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Riemann Representation Method in Viscoelasticity

I. Characterization and Construction of the Riemann Function. Solution of Problems with Prescribed Body Forces

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1. Introduction

Recent work on the theory of materials with memory [*1*–*5*] concerned with the study of singular surfaces in such materials, has very much enlightened the basic structure of the governing equations of motion. It has shown that under appropriate smoothness hypotheses, they behave in many respects like hyperbolic partial differential equations. It is therefore desirable to study the dynamics of materials with memory by methods similar to the methods used in the study of hyperbolic partial differential equations.

However, a systematic theory of functional partial differential equations has not been developed so far. A first step in this direction has been given recently by the author [*6*]. However, the paper is concerned mainly with questions of uniqueness and existence, and many other aspects of the theory of hyperbolic differential equations remain to be extended to functional partial differential equations of hyperbolic type.

It is well known [*7*] that the Riemann representation of solutions is extremely helpful in understanding the basic structure of solutions of linear hyperbolic differential equations. As a matter of fact, this elegant method has the advantage of exhibiting what may be called the microscopic structure of the solutions which gives insight not only to the study of the linear problems, but to the non-linear problems as well.

This is the first of a series of papers whose purpose is to extend Riemann's representation method to functional partial differential equations of hyperbolic type. However, to be more specific, Riemann's representation will be formulated only for the equations of motion of linear viscoelasticity, although similar ideas can be applied to functional partial differential equations in general, and it seems desirable to develop such a general theory.

In this paper a Riemann representation of solutions for two dimensional problems (position and time) is formulated when the body forces are known in the whole space. The Riemann function is characterized in terms of its values on two characteristic lines and some jump discontinuities for its spatial derivatives across the time line passing through the singular point. An iteration scheme to construct the Riemann function is formulated. This scheme is shown to be convergent, and by means of it the existence of the Riemann's function is proved. Uniqueness of solution when the body forces are prescribed is also shown.

It is well known [7] that Riemann's function for a single hyperbolic equation in two independent variables is characterized by its values on two characteristics. Thus, Riemann's function for the equations of dynamic viscoelasticity exhibit a new feature. This is a line of discontinuity for the first derivatives along the line of "memory transmission". This fits quite well with the known facts about Riemann's functions. The processes controlling the phenomena which the partial differential equations describe are manifested in the Riemann function through singular lines or surfaces. For linear elasticity the only basic process is that of wave transmission which is manifested by means of discontinuities along characteristic lines or surfaces. For one-dimensional motions in linear viscoelasticity, in addition to wave transmission, we have memory transmission; thus, in addition to the discontinuities along two characteristic lines manifesting the process of wave transmission, we have discontinuities across the line of memory transmission.

The existence of the Riemann function is shown under the hypotheses that the coefficients of the equations of motion are C^3. It seems that to assure existence it is enough to assume them to be C^2. Apparently, however, the methods used in this paper are not appropriate for obtaining such a refinement, and methods similar to those used by FRIEDRICHS [8] should be used. However, the importance of that refinement in the theory of hyperbolic differential equations lies in the fact that, by means of it, existence can be shown in the non-linear case [8]. Nevertheless for the type of equations discussed here, existence in the non-linear case can be shown by a direct procedure based on the results already known for linear partial differential equations, and this has already been done [6]. Since methods of the same type are used throughout the paper, it has not to convenient seemed break the homogeneity for the purpose of achieving the refinement mentioned above.

For partial differential equations there are reciprocal theorems which are closely connected with Riemann's representation method. The same is true for the theory developed in this paper. A reciprocal theorem of this type is presented in section 8. It is obviously related to a reciprocal theorem given by VOLTERRA in 1909 [11].

The obvious applicability of the results for the formulation of numerical methods must also be stressed because the integral equation obtained in section 4 is well suited to be treated numerically.

2. Notation

We shall be concerned in this paper with the dynamic equations of linear viscoelasticity for one-dimensional motions. If $u(x, t) \in C^2$ is the displacement field, they are

$$\frac{\partial}{\partial x}\left\{E(t)\frac{\partial u}{\partial x}(t)\right\}+\frac{\partial}{\partial x}\int_{-\infty}^{t} G(t,\tau)\frac{\partial u}{\partial x}(\tau)\,d\tau-\rho\frac{\partial^2 u}{\partial t^2}(t)=f(x,t) \tag{2.1}$$

where

$E(x, t)>0$ is the "initial value" of the relaxation tensor, defined for every $(x, t)\in R_2$.

$G(x, t, \tau)$ is the relaxation tensor defined for every $(x, t)\in R_2$, $\tau\leqq t$. It is convenient to extend the definition of $G(t, \tau)$ to be zero when $\tau>t$.

$\rho(x)>0$ is the density, defined on R_1.

$f(x, t)$ are the body forces defined on R_2.

The speeds of propagation of acceleration waves [1, 2] $c(x, t)$ at every point satisfy

$$c^2=\frac{E}{\rho}. \tag{2.2}$$

We systematically assume that

$$G(x,t,\tau)\in C^2 \quad \text{when } (x,t)\in R_2,\ \tau\leqq t,$$

$$E(x,t)\in C^2 \quad \text{on } R_2,$$

$$\rho(x)\in C^2 \quad \text{on } R_1.$$

With every point (x_0, t_0) we associate two lines

$$t=t_a(x_0, t_0, x), \tag{2.3.a}$$

$$t=t_r(x_0, t_0, x) \tag{2.3.b}$$

such that

$$\frac{dt_a}{dx}=c_+^{-1}(x,t), \tag{2.4.a}$$

$$\frac{dt_r}{dx}=c_-^{-1}(x,t), \tag{2.4.b}$$

$$t_a(x_0, t_0, x_0)=t_r(x_0, t_0, x_0)=t_0 \tag{2.4.c}$$

where c_+ and c_- are the positive and negative roots of (2.2) respectively.

For one-dimensional motions governed by equations (2.1) it is possible to show uniqueness theorems analogous to the uniqueness theorems established for the three-dimensional dynamic theory [9]. This justifies defining the domain of dependence $D_-(x_0, t_0)$ of the point (x_0, t_0) as the closed region given by

$$D_-(x_0, t_0)=\{(x,t)\mid t\leqq t_r(x_0, t_0, x),\ t\leqq t_a(x_0, t_0, x)\} \tag{2.5.a}$$

and the domain of influence $D_+(x_0, t_0)$ of (x_0, t_0) by

$$D_+(x_0, t_0)=\{(x,t)\mid t\geqq t_r(x_0, t_0, x),\ t\geqq t_a(x_0, t_0, x)\}. \tag{2.5.b}$$

Observe that c_+ and c_- are positive and negative definite, respectively, and therefore t_a and t_r are monotonically increasing and monotonically decreasing functions of x, respectively. Thus, the only point of intersection of t_a and t_r is (x_0, t_0). The boundary of $D_+(x_0, t_0)$ will be denoted by $S_+(x_0, t_0)$. Its equation is

$$t = t_+(x_0, t_0, x) \tag{2.6.a}$$

where

$$t_+(x_0, t_0, x) = \begin{cases} t_r(x_0, t_0, x), & \text{if } x \leqq x_0 \\ t_a(x_0, t_0, x), & \text{if } x \geqq x_0 . \end{cases} \tag{2.6.b}$$

Similarly, the boundary of $D_-(x_0, t_0)$ is $S_-(x_0, t_0)$ whose equation is

$$t = t_-(x_0, t_0, x) \tag{2.7.a}$$

where

$$t_-(x_0, t_0, x) = \begin{cases} t_a(x_0, t_0, x), & \text{if } x \leqq x_0 \\ t_r(x_0, t_0, x), & \text{if } x \geqq x_0 . \end{cases} \tag{2.7.b}$$

At every point of S_+ (except at x_0, t_0) we define the unit normal vector $\boldsymbol{n} = (n_x, n_t)$, outward to D_+. Similarly $\boldsymbol{n}$ on S_- is taken outward to D_-. We associate with $\boldsymbol{n}$ the unit tangent vector $(n_t, -n_x)$ which is obtained rotating $\boldsymbol{n}$, 90° clockwise. Observe that

$$n_t = -c\, n_x, \tag{2.8.a}$$

where we assume c defined by

$$c = \begin{cases} c_+ & \text{on } t_a \\ c_- & \text{on } t_r . \end{cases} \tag{2.8.b}$$

The adjoint equation of (2.1) is defined by

$$\frac{\partial}{\partial x}\left(E \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x} \int_t^\infty G(\tau, t) \frac{\partial v}{\partial x}(\tau)\, d\tau - \rho \frac{\partial^2 v}{\partial t^2} = f^* . \tag{2.9}$$

It is convenient to decompose D_+ and D_- into two sub-regions. Given (x_0, t_0), we define

$$D_{l\pm}(x_0, t_0) = \{(x, t) \mid (x, t) \in D_\pm(x_0, t_0),\ x \leqq x_0\}, \tag{2.10.a}$$

$$D_{r\pm}(x_0, t_0) = \{(x, t) \mid (x, t) \in D_\pm(x_0, t_0),\ x \geqq x_0\}. \tag{2.10.b}$$

Also

$$d_\pm(x_0, t_0) = D_{l\pm}(x_0, t_0) \cap D_{r\pm}(x_0, t_0) = \begin{cases} \{(x_0, t) \mid t \geqq t_0\} \\ \{(x_0, t) \mid t \leqq t_0\} . \end{cases} \tag{2.11}$$

Sometimes it will be convenient to consider the vector

$$\boldsymbol{X} = (x, t) . \tag{2.12}$$

With this notation

$$u(x,t)=u(X).$$

In the x, t-plane we shall consider line and surface integrals. The element of integration in both cases will be denoted by dX and the character of the integral will be specified by a subindex under the sign of integration.

When we consider a piecewise continuous function and restrict attention to some closed domain in the interior of which the function is continuous, it is possible to define a new function which is continuous in the entire domain and differs from the previous one, at most, on the boundary. In many instances it is possible to replace the former by the latter without altering the validity of the relations considered. Therefore we frequently shall do so, although we shall not use a new symbol for the new function.

When dealing with functions of several variables, we write explicitly only those arguments whose values in the expressions considered are not obvious.

We shall restrict attention mainly to solutions $u(x, t)$ of (2.1) for which there is a number T, such that

$$u(x,t)=0, \qquad \text{whenever } t\leqq T, \tag{2.13}$$

and we shall say that functions satisfying (2.13) possess "finite history". Functions which vanish outside a bounded domain will be called of "bounded support".

The notation R_n for the n-dimensional Euclidean space will be used throughout the paper.

3. Characterization of the Riemann's Function

Given a point (x_0, t_0), we say that the function $Q(x, t, x_0, t_0)$ is a quasi-Riemann function of (2.1) with singularity at (x_0, t_0) if as a function of (x, t), it is continuous on $D_-(x_0, t_0)$ and possesses piecewise continuous first and second derivatives on $D_-(x_0, t_0)$, whose only line of discontinuity is $d_-(x_0, t_0)$. In addition

$$Q(x,t,x_0,t_0)=0 \qquad \text{for every } (x,t)\notin D_-(x_0,t_0). \tag{3.1}$$

On $S_-(x_0, t_0)$

$$Q(x,t,x_0,t_0)=A\{\rho(x)\,c_+(x,t)\}^{-\frac{1}{2}}\, e^{-\int_{t_0}^{t}\frac{G(x',t',t')}{2E(x',t')}dt'} \tag{3.2.a}$$

where

$$A=-\tfrac{1}{2}\{\rho(x_0)\,c_+(x_0,t_0)\}^{-\frac{1}{2}}. \tag{3.2.b}$$

On the line $d_-(x_0, t_0)$, it is required that

$$E(x_0,t)\left[\frac{\partial Q}{\partial x}(x_0,t,x_0,t_0)\right]=-\frac{G(x_0,t_0,t)}{E(x_0,t_0)}-\int_{t}^{t_0} G(x_0,\tau,t)\left[\frac{\partial Q}{\partial x}(x_0,\tau,x_0,t_0)\right]d\tau, \tag{3.3}$$

where the square brackets stand for the jump discontinuity (the value on the right minus the value on the left).

It is convenient to associate with Q the function $g(x, t, x_0, t_0)$ defined by

$$\frac{\partial}{\partial x}\left(E\frac{\partial Q}{\partial x}\right)+\frac{\partial}{\partial x}\int_t^{t_-} G(\tau,t)\frac{\partial Q}{\partial x}(\tau)\,d\tau+\frac{\partial}{\partial x}\left\{\frac{G(t_-(x),t)\,Q(t_-(x))}{c(t_-(x))}\right\}- \\ -\rho\frac{\partial^2 Q}{\partial t^2}=g(x,t,x_0,t_0), \qquad (x,t)\notin D_-(x_0,t_0), \tag{3.4.a}$$

$$g(x,t,x_0,t_0)=0, \qquad (x,t)\notin D_-(x_0,t_0). \tag{3.4.b}$$

From the definition of Q and the assumptions about E, G and ρ stated in section 1, it follows that $g(x, t, x_0, t_0)$ as a function of (x, t) is piecewise continuous on $D_-(x_0, t_0)$ and $d_-(x_0, t_0)$ is the only line across which g may have jump discontinuities.

When

$$g(x,t,x_0,t_0)=0 \qquad \text{for every } (x,t)\in D_-(x_0,t_0), \tag{3.5}$$

we say that Q is a Riemann function of (2.1) with singularity at (x_0, t_0). Such a function will be denoted by $R(x, t, x_0, t_0)$.

Similarly, we say that $Q^*(x, t, x_1, t_1)$ is a quasi-Riemann function of the adjoint equation (2.9) with singularity at (x_1, t_1), if, as a function of (x, t), it is continuous on $D_+(x_1, t_1)$ and possesses piecewise continuous first and second derivatives on $D_+(x_1, t_1)$, whose only possible line of discontinuity is $d_+(x_1, t_1)$. In addition,

$$Q^*(x,t,x_1,t_1)=0 \qquad \text{for every } (x,t)\notin D_+(x_1,t_1). \tag{3.6}$$

On $S_+(x_1, t_1)$

$$Q^*(x,t,x_1,t_1)=A^*\{\rho(x)\,c_+(x,t)\}^{-\frac{1}{2}}\,e^{\int_{t_1}^{t}\frac{G(x',t',t')}{2E(x',t')}dt'} \tag{3.7.a}$$

where

$$A^*=-\tfrac{1}{2}\{\rho(x_1)\,c_+(x_1,t_1)\}^{-\frac{1}{2}}. \tag{3.7.b}$$

On the line of discontinuity $d_+(x_1, t_1)$, it is required that

$$E(x_1,t)\left[\frac{\partial Q^*}{\partial x}(t)\right]=-\frac{G(t,t_1)}{E(x_1,t_1)}-\int_{t_1}^{t} G(t_1,\tau)\left[\frac{\partial Q^*}{\partial x}(\tau)\right]d\tau. \tag{3.8}$$

We associate with Q^* the function g^* defined by

$$\frac{\partial}{\partial x}\left(E\frac{\partial Q^*}{\partial x}\right)+\frac{\partial}{\partial x}\int_{t_+}^{t} G(t,\tau)\frac{\partial Q^*}{\partial x}(\tau)\,d\tau-\frac{\partial}{\partial x}\left\{\frac{G(t,t_+)\,Q^*(t_+)}{c(t_+)}\right\}- \\ -\rho\frac{\partial^2 Q^*}{\partial t^2}=g^*(x,t,x_1,t_1), \qquad (x,t)\in D_+(x_1,t_1), \tag{3.9.a}$$

$$g^*(x,t,x_1,t_1)=0, \qquad (x,t)\notin D_+(x_1,t_1). \tag{3.9.b}$$

Again, g^* is a piecewise continuous function of (x, t) on $D_+(x_1, t_1)$ whose only line of discontinuity is $d_+(x_1, t_1)$.

When

$$g^*(x, t, x_1, t_1) = 0 \qquad \text{for every } (x, t) \in D_+(x_1, t_1), \tag{3.10}$$

we say that Q is a Riemann function of the adjoint equation (2.9) with singularity at (x_1, t_1). This Riemann function will be denoted by $R^*(x, t, x_1, t_1)$.

With this notation, we have the following

Lemma 3.1. *Let $u(x, t) \in C^2$ on R_2 possess a finite history. Define $f(x, t)$ by means of* (2.1) *on R_2. Then if $Q(x, t, x_0, t_0)$ is a quasi-Riemann function of* (2.1) *with singularity at (x_0, t_0), we have*

$$u(x_0, t_0) = \int\limits_{D_-(x_0, t_0)} f(X)\, Q(X, x_0, t_0)\, dX - \int\limits_{D_-(x_0, t_0)} g(X, x_0, t_0)\, u(X)\, dX. \tag{3.11}$$

Proof. Observe first that $f(x, t) \in C$ necessarily because it is a linear combination of continuous functions.

Now define

$$\begin{aligned} I = \int\limits_{D_-(x_0, t_0)} \Bigg\{ & Q(X) \frac{\partial}{\partial x} \int\limits_{-\infty}^{t} G(t, \tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u(X) \frac{\partial}{\partial x} \int\limits_{t}^{t_-} G(\tau, t) \frac{\partial Q}{\partial x}(\tau)\, d\tau - \\ & - u(X) \frac{\partial}{\partial x} \left(\frac{G(t_-, t)\, Q(t_-)}{c(t_-)} \right) \Bigg\}\, dX, \end{aligned} \tag{3.12}$$

which by using the divergence theorem can be written

$$\begin{aligned} I = & \int\limits_{S_-(x_0, t_0)} \left\{ Q(X) \int\limits_{-\infty}^{t_-} G(t_-, \tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u(X) \int\limits_{t}^{t_-} G(\tau, t_-) \frac{\partial Q}{\partial x}(\tau)\, d\tau \right\} n_x\, dX - \\ & - \int\limits_{D_-(x_0, t_0)} u(X) \frac{\partial}{\partial x} \left\{ \frac{G(t_-, t)\, Q(t_-)}{c(t_-)} \right\} dX + \\ & + \int\limits_{-\infty}^{t_0} \left\{ u(x_0, t) \int\limits_{t}^{t_0} G(x_0, \tau, t) \left[\frac{\partial Q}{\partial x}(x_0, \tau) \right] d\tau \right\} dt. \end{aligned} \tag{3.13}$$

The fact that

$$\begin{aligned} & \int\limits_{D_-} \left\{ \frac{\partial Q}{\partial x}(X) \int\limits_{-\infty}^{t} G(t, \tau) \frac{\partial u}{\partial x}(x, \tau)\, d\tau \right\} dX \\ & = \int\limits_{S_-} \left\{ \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{\tau'} G(\tau', \tau) \frac{\partial Q}{\partial x}(\tau') \frac{\partial u}{\partial x}(\tau)\, d\tau\, d\tau' \right\} n_t\, dX \\ & = \int\limits_{D_-} \left\{ \frac{\partial u}{\partial x}(X) \int\limits_{t}^{t_-} G(\tau, t) \frac{\partial Q}{\partial x}(\tau)\, d\tau \right\} dX \end{aligned}$$

was used to obtain (3.13). Observe that in (3.13) we have

$$u(X)\int_{t_-}^{t_-} G(\tau, t_-)\,\frac{\partial Q}{\partial x}(\tau)\,d\tau=0$$

and

$$\begin{aligned}\int_{S_-}&\left\{Q(X)\int_{-\infty}^{t_-} G(t_-,\tau)\,\frac{\partial u}{\partial x}(\tau)\,d\tau\right\}n_x\,dX\\ &=-\int_{S_-}\left\{\int_{-\infty}^{t_-}\frac{Q(x,t_-)\,G(t_-,\tau)}{c(t_-)}\,\frac{\partial u}{\partial x}(\tau)\,d\tau\right\}n_t\,dX\\ &=-\int_{D_-}\frac{Q(x,t_-)\,G(t_-,t)}{c(t_-)}\,\frac{\partial u}{\partial x}(X)\,dX,\end{aligned}$$

where (2.8.a) and the divergence theorem have been used. Therefore from (3.13) it follows that

$$\begin{aligned}I=&\int_{-\infty}^{t_0}\left\{u(x_0,t)\int_t^{t_0} G(\tau,t)\left[\frac{\partial Q}{\partial x}(x_0,\tau)\right]d\tau\right\}dt-\\ &-\int_{D_-}\frac{\partial}{\partial x}\left\{\frac{u(X)\,G(x,t_-,t)\,Q(x,t_-)}{c(t_-)}\right\}dX\\ =&-\int_{-\infty}^{t_0}u(t)\left\{2\,\frac{G(t_0,t)\,Q(x_0,t_0)}{c_+(t_0)}-\int_t^{t_0}G(\tau,t)\left[\frac{\partial Q}{\partial x}(\tau)\right]d\tau\right\}dt-\\ &-\int_{S_-}\frac{u(X)\,G(t_-,t_-)\,Q(X)\,n_x}{c(X)}\,dX.\end{aligned}\tag{3.14}$$

On the other hand, using the divergence theorem, we have

$$\begin{aligned}&\int_{D_-(x_0,t_0)}\left\{Q\,\frac{\partial}{\partial x}\left(E\,\frac{\partial u}{\partial x}\right)-u\,\frac{\partial}{\partial x}\left(E\,\frac{\partial Q}{\partial x}\right)-\rho\left(Q\,\frac{\partial^2 u}{\partial t^2}-u\,\frac{\partial^2 Q}{\partial t^2}\right)\right\}dX\\ &=\int_{D_-(x_0,t_0)}\left\{\frac{\partial}{\partial x}\left(Q\,E\,\frac{\partial u}{\partial x}-u\,E\,\frac{\partial Q}{\partial x}\right)-\frac{\partial}{\partial t}\left(\rho\,Q\frac{\partial u}{\partial t}-\rho\,u\,\frac{\partial Q}{\partial t}\right)\right\}dX\\ &=\int_{S_-}\left\{\left(Q\,E\,\frac{\partial u}{\partial x}-u\,E\,\frac{\partial Q}{\partial x}\right)n_x-\rho\left(Q\,\frac{\partial u}{\partial t}-u\,\frac{\partial Q}{\partial t}\right)n_t\right\}dX+\\ &+\int_{-\infty}^{t_0}u(\tau)\,E(\tau)\left[\frac{\partial Q}{\partial x}(\tau)\right]d\tau=-\int_{S_-}\rho\left(Q\,c\,\frac{du}{ds}-u\,c\,\frac{dQ}{ds}\right)dX+\\ &+\int_{-\infty}^{t_0}u(\tau)\,E(\tau)\left[\frac{\partial Q}{\partial x}(x_0,\tau)\right]d\tau,\end{aligned}\tag{3.15}$$

where

$$\frac{d}{ds}=n_t\,\frac{\partial}{\partial x}-n_x\,\frac{\partial}{\partial t}.\tag{3.16}$$

Equations (3.12), (3.14) and (3.15) together imply that if (2.1) is multiplied by Q, (3.4) by u, and the resulting equations subtracted from each other, an equation is obtained which, after integration over $D_-(x_0, t_0)$, yields

$$\begin{aligned}
&\int_{D_-(x_0,t_0)} f(X)\,Q(X, x_0, t_0)\,dX - \int_{D_-(x_0,t_0)} g(X, x_0, t_0)\,u(X)\,dX \\
&= \int_{S_-} \left\{\rho c u \frac{dQ}{ds} - \frac{u(X)\,G(t_-, t_-)\,Q(X)}{c(X)}\, n_x - \rho c Q \frac{du}{ds}\right\} dX + \\
&+ \int_{-\infty}^{t_0} u(t) \left\{E(t)\left[\frac{\partial Q}{\partial x}(x_0, t)\right] - 2\,\frac{G(t_0, t)\,Q(t_0)}{c_+(t_0)} + \right. \\
&\left. + \int_t^{t_0} G(\tau, t)\left[\frac{\partial Q}{\partial x}(x_0, \tau)\right] d\tau\right\} dt \\
&= -2(\rho c_+ Q u)_{x_0, t_0} - \int_{S_-} \left\{2\rho c \frac{dQ}{dt} + \left(\frac{d\rho c}{dt} + \frac{G(t_-, t_-)}{c}\right) Q\right\} u\, n_x\, dX + \\
&+ \int_{-\infty}^{t_0} u(t) \left\{E(t)\left[\frac{\partial Q}{\partial x}(x_0, t)\right] - 2\,\frac{G(t_0, t)\,Q(t_0)}{c_+(t_0)} + \right. \\
&\left. + \int_t^{t_0} G(\tau, t)\left[\frac{\partial Q}{\partial x}(x_0, \tau)\right] d\tau\right\} dt,
\end{aligned} \tag{3.17}$$

where we have used (2.8.a) and have defined

$$\frac{d}{dt} = -\frac{1}{n_x}\frac{d}{ds}. \tag{3.18}$$

Now from (3.2) it follows that

$$-2(\rho c_+ Q u)_{x_0, t_0} = u(x_0, t_0) \tag{3.19.a}$$

and

$$2\rho c \frac{dQ}{dt} + \left(\frac{d\rho c}{dt} + \frac{G(t_-, t_-)}{c}\right) Q = 0 \qquad \text{on } S_-(x_0, t_0). \tag{3.19.b}$$

Hence, (3.11) follows from (3.2), (3.3), (3.17) and (3.19).

Lemma 3.1 implies the

Integral Representation Theorem 3.1. *Let $Q(x, t, x_0, t_0)$ be a quasi-Riemann function of* (2.1) *with singularity at (x_0, t_0). Associate with every $u(x, t) \in C^2$ on R_2 of finite history the function $f(x, t)$ by means of* (2.1). *Then $Q(x, t, x_0, t_0)$ is a Riemann function if and only if we have*

$$u(x_0, t_0) = \int_{D_-(x_0,t_0)} f(X)\,Q(X, x_0, t_0)\,dX \tag{3.20}$$

for every $u \in C^2$ of finite history.

Proof. Equations (3.11) and (3.20) hold simultaneously if and only if

$$\int_{D_-(x_0,t_0)} g(X, x_0, t_0)\,u(X)\,dX = 0$$

for every u of finite history; thus, if and only if

$$g(X, x_0, t_0) = 0$$

for every X.

Corresponding to the lemma and theorem above there are dual results. They are

Lemma 3.2. *Let* $v(x, t) \in C^2$ *on* R_2, *of bounded support, satisfy the adjoint equation* (2.9). *Then if* $Q^*(x, t, x_1, t_1)$ *is a quasi-Riemann function of* (2.9) *with singularity at* (x_1, t_1), *we have*

$$v(x_1, t_1) = \int_{D_+(x_1, t_1)} f^*(X)\, Q^*(X, x_1, t_1)\, dX - \int_{D_+(x_1, t_1)} g^*(X, x_1, t_1)\, v(X)\, dX. \quad (3.21)$$

Theorem 3.2. *Let* $Q^*(x, t, x_1, t_1)$ *be a quasi-Riemann function of the adjoint equation* (2.9). *Associate with every* $v(x, t) \in C^2$ *on* R_2 *of bounded support the function* $f^*(x, t)$, *by means of* (2.9). *Then* $Q^*(x, t, x_1, t_1)$ *is a Riemann function of* (2.9) *if and only if we have*

$$v(x_1, t_1) = \int_{D_+(x_1, t_1)} f^*(X)\, Q^*(X, x_1, t_1)\, dX \quad (3.22)$$

for every $v \in C^2$ *of bounded support.*

Proof. The proof of these results can be constructed if we interchange the roles of equation (2.1), Q, R with those of the adjoint equation (2.9), Q^*, R^*, respectively, in the proofs of Lemma 3.1 and the Integral Representation Theorem.

4. An Integral Equation Satisfied by the Riemann Function

The construction of the Riemann function can be reduced to solving an integral equation. It is the purpose of this section to establish such an integral equation. It may be deduced from the following

Theorem 4.1. *Let* $Q(x, t, x_0, t_0)$ *and* $Q^*(x, t, x_1, t_1)$ *be a quasi-Riemann function of* (2.1) *and* (2.9) *respectively. Then*

$$\begin{aligned} Q^*(x_0, t_0, x_1, t_1) &+ \int_{D(x_1, t_1, x_0, t_0)} g(X, x_0, t_0)\, Q^*(X, x_1, t_1)\, dX \\ &= Q(x_1, t_1, x_0, t_0) + \int_{D(x_1, t_1, x_0, t_0)} g^*(X, x_1, t_1)\, Q(X, x_0, t_0)\, dX \end{aligned} \quad (4.1)$$

where

$$D(x_1, t_1, x_0, t_0) = D_+(x_1, t_1) \cap D_-(x_0, t_0), \quad (4.2)$$

where g *and* g^* *are given by* (3.4) *and* (3.9) *respectively.*

Proof. When D is void, equation (4.1) is obvious because D is void if and only if $(x_1, t_1) \notin D_-(x_0, t_0)$ and $(x_0, t_0) \notin D_+(x_1, t_1)$. In such a case

$$Q^*(x_0, t_0, x_1, t_1) = Q(x_1, t_1, x_0, t_0) = 0.$$

If (x_1, t_1) lies on the boundary of $D_-(x_0, t_0)$, then (x_0, t_0) also lies on the boundary of $D_+(x_1, t_1)$. In this case the integrals appearing in (4.1) vanish, and (4.1) follows from (3.2) and (3.7).

Thus we restrict attention to the case when $D(x_1, t_1, x_0, t_0)$ has a non-vanishing area. Let S be the boundary of D, and define

$$S'_+ = D \cap S_+(x_1, t_1), \quad (4.3.a)$$

$$S'_- = D \cap S_-(x_0, t_0). \quad (4.3.b)$$

Then

$$S = S'_+ \cup S'_- . \tag{4.3.c}$$

Also, to save space, d_+ will stand for $d_+(x_1, t_1)$ and d_- for $d_-(x_0, t_0)$. Similar conventions will be used for $t_+(x_1, t_1, x)$ and $t_-(x_0, t_0, x)$.

Observe that by virtue of divergence theorem, denoting the following integral by I, we have

$$\begin{aligned}
I = & \int_D \left\{ Q(X) \frac{\partial}{\partial x} \int_{t_+}^{t} G(X,\tau) \frac{\partial Q^*}{\partial x}(\tau)\, d\tau - Q^*(X) \frac{\partial}{\partial x} \int_{t}^{t_-} G(X,t) \frac{\partial Q}{\partial x}(\tau)\, d\tau - \right.\\
& \left. - Q(X) \frac{\partial}{\partial x}\left(\frac{G(t,t_+)\, Q^*(t_+)}{c(t_+)}\right) - Q^*(X) \frac{\partial}{\partial x}\left(\frac{G(t_-,t)\, Q(t_-)}{c(t_-)}\right)\right\} dX \\
= & \int_S \left\{ Q(X) \int_{t_+}^{t} G(t,\tau) \frac{\partial Q^*}{\partial x}(\tau)\, d\tau - Q^*(X) \int_{t}^{t_-} G(\tau,t) \frac{\partial Q}{\partial x}(\tau)\, d\tau \right\} n_x\, dX - \\
& - \int_D \left\{ Q(X) \frac{\partial}{\partial x}\left(\frac{G(t,t_+)\, Q^*(t_+)}{c(t_+)}\right) + Q^*(X) \frac{\partial}{\partial x}\left(\frac{G(t_-,t)\, Q(t_-)}{c(t_-)}\right)\right\} dX + \\
& + \int_{t_+}^{t_0} \left\{ Q^*(x,t) \int_{t}^{t_0} G(x_0,\tau,t) \left[\frac{\partial Q}{\partial x}(x_0,\tau)\right] d\tau \right\} dt - \\
& - \int_{t_1}^{t_-} \left\{ Q(x_1,t) \int_{t_1}^{t} G(x_1,t,\tau) \left[\frac{\partial Q^*}{\partial x}(x_1,\tau)\right] d\tau \right\} dt .
\end{aligned} \tag{4.4}$$

On the other hand, using (2.8.a) and the divergence theorem, we can write

$$\begin{aligned}
\int_S \left\{ Q(X) \int_{t_+}^{t} G(t,\tau) \frac{\partial Q^*}{\partial x}(\tau)\, d\tau \right\} n_x\, dX &= - \int_S \left\{ \frac{Q(t_-)}{c(t_-)} \int_{t_+}^{t} G(t_-,\tau) \frac{\partial Q^*}{\partial x}(\tau)\, d\tau \right\} n_t\, dX \\
&= - \int_D \frac{Q(t_-)\, G(t_-,t)}{c(t_-)} \frac{\partial Q^*}{\partial x}(t)\, dX
\end{aligned}$$

and

$$\begin{aligned}
\int_S \left\{ Q^*(X) \int_{t}^{t_-} G(\tau,t) \frac{\partial Q}{\partial x}(\tau)\, d\tau \right\} n_x\, dX &= - \int_S \left\{ \frac{Q^*(t_+)}{c(t_+)} \int_{t}^{t_-} G(\tau,t_+) \frac{\partial Q}{\partial x}(\tau)\, d\tau \right\} n_t\, dX \\
&= \int_D \left\{ \frac{Q(t_+)\, G(t,t_+)}{c(t_+)} \frac{\partial Q}{\partial x}(t) \right\} dX .
\end{aligned}$$

Again using the divergence theorem, we have

$$\begin{aligned}
I = & \int_{t_+}^{t_0} \left\{ Q^*(x_0,t) \int_{t}^{t_0} G(x_0,\tau,t) \left[\frac{\partial Q}{\partial x}(x_0,\tau)\right] d\tau \right\} dt - \\
& - \int_{t_1}^{t_-} \left\{ Q(x_1,t) \int_{t_1}^{t} G(x_1,t,\tau) \left[\frac{\partial Q^*}{\partial x}(x_1,\tau)\right] d\tau \right\} dt - \\
& - \int_S \left\{ \frac{Q(t)\, G(t,t_+)\, Q^*(t_+)}{c(t_+)} + \frac{Q^*(t)\, G(t_-,t)\, Q(t_-)}{c(t_-)} \right\} n_x\, dX + \\
& + 2 \int_{t_1}^{t_-} \frac{Q(x_1,t)\, G(x_1,t,t_1)\, Q^*(x_1,t_1)}{c_+(x_1,t_1)}\, dt - 2 \int_{t_+}^{t_0} \frac{Q^*(x_0,t)\, G(x_0,t_0,t)\, Q(x_0,t_0)}{c_+(x_0,t_0)}\, dt .
\end{aligned} \tag{4.5}$$

Now, use (2.8.a) to get

$$\begin{aligned}
&-\int_S \left\{\frac{Q(t)\,G(t,t_+)\,Q^*(t_+)}{c(t_+)}+\frac{Q^*(t)\,G(t_-,t)\,Q(t_-)}{c(t_-)}\right\} n_x\,d\boldsymbol{X}\\
&=\int_S \left\{\frac{Q(t)\,G(t,t_+)\,Q^*(t_+)}{c(t_+)\,c(t)}+\frac{Q^*(t)\,G(t_-,t)\,Q(t_-)}{c(t_-)\,c(t)}\right\} n_t\,d\boldsymbol{X}\\
&=\int_{S'_+} \left\{\frac{Q(t_+)\,G(t_+,t_+)\,Q^*(t_+)}{c(t_+)\,c(t_+)}+\frac{Q^*(t_+)\,G(t_-,t_+)\,Q(t_-)}{c(t_-)\,c(t_+)}\right\} n_t\,d\boldsymbol{X}+\\
&+\int_{S'_-} \left\{\frac{Q(t_-)\,G(t_-,t_+)\,Q^*(t_+)}{c(t_+)\,c(t_-)}+\frac{Q^*(t_-)\,G(t_-,t_-)\,Q(t_-)}{c(t_-)\,c(t_-)}\right\} n_t\,d\boldsymbol{X}\\
&=\int_S \left\{\frac{Q^*(t_+)\,G(t_-,t_+)\,Q(t_-)}{c(t_-)\,c(t_+)}+\frac{Q^*(t)\,G(t,t)\,Q(t)}{c^2(t)}\right\} n_t\,d\boldsymbol{X}\\
&=\int_S \frac{Q^*(t)\,G(t,t)\,Q(t)}{c^2(t)}\,n_t\,d\boldsymbol{X},
\end{aligned}\tag{4.6}$$

where we have used the fact that

$$\int_S \frac{Q^*(t_+)\,G(t_-,t_+)\,Q(t_-)}{c(t_-)\,c(t_+)}\,n_t\,d\boldsymbol{X}=0,$$

implied by divergence theorem because the integrand is a function for x only. Equations (4.5) and (4.6) together imply

$$\begin{aligned}
I=&-\int_S \frac{Q^*(t)\,G(t,t)\,Q(t)}{c(t)}\,n_x\,d\boldsymbol{X}+\int_{t_1}^{t_-} Q(x_1,t)\,\times\\
&\times\left\{\frac{2G(x_1,t,t_1)\,Q^*(x_1,t_1)}{c_+(x_1,t_1)}-\int_{t_1}^{t} G(x_1,t,\tau)\left[\frac{\partial Q^*}{\partial x}(x_1,\tau)\right]d\tau\right\}dt-\\
&-\int_{t_+}^{t_0} Q^*(x_0,t)\left\{\frac{2G(x_0,t_0,t)\,Q(x_0,t_0)}{c_+(x_0,t_0)}-\int_{t}^{t_0} G(x_0,\tau,t)\left[\frac{\partial Q}{\partial x}(x_0,\tau)\right]d\tau\right\}dt
\end{aligned}\tag{4.7}$$

where we have used (2.8.a). Finally, write

$$\begin{aligned}
I'=&\int_D \left\{Q\frac{\partial}{\partial x}\left(E\frac{\partial Q^*}{\partial x}\right)-Q^*\frac{\partial}{\partial x}\left(E\frac{\partial Q}{\partial x}\right)-\rho\left(Q\frac{\partial^2 Q^*}{\partial t^2}-Q^*\frac{\partial^2 Q}{\partial t^2}\right)\right\}d\boldsymbol{X}\\
=&\int_D \left\{\frac{\partial}{\partial x}\left(Q\,E\frac{\partial Q^*}{\partial x}-Q^*\,E\frac{\partial Q}{\partial x}\right)-\frac{\partial}{\partial t}\left(\rho\,Q\frac{\partial Q^*}{\partial t}-\rho\,Q^*\frac{\partial Q}{\partial t}\right)\right\}d\boldsymbol{X}\\
=&\int_S \left\{\left(Q\,E\frac{\partial Q^*}{\partial x}-Q^*\,E\frac{\partial Q}{\partial x}\right)n_x-\rho\left(Q\frac{\partial Q^*}{\partial t}-Q^*\frac{\partial Q}{\partial t}\right)n_t\right\}d\boldsymbol{X}+\\
&+\int_{t_+}^{t_0} Q^*(x_0,t)\,E(x_0,t)\left[\frac{\partial Q}{\partial x}(x_0,t)\right]dt-\int_{t_1}^{t_-} Q(x_1,t)\,E(x_1,t)\left[\frac{\partial Q^*}{\partial x}(x_1,t)\right]dt\\
=&-\int_S \rho\left(Q\,c\frac{dQ^*}{ds}-Q^*\,c\frac{dQ}{ds}\right)d\boldsymbol{X}+\int_{t_+}^{t_0} Q^*(x_0,t)\,E(x_0,t)\left[\frac{\partial Q}{\partial x}(x_0,t)\right]dt-\\
&-\int_{t_1}^{t_-} Q(x_1,t)\,E(x_1,t)\left[\frac{\partial Q^*}{\partial x}(x_1,t)\right]dt
\end{aligned}\tag{4.8}$$

where the convention (3.16) has been used. Adding (4.7) and (4.8) and integrating by parts, we get

$$\begin{aligned}I+I'=&\int_{t_1}^{t_-} Q(x_1,t)\left\{2\frac{G(t,t_1)Q^*(x_1,t_1)}{C_+(x_1,t_1)}-E(t)\left[\frac{\partial Q^*}{\partial x}(x_1,t)\right]-\right.\\&\left.-\int_{t_1}^{t} G(t,\tau)\left[\frac{\partial Q^*}{\partial x}(x_1,\tau)\right]d\tau\right\}dt-\\&-\int_{t_+}^{t_0} Q^*(x_0,t)\left\{\frac{2G(x_0,t_0,t)Q(x_0,t_0)}{c_+(x_0,t_0)}-E(t)\left[\frac{\partial Q}{\partial x}(x_0,t)\right]-\right.\\&\left.-\int_{t}^{t_0} G(x_0,\tau,t)\left[\frac{\partial Q}{\partial x}(x_0,\tau)\right]d\tau\right\}dt+2(\rho c_+ Q Q^*)_{(x_1,t_1)}-\\&-2(\rho c_+ Q Q^*)_{(x_0,t_0)}-\int_{S'_-}\left\{2\rho c\frac{dQ}{dt}+\left(\frac{d\rho c}{dt}+\frac{G(t,t)}{c(t)}\right)Q\right\}Q^* n_x\,dX+\\&+\int_{S'_+}\left\{2\rho c\frac{dQ^*}{dt}+\left(\frac{d\rho c}{dt}-\frac{G(t,t)}{c(t)}\right)Q^*\right\}Q\,n_x\,dX\\=&\,Q^*(x_0,t_0,x_1,t_1)-Q(x_1,t_1,x_0,t_0)\end{aligned}\tag{4.9}$$

where (3.16), (3.18), (3.2), (3.3), (3.7) and (3.8) have been used.

Therefore, from equations (3.4), (3.9), (4.4), (4.8) and (4.9), relation (4.1) follows. This completes the proof of the theorem.

Corollary 4.1. *Let* $R(x, t, x_0, t_0)$ *be a Riemann function of* (2.1), *and let* $Q^*(x, t, x_1, t_1)$ *be a quasi-Riemann function of* (2.9), *respectively. Then*

$$\begin{aligned}R(x_1,t_1,x_0,t_0)=&\,Q^*(x_0,t_0,x_1,t_1)-\\&-\int_{D(x_1,t_1,x_0,t_0)} g^*(X,x_1,t_1)R(X,x_0,t_0)\,dX.\end{aligned}\tag{4.10}$$

Proof. The proof follows from (4.1) if we set $g=0$. If we permit x_1, t_1 to vary, (4.10) may be interpreted as an integral equation satisfied by R which will be the basis for the proof of uniqueness and existence of the Riemann function.

Corollary 4.2. *If the Riemann functions for* (2.1) *and its adjoint exist, then*

$$R(x_1,t_1,x_0,t_0)=R^*(x_0,t_0,x_1,t_1).\tag{4.11}$$

Proof. Set

$$g^*=0$$

in (4.10).

5. Discussion of the Integral Equations Satisfied by the Jump Discontinuities

Let $K^*(x_1, t_1, t)$ $(t\geqq t_1)$ be a solution of the integral equation (3.8), *i.e.*

$$E(x_1,t)K^*(x_1,t_1,t)=-\frac{G(x_1,t,t_1)}{E(x_1,t_1)}-\int_{t_1}^{t} G(x_1,t,\tau)K^*(x_1,t_1,\tau)\,d\tau.\tag{5.1.a}$$

Similarly, let $K(x_0, t_0, t)$ $(t_0 \geqq t)$ be a solution of (3.3), *i.e.*

$$E(x_0, t) K(x_0, t_0, t) = -\frac{G(x_0, t_0, t)}{E(x_0, t_0)} - \int_t^{t_0} G(x_0, \tau, t) K(x_0, t_0, \tau)\, d\tau. \quad (5.1.b)$$

These are VOLTERRA's integral equations for K^* and K respectively. It is well known and not difficult to show that when the functions E and G are C^2 as functions of their arguments, the functions K^* and K are uniquely defined by (5.1), and they are also C^2 as functions of their arguments for $t \geqq t_1$ and $t \leqq t_0$, respectively.

If E and G are C^3 as functions of their arguments, then K and K^* are also C^3 as functions of (x, t, τ) in the range mentioned above.

Finally, we prove the following

Lemma 5.1. *For every x the functions K and K^* satisfy the relation*

$$K(x, t, t') = K^*(x, t', t); \qquad t \geqq t'. \quad (5.2)$$

Proof. We have

$$E(x, t') E(x, t) K(x, t, t') = -G(x, t, t') - E(x, t) \int_{t'}^{t} G(x, \tau, t') K(x, t, \tau)\, d\tau,$$

$$E(x, t') E(x, t) K^*(x, t', t) = -G(x, t, t') - E(x, t') \int_{t'}^{t} G(x, t, \tau) K^*(x, t', \tau)\, d\tau.$$

From these relations it follows that

$$K(x, t, t') = K^*(x, t', t)$$

if and only if

$$\int_{t'}^{t} \{E(x, t) G(x, \tau, t') K(x, t, \tau) - E(x, t') G(x, t, \tau) K^*(x, t', \tau)\}\, d\tau = 0. \quad (5.3)$$

Now write equations (5.1) in the form

$$E(x, t) E(x, \tau) K(x, t, \tau) = -G(x, t, \tau) - E(x, t) \int_{\tau}^{t} G(x, \tau', \tau) K(x, t, \tau')\, d\tau', \quad (5.4.a)$$

$$E(x, \tau) E(x, t') K^*(x, t', \tau) = -G(x, \tau, t') - E(x, t') \int_{t'}^{\tau} G(x, \tau, \tau') K^*(x, t', \tau')\, d\tau'. \quad (5.4.b)$$

Multiply (5.4.a) by $E(x, t') K^*(x, t', \tau)$ and (5.4.b) by $E(x, t) K(x, t, \tau)$, substract, and integrate the resulting equation with respect to τ from t' to t to get (5.3).

6. Some Remarks on the Construction of a Quasi-Riemann Function

It is convenient to define in R_4 the sets

$$\hat{D}_r = \{(X, X') \mid X \in D_{r+}(X')\}, \quad (6.1.a)$$

$$\hat{D}_l = \{(X, X') \mid X \in D_{l+}(X')\}, \quad (6.1.b)$$

$$\hat{d} = \{(x, t, x', t') \mid x = x', t \geqq t'\} \quad (6.1.c)$$

and to let $\hat{S}$ be the boundary of $\hat{D}_r \cup \hat{D}_l$, *i.e.*

$$\hat{S} = \{(X, X') \mid X \in S_+(X')\}. \quad (6.1.d)$$

Observe that $\hat{S}$, under the smoothness hypothesis of section 1 for E and ρ, is a piecewise C^2 three-dimensional variety for which the "edge" formed by the set $\{(X, X)\}$ is the only sub-variety of discontinuity. However, if E and ρ are assumed to be C^3, then C^2 may be replaced by C^3 in the above statement.

In this section we discuss the construction of a family of quasi-Riemann functions which will meet some special smoothness requirements.

We assume throughout the section that E, G and ρ are C^3 so that the boundary conditions (3.2) and (3.7) are continuous on $\hat{S}$ and C^3 on each of the two smooth components of $\hat{S}$ as functions of the four variables (x_1, t_1, x_0, t_0). The jump discontinuities, satisfied by the first derivatives with respect to x of the quasi-Riemann function, which are given by $K(x_1, t_1, t)$ and $K^*(x_0, t_0, t)$, are also C^3 on $\hat{d}$. Under these conditions, it is easy to show that it is possible to construct a family of quasi-Riemann functions $Q^*(x, t, x_1, t_1)$ of the adjoint equation which is continuous on $\hat{D}_r \cup \hat{D}_l$ and C^3 on $\hat{D}_r$ and on $\hat{D}_l$ separately.

Let $Q^*(x, t, x_1, t_1)$ be such a quasi-Riemann function; then the function

$$Q(x, t, x_0, t_0) = Q^*(x_0, t_0, x, t) \tag{6.2}$$

by virtue of (3.2), (3.7) and (5.2), is a quasi-Riemann function of (2.1), which obviously meets the same smoothness requirements as Q^*.

If we associate with $Q(x, t, x_0, t_0)$ the function $g(x, t, x_0, t_0)$ by means of (3.4) and with $Q^*(x, t, x_1, t_1)$ the function $g^*(x, t, x_1, t_1)$ by means of (3.9), it follows from (4.1) and (6.2) that

$$\begin{aligned} &\int_{D(x_1, t_1, x_0, t_0)} g(X, x_0, t_0)\, Q^*(X, x_1, t_1)\, dX \\ &\qquad = \int_{D(x_1, t_1, x_0, t_0)} g^*(X, x_1, t_1)\, Q(X, x_0, t_0)\, dX . \end{aligned} \tag{6.3}$$

It must be observed that g^* as well as g belong to C^1 on $\hat{D}_r$ and on $\hat{D}_l$ separately, by virtue of (3.4.a) and (3.9.a).

7. Construction and Uniqueness of the Riemann Function

In this section we establish the following

Existence and Uniqueness Theorem 7.1. *Let E, G, and $\rho \in C^3$. Then for every point $(x_0, t_0) \in R_2$, there is one and only one Riemann function of equation (2.1) with singularity at (x_0, t_0).*

Further, for every $(x, t) \in D_-(x_0, t_0)$, it is given by

$$\begin{aligned} R(x, t, x_0, t_0) &= Q^*(x_0, t_0, x, t) + \\ &+ \sum_{n=1}^{\infty} (-1)^n \underbrace{\int_{D_-} \cdots \int_{D_-}}_{n\text{-integrals}} g^*(X', x, t) \ldots g^*(X^n, X^{n-1})\, Q^*(x_0, t_0, X^n)\, dX' \ldots dX^n \\ &= Q(x, t, x_0, t_0) + \\ &+ \sum_{n=1}^{\infty} (-1)^n \underbrace{\int_{D_-} \cdots \int_{D_-}}_{n\text{-integrals}} Q(x, t, X')\, g(X', X'') \ldots g(X^n, x_0, t_0)\, dX' \ldots dX^n \end{aligned} \tag{7.1}$$

where $D_- = D_-(x_0, t_0)$, $Q(X, X')$ *and* $Q^*(X, X')$ *are any quasi-Riemann functions of equation* (2.1) *and its adjoint equation respectively, which as functions of* (X, X') *are* C^2 *on* $\hat{D}_r$ *and* $\hat{D}_l$ *separately*; $g(X, X')$ *and* $g^*(X, X')$ *are given by* (3.4) *and* (3.9)) *respectively.*

Proof. Because of the definition of the Riemann function we have

$$R(x, t, x_0, t_0) = 0 \qquad \text{for every } (x, t) \notin D_-(x_0, t_0). \tag{7.2}$$

Therefore, we restrict attention to $D_-(x_0, t_0)$. Let (x_1, t_1) be any interior point of $D_-(x_0, t_0)$, and define $\mathscr{D}$ as the set of real valued continuous functions defined on $D(x_1, t_1, x_0, t_0)$. It is convenient to introduce for every function $u(x, t)$ belonging to this set, the norm $\| \ \|$ defined by

$$\| u \| = \max_{t_0 \geqq t \geqq t_1} |u|_t \, e^{k(t - t_0)}, \tag{7.3.a}$$

where k is some positive real number that will be conveniently chosen later. Here

$$|u|_t = \max \{ |u(x, \tau)| \mid t \leqq \tau \leqq t_0, (x, \tau) \in D(x_1, t_1, x_0, t_0) \}. \tag{7.3.b}$$

The sets

$$\begin{aligned} \hat{\hat{D}}_r(x_1, t_1, x_0, t_0) = \{ (X, X') \mid (X, X') \in \hat{D}_r, \ X \in D(x_1, t_1, x_0, t_0) \\ \text{and } X' \in D(x_1, t_1, x_0, t_0) \}, \end{aligned} \tag{7.4.a}$$

$$\begin{aligned} \hat{\hat{D}}_l(x_1, t_1, x_0, t_0) = \{ (X, X') \mid (X, X') \in \hat{D}_l, \ X \in D(x_1, t_1, x_0, t_0) \\ \text{and } X' \in D(x_1, t_1, x_0, t_0) \}, \end{aligned} \tag{7.4.b}$$

$$D_r(x_1, t_1, x_0, t_0) = D_{r+}(x_1, t_1) \cap D(x_1, t_1, x_0, t_0), \tag{7.4.c}$$

$$D_l(x_1, t_1, x_0, t_0) = D_{l+}(x_1, t_1) \cap D(x_1, t_1, x_0, t_0) \tag{7.4.d}$$

will also be used in the discussion to follow. Equations (7.4.a) and (7.4.b) are closed and bounded subsets of R_4; equations (7.4.c) and (7.4.d) of R_2.

Let $Q^*(x_0, t_0, x, t)$ meet the smoothness hypothesis of the theorem. By section 6 such a Q^* exists. Define the transformation τ by

$$\tau(u)(x, t) = Q^*(x_0, t_0, x, t) - \int_{D(x, t, x_0, t_0)} g^*(X', x, t) \, u(X') \, dX'. \tag{7.5}$$

Then this transformation maps the set $\mathscr{D}$ into itself. This assertion would be obvious if g^* were continuous, but it is only piecewise continuous. However according to the remarks made in section 6, g^* is continuous on each of the sets $\hat{D}_r$ and $\hat{D}_l$. Hence

$$\begin{aligned} \int_{D(x, t, x_0, t_0)} g^*(X', x, t) \, u(X') \, dX' = \int_{D_r(x, t, x_0, t_0)} g^*(X', x, t) \, u(X') \, dX' + \\ + \int_{D_l(x, t, x_0, t_0)} g^*(X', x, t) \, u(X') \, dX' \end{aligned} \tag{7.6}$$

is the sum of two continuous functions of (x, t) and therefore is continuous.

For any $k > 0$ the set $\mathscr{D}$ is a complete metric space with respect to the norm $\| \ \|$. Our goal will be to prove that for some $k > 0$ the mapping τ is a contraction in this metric space. The so-called Principle of Contraction Mappings [10] will assure us of the existence of one and only one fixed point of the contraction belonging to the space. By (4.10) the restriction to $D(x_1, t_1, x_0, t_0)$ of any Riemann function is a fixed point of this mapping belonging to $\mathscr{D}$. Since (x_1, t_1) is arbitrary, this will show uniqueness of the Riemann function in the whole of R_2. We will show later that this fixed point is indeed a Riemann function.

Let u and v belong to $\mathscr{D}$. Then by (7.5), using relation (7.6), we have

$$\begin{aligned}\tau(u)(x,t)-\tau(v)(x,t)= &-\int_{D_r(x,t,x_0,t_0)} g^*(X',x,t)\{u(X')-v(X')\}\,dX-\\ &-\int_{D_l(x,t,x_0,t_0)} g^*(X',x,t)\{u(X')-v(X')\}\,dX'.\end{aligned} \tag{7.7}$$

Observe now that in the range covered by the first integral $(X, x, t)\in\hat{\hat{D}}_r(x_1, t_1, x_0, t_0)\subset \hat{D}_r$, while in the second one $(X, x, t)\in\hat{\hat{D}}_l(x_1, t_1, x_0, t_0)\subset\hat{D}_l$. Since g^* is continuous on each of $\hat{D}_r$ and $\hat{D}_l$, and $\hat{\hat{D}}_r$ and $\hat{\hat{D}}_l$ are closed and bounded, it follows that there exists an $M>0$ such that $|g^*(X, x, t)|<M$ whenever $(X, x, t)\in \hat{\hat{D}}_r\cup\hat{\hat{D}}_l$. Hence, from (7.7) and (7.3.b), it follows that

$$|\tau(u)-\tau(v)|_t\leqq M\,m\int_t^{t_0}|u-v|_\tau\,d\tau \tag{7.8}$$

where $m>0$ may be taken as the diameter of the set $D(x_1, t_1, x_0, t_0)$.

Multiplying (7.8) by $e^{k(t-t_0)}$ and using (7.3), we get

$$\begin{aligned}\|\tau(u)-\tau(v)\|&\leqq M\,m\,e^{k(t_1-t_0)}\int_{t_1}^{t_0}|u-v|_\tau\,e^{k(\tau-t_0)}\,e^{-k(\tau-t_0)}\,d\tau\\ &\leqq M\,m\,e^{k(t_1-t_0)}\,\|u-v\|\int_{t_1}^{t_0}e^{-k(\tau-t_0)}\,d\tau\leqq\frac{M\,m}{k}\,\|u-v\|\,.\end{aligned}$$

Taking $k>Mm$, we have

$$\|\tau(u)-\tau(v)\|\leqq\mu\,\|u-v\|$$

for some $\mu<1$. This shows that τ is indeed a contraction.

The proof of uniqueness is therefore complete. The above argument shows also that the integral equation

$$w(x,t)=Q^*(x_0,t_0,x,t)-\int_{D(x,t,x_0,t_0)} g^*(X',x,t)\,w(X')\,dX' \tag{7.9}$$

has one and only one continuous solution on $D_-(x_0, t_0)$.

Using a particular approximating sequence to the fixed point of the contraction, we get

$$\begin{aligned}w(x,t)=&\,Q^*(x_0,t_0,x,t)+\\ &+\sum_{n=1}^{\infty}(-1)^n\underbrace{\int_{D_-}\cdots\int_{D_-}}_{n\text{-integrals}} g^*(X',x,t)\dots g^*(X^n,X^{n-1})\,Q^*(x_0,t_0,X^n)\,dX'\dots dX^n\end{aligned} \tag{7.10}$$

where we have used the fact that $g^*(X^r, X^{r-1})$ vanishes when $X^r\notin D_+(X^{r-1})$, and $D_-=D_-(x_0, t_0)$.

On the other hand, associating with Q^* the function Q by means of (6.2), by (6.3) we have for every r

$$\begin{aligned}\int_{D_-} g^*(X^r, X^{r-1})\, Q^*(X^{r+1}, X^r)\, dX^r &= \int_{D_-} g^*(X^r, X^{r-1})\, Q(X^r, X^{r+1})\, dX^r \\ &= \int_{D_-} g(X^r, X^{r+1})\, Q^*(X^r, X^{r-1})\, dX^r \qquad (7.11)\\ &= \int_{D_-} g(X^r, X^{r+1})\, Q(X^{r-1}, X^r)\, dX^r\end{aligned}$$

when the X's are in the range of integration of the integrals appearing in (7.10). By repeated use of this relation (7.10) becomes

$$w(x,t) = Q(x,t,x_0,t_0) + \\ + \sum_{n=1}^{\infty} (-1)^n \underbrace{\int_{D_-} \dots \int_{D_-}}_{n\text{-integrals}} Q(x,t,X')\, g(X', X'') \dots g(X^n, x_0, t_0)\, dX' \dots dX^n. \qquad (7.12)$$

Thus we have shown that (7.1) is valid whenever a Riemann function exists.

In the proof of existence of the Riemann function we will use a quasi-Riemann function $Q^*(x, t, x_1, t_1)$ of the adjoint equation, which, as a function of (x, t, x_1, t_1), is C^3 on $\hat{D}_r$ and on $\hat{D}_l$, separately. The existence of such a quasi-Riemann function under the assumptions of the theorem is guaranteed by the remarks in Section 6. Define for every $X \in D_- = D_-(x_0, t_0)$ the function

$$p(X) = g(X, x_0, t_0) + \sum_{n=1}^{\infty} (-1)^n \int_{D_-} \dots \int_{D_-} g(X, X') \dots g(X^n, x_0, t_0)\, dX' \dots dX^n, \qquad (7.13.a)$$

and observe that

$$p(X) = g(X, x_0, t_0) - \int_{D(X, x_0, t_0)} g(X, X')\, p(X')\, dX'. \qquad (7.13.b)$$

Using arguments similar to those used to prove that the left-hand member of (7.5) is continuous, it may be shown that every term of the sum appearing in (7.13.a) is a piecewise continuous function of X on $D_-(x_0, t_0)$ and that the only line on which it has jump discontinuities is $d_-(x_0, t_0)$. Therefore, $p(X)$ has the same property because the series in (7.13.a) is uniformly and absolutely convergent on $D(X, x_0, t_0)$ for every $X \in D_-(x_0, t_0)$.

The function $g(X, X') \in C^1$ on $\hat{D}_r$ and on $\hat{D}_l$ by virtue of (3.4.a) because $Q(X, X') \in C^3$ on those sets. Also, it is possible to introduce a system of "characteristic coordinates" $\xi(x, t)$, $\eta(x, t)$ such that

$$c_+ \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial t} = 0,$$

$$c_- \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0$$

and $\xi, \eta \in C^3$. This is possible because $c_+(x, t)$ and $c_-(x, t)$ are C^3 on R_2. Let $\xi_0 = \xi(x_0, t_0)$, $\eta_0 = \eta(x_0, t_0)$, and choose the coordinate system so that $\xi(X) \leqq \xi_0$ and $\eta(X) \leqq \eta_0$ whenever $X \in D_-(x_0, t_0)$. Then the domain $D(X, x_0, t_0)$ maps into the region determined by

$$\xi(X) \leqq \xi \leqq \xi_0, \quad \eta(X) \leqq \eta \leqq \eta_0.$$

From these remarks it follows easily that $p(X) \in C^1$ on $D_{l-}(x_0\ t_0)$ and on $D_{r-}(x_0, t_0)$ separately.

In the same manner we can show that the function

$$\int_{D(x,t,x_0,t_0)} Q(x,t,X')\, p(X')\, dX' \tag{7.14}$$

possesses second order derivatives with respect to x and t which are jointly continuous on $D_{l-}(x_0, t_0)$ and on $D_{r-}(x_0, t_0)$. In this proof the fact, already proved, that $p(X) \in C^1$ on $D_{r-}(x_0, t_0)$ and on $D_{l-}(x_0, t_0)$ has to be used. Furthermore, the first partial derivatives of the function of (x, t) defined by (7.14) are continuous on the whole $D_-(x_0, t_0)$ because they are

$$\begin{aligned}\int_{D(x,t,x_0,t_0)} \frac{\partial Q}{\partial x}(x,t,X')\, p(X')\, dX' + \frac{\partial \xi}{\partial x}(x,t) \int_{S_{r+}(x,t)} Q(x,t,X')\, p(X')\, \delta_1(X')\, dX' \\ + \frac{\partial \eta}{\partial x}(x,t) \int_{S_{l+}(x,t)} Q(x,t,X')\, p(X')\, \delta_2(X')\, dX'\end{aligned} \tag{7.15.a}$$

and

$$\begin{aligned}\int_{D(x,t,x_0,t_0)} \frac{\partial Q}{\partial t}(x,t,X')\, p(X')\, dX' - \frac{\partial \xi}{\partial t}(x,t) \int_{S_{r+}} Q(x,t,X')\, p(X')\, \delta_1(X')\, dX' - \\ - \frac{\partial \eta}{\partial t}(x,t) \int_{S_{l+}} Q(x,t,X')\, p(X')\, \delta_2(X')\, dX'\end{aligned} \tag{7.15.b}$$

where

$$\delta_1 = \left\{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial t}\right)^2\right\}^{-\frac{1}{2}}; \quad \delta_2 = \left\{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial t}\right)^2\right\}^{-\frac{1}{2}}.$$

To obtain (7.15), the fact that $Q(x, t, X')$ for fixed (x, t) is a continuous function of X' in the whole domain $D_+(x, t)$ was taken into account.

Finally from (7.12) and (7.13) it follows that

$$w(x,t) = Q(x,t,x_0,t_0) - \int_{D(x,t,x_0,t_0)} Q(x,t,X')\, p(X')\, dX'. \tag{7.16}$$

Hence $w(x, t) \in C^2$ on $D_{r-}(x_0, t_0)$ and on $D_{l-}(x_0, t_0)$ separately and is continuous on $D_-(x_0, t_0)$. Since the function given by (7.14) is C^1 with respect to (x, t) on $D_-(x_0, t_0)$, it follows from (7.16) that

$$\left[\frac{\partial w}{\partial x}\right] = \left[\frac{\partial Q}{\partial x}\right]. \tag{7.17.a}$$

The integral appearing in (7.16) vanishes when $(x, t) \in S_-(x_0, t_0)$. Thus

$$w(x,t) = Q(x,t,x_0,t_0) \qquad \text{on } S_-(x_0,t_0). \tag{7.17.b}$$

All this shows that $w(x, t)$ is a quasi-Riemann function with singularity at (x_0, t_0). By the Integral Representation Theorem of Section 2, used to prove theorem 7.1, it remains only to show that for any $u(x, t) \in C^2$ of finite history we have

$$u(x_0, t_0) = \int_{D_-(x_0, t_0)} f(X)\, w(X)\, dX \tag{7.18}$$

where $f(X)$ is given by (2.1).

Now by (7.16) and (3.11)

$$\begin{aligned}\int_{D_-} f(X)\, w(X)\, dX &= \int_{D_-} f(X)\, Q(X, x_0, t_0)\, dX - \int_{D_-} \int_{D_-} Q(X, X')\, p(X')\, f(X)\, dX\, dX' \\ &= -\int_{D_-} p(X') \{\int_{D_-} Q(X, X')\, f(X)\, dX\}\, dX' + \\ &\quad + \int_{D_-} f(X)\, Q(X, x_0, t_0)\, dX \\ &= u(x_0, t_0) + \\ &\quad + \int_{D_-} \{g(X, x_0, t_0) - p(X) - \int_{D_-} p(X')\, g(X, X')\, dX'\}\, u(X)\, dX\end{aligned} \tag{7.19}$$

which by (7.13.b) yields (7.18).

This completes the proof of existence of the Riemann function. Thus the proof of the theorem is also complete.

A dual argument shows the following:

Theorem 7.2. *Under the hypothesis of theorem* 7.1, *for every point* $(x_1, t_1) \in R_2$, *there is one and only one Riemann function of the adjoint equation* (2.9) *with singularity at* (x_1, t_1). *It is given by*

$$R^*(x, t, x_1, t_1) = R(x_1, t_1, x, t). \tag{7.20}$$

Proof. The existence and uniqueness may be shown in the same manner as theorem 7.1 was shown. Once existence has been shown we can apply (4.11) to obtain (7.20).

8. A Reciprocal Theorem

In this section we prove the following

Theorem 8.1. *Let* $u(x, t) \in C^2$ *be a motion of finite history* (*i.e. u satisfies* (2.1) *and* (2.13)), *and let* $v(x, t) \in C^2$, *of bounded support, satisfy* (2.9); *then*

$$\int_{R_2} f(X)\, v(X)\, dX = \int_{R_2} f^*(X)\, u(X)\, dX. \tag{8.1}$$

Proof. Multiply (2.1.1) by v and (2.9) by u; subtract the resulting equations, and integrate over the entire space R_2 to obtain

$$\begin{aligned}&\int_{R_2} \frac{\partial}{\partial x} \Big\{v(X)\, E \frac{\partial u}{\partial x}(X) + v(X) \int_{-\infty}^{t} G(t, \tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u(X)\, E \frac{\partial v}{\partial x}(X) - \\ &\qquad - u(X) \int_{t}^{\infty} G(\tau, t) \frac{\partial v}{\partial x}(\tau)\, d\tau\Big\}\, dX - \int_{R_2} \rho \frac{\partial}{\partial t} \Big\{v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t}\Big\}\, dX \\ &= \int_{R_2} \{f(X)\, v(X) - f^*(X)\, u(X)\}\, dX\end{aligned} \tag{8.2}$$

where we have used the fact that

$$\int_{R_2} \left\{ \frac{\partial v}{\partial x}(X) \int_{-\infty}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau \right\} dX = \int_{R_2} \left\{ \frac{\partial u}{\partial x}(X) \int_{t}^{\infty} G(\tau,t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX .$$

By means of divergence theorem, where we have taken into account the fact that $v(X)$ is of bounded support, it follows that the left-hand member of equation (8.2) vanishes, and therefore the proof of the theorem is complete.

Corollary 8.1. *Let $u(x,t) \in C^2$ possess finite history. Assume $f(x,t) \in C$ is any function. Associate with every function $v(x,t) \in C^2$ of bounded support a function $f^*(x,t)$ by means of* (2.9). *Suppose that for every $v(x,t) \in C^2$ of bounded support* (8.1) *holds. Then $u(x,t)$ satisfies* (2.1) *with $f(x,t)$ as body forces.*

Proof. Let us call $F(x,t)$ the value of the left-hand member of (2.1). Then by theorem 8.1 we have

$$\int_{R_2} F(X)\, v(X)\, dX = \int_{R_2} f^*(X)\, u(X)\, dX . \tag{8.4}$$

Subtracting (8.1) from (8.4), we obtain

$$\int_{R_2} \{F(X) - f(X)\}\, v(X)\, dX = 0 . \tag{8.5}$$

Since this is true for any function of bounded support, it follows that

$$F(X) = f(X),$$

and the corollary follows.

9. Construction of the Solution when the Body Forces are Prescribed

As an obvious consequence of the integral representation theorem 3.1 and the existence and uniqueness theorem 7.1 for the Riemann function, we have the following

Theorem 9.1. *Under the hypothesis of theorem* 7.1, *given any $f(X) \in C$, there is at most one function $u(x,t) \in C^2$ with finite history for which the right-hand member of* (2.1) *is $f(x,t)$. Furthermore, if it exists, it is given by*

$$u(x,t) = \int_{D_-(x,t)} f(X')\, R(X',x,t)\, dX' = \int_{D_-(x,t)} f(X')\, R^*(x,t,X')\, dX . \tag{9.1}$$

Proof. It follows from theorems 3.1, 7.1 and 7.2.

This theorem enables us to construct the only motion with finite history for which the body forces are $f(x,t)$, when such a motion exists. However, it does not deliver conditions under which the integrals appearing in (9.1) represent a solution of (2.1). It is the purpose of this section to elucidate this matter.

Existence Theorem 9.2. *Under hypothesis of theorem* 7.1, *for any $f(x,t) \in C^1$ of finite history, the function $u(x,t)$ defined by* (9.1) *is C^2 on R_2, and it is a solution of* (2.1) *with body forces $f(x,t)$.*

Proof. The Riemann function $R(X,x,t)$ has been shown to have second order continuous derivatives with respect to x and t on D_r and D_l, separately. Therefore, from (9.1) it follows that $u(x,t) \in C^2$, because $f \in C^1$. Obviously u possesses finite history.

We show now that for any function $v \in C^2$ of bounded support equation (8.1) holds. In this manner the theorem follows from Corollary 8.1.

By (9.1) we have

$$\begin{aligned}\int_{R_2} f^*(X)\,u(X)\,dX &= \int_{R_2} f^*(X)\Big\{\int_{D_-(X)} f(X')\,R(X',X)\,dX'\Big\}\,dX \\ &= \int_{R_2}\int_{R_2} f^*(X)\,f(X')\,R(X',X)\,dX'\,dX\end{aligned} \tag{9.2}$$

where f^* is given by (2.9) and where we have used the fact that

$$R(X',X)=0, \qquad \text{whenever } X' \notin D_-(X). \tag{9.3}$$

Therefore, from (3.22) it follows that

$$\begin{aligned}\int_{R_2} f^*(X)\,u(X)\,dX &= \int_{R_2} f(X')\Big\{\int_{R_2} f^*(X)\,R(X',X)\,dX\Big\}\,dX' \\ &= \int_{R_2} f(X')\Big\{\int_{D_+(X')} f^*(X)\,R^*(X,X')\,dX\Big\}\,dX' \\ &= \int_{R_2} f(X')\,v(X')\,dX'\end{aligned}$$

where we have used (7.20) and the fact that

$$R^*(X,X')=0, \qquad \text{whenever } X \notin D_+(X').$$

Thus, the proof of the theorem is complete.

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