n=1}^{\infty} L^n \mathcal{L} u \right| \leq |\mathcal{L} u|_t \sum_{n=1}^{\infty} \frac{M^n t^n}{n!} \leq \left(k \sum_{n=1}^{\infty} \frac{M^n t^n}{n!} \right) |u|_t$$

because ba^{-1}_t is a fixed function. Finally

$$\begin{aligned} L u_t &= -\int_0^t b(x,t,\tau)\, a^{-1}(x,\tau)\, u_t(x,\tau)\, d\tau \\ &= b(x,t,0)\, a^{-1}(x,0)\, u(x,0) - b(x,t,t)\, a^{-1}(x,t)\, u(x,t) \\ &\qquad + \int_0^t (ba^{-1})_\tau\, u(x,\tau)\, d\tau \end{aligned} \tag{9.4}$$

and therefore

$$\| L u \| \leq k' |u|$$

for some k', whenever $u \in \mathcal{F}^1(\beta, \delta_0)$

Thus

$$\left|\sum_{n=1}^{\infty} L^n u_t\right| \leqslant \left(k' \sum_{n=0}^{\infty} \frac{M^n t^n}{n!}\right) |u|_t$$

Similar bounds can be obtained for the first order derivatives differentiating term by term (9.3.a, b), using (9.4). The existence of the second derivatives can also be shown differentiating term by term the series in (9.3). The occurrence of higher order derivatives of u is avoided by means of (8.15). Once the hypothesis of theorem 7.2 have been verified, theorem 9.1 follows from that theorem and theorem 8.1.

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