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The Speed of Propagation of Signals in Viscoelastic Materials

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1. Introduction

Recent work on the propagation of wavefronts in materials with memory by Herrera and Gurtin,[1] Coleman *et al.*,[2] Coleman and Gurtin[3,4] and Varley[5] has shown that the dynamic equations for such materials exhibit many of the properties of hyperbolic partial differential equations.

However, it seems that so far this fact has not been exploited in the solution of problems. The only case reported in the literature of which I am aware are some papers by Glanx and Lee,[6] and Lee and Kanter[7] in which they used the method of characteristics to solve some problems. But this was done only for a very special type of strain-stress relations for which it was possible to reduce the equations of motion to partial differential equations of hyperbolic type without memory.

Thus, it is natural to try to formulate a general mathematical theory of functional partial differential equations of hyperbolic type which would include as a particular case the equations of motion for materials with memory.

Within this program, I have recently formulated a theory for nonlinear systems of functional partial differential equations of hyperbolic type in two variables and proved existence and uniqueness of solution in a domain of determinacy.

To extend such a theory to equations for functions in four variables (three spatial variables and time) does not seem to be an easy task. Therefore, I have recently restricted my attention to linear systems. One of the simplest problems that may be considered refers to questions of uniqueness. In this paper, attention will be restricted to the equations of motion for the linear dynamic theory of viscoelasticity.

Uniqueness questions for quasi-static linear viscoelasticity have been extensively studied.[8–10] However, the equations of quasi-static viscoelasticity differ from those of dynamic viscoelasticity in one essential respect: for any given past history the former are elliptic while the latter are hyperbolic. Correspondingly, the well-posed problems in both cases are different.

Initial boundary value problems for the dynamic theory have already been investigated.[11,12] On the other hand, Cauchy's initial value problems apparently have not been investigated at all, in spite of the fact that this type of problem is specially relevant in connection with hyperbolic equations.

Uniqueness questions for Cauchy problems are connected with the determination of appropriate domains of determinacy whose knowledge in turn gives information about the speed of propagation of viscoelastic signals, because the speed with which the boundary of a domain of determinacy moves is a bound for the speed of propagation of signals. If the domain of determinacy is maximum in some sense, then the speed with which its boundary moves equals the speed of propagation of viscoelastic signals.

The purpose of this paper is to report a uniqueness theorem for a Cauchy's initial value problem in the domain of determinacy defined by the speed of propagation of acceleration waves.[1]

When formulating Cauchy's initial value problems for equations with memory, not only the initial values of displacements and velocities must be given, but the whole past history must be given as well. Then, the problem consists in determining a solution in the domain of determinacy, having the given past history.

Roughly speaking, a theorem reported in this paper shows that if the past history of a material is prescribed in a region R_0 of E^3, then there is at most one solution of the equations of motion in any domain obtained by shrinking R_0 at a speed which is everywhere larger than the greatest speed of propagation of acceleration waves. This tends to give support to the widespread suspicion that the speed of propagation of signals equals the maximum speed of propagation of weak discontinuities; a suspicion which is also supported by what is known in some other fields such as the theory of elasticity.[13]

It is assumed that the elastic tensor is initially symmetric and initially positive definite but anisotropic and inhomogeneous. Another good feature of the theorem is that it does not assume that the elastic tensor is time translation invariant, an assumption which has been used in almost every previous discussion of uniqueness questions.

The method used to prove the theorem mentioned above, with slight modifications, can be applied to boundary value problems. Therefore, it has also been used to generalize in one direction a result originally due to Eldestein and Gurtin (first uniqueness theorem of Eldestein and Gurtin[11]) by removing the requirement that the elastic tensor be time translation invariant.

2. Notation

Let $\mathbf{x}$ be a point of the Euclidean space E^3 and t the time. Let R_0 be a compact region of E^3 whose boundary is R_0. Even more, we assume that R_0 is the closure of a normal domain, i.e., we assume that R_0 is closed and bounded in E^3 and that the divergence theorem can be applied in R_0. B_u and B_τ will be complementary subsets of ∂R_0, i.e.,

$$\partial R_0 = B_u \cup B_\tau \qquad B_u \cap B_\tau = \phi$$

We use index notation, and Latin indices will run from 1 to 3 unless otherwise explicitly stated. Summation will be understood over repeated indices.

For a function $f(\mathbf{x}, t)$ we use the notation

$$\underbrace{f^{(n)}_{,ij\cdots k}}_{m\text{-indices}}(\mathbf{x}, t) = \frac{\partial^{m+n} f(\mathbf{x}, t)}{\partial x_i \, \partial x_j \cdots \partial x_k \, \partial t^n}$$

We define the norm of a tensor $A_{ij\cdots k}$ by

$$\|\mathbf{A}\|^2 = \|A_{ij\cdots k}\|^2 = A_{ij\cdots k} A_{ij\cdots k} \tag{2.1a}$$

Observe that if $A_{ij\cdots k}$ and $B_{ij\cdots k\cdots n}$ are two tensors and

$$C_{l\cdots n} = A_{ij\cdots k} B_{ij\cdots kl\cdots n}$$

then

$$\|\mathbf{C}\| \leqslant \|\mathbf{A}\| \, \|\mathbf{B}\| \tag{2.1b}$$

The Cartesian components of the relaxation tensor will be denoted by $G_{ijpq}(\mathbf{x}, t, \tau)$; they are assumed to be in general a function of $\mathbf{x}, t, \tau$ so that the subject of our discussion will be an inhomogeneous, anisotropic solid whose stress-strain relation is not restricted to be time translation invariant, $u_i(\mathbf{x}, t)$ will be the Cartesian components of the displacement vector; $\tau_{ij}(\mathbf{x}, t)$ the Cartesian components of the stress tensor; $\rho(\mathbf{x})$ the mass density; and $C_{ijpq}(\mathbf{x}, t) \equiv G_{ijpq}(\mathbf{x}, t, \tau)$ the "initial value" of G_{ijpq}.

We denote by $\xi = (\mathbf{x}, t) = (x_1, x_2, x_3, t)$ a point in the space-time $(E^3 \times (-\infty, \infty))$. All integrals will be in the four-dimensional space-time in which $d\xi$ denotes the element of integration. The character of the integrals (volume or surface integral) will be denoted by a subindex under the integral sign.

The symbols $\dot{G}_{ijpq}$ and $\bar{G}_{ijpq}$ are defined by

$$\dot{G}_{ijpq}(\mathbf{x}, t, \tau) = \frac{\partial}{\partial t} G_{ijpq}(\mathbf{x}, t, \tau) \tag{2.2}$$

$$\bar{G}_{ijpq}(\mathbf{x}, t, \tau) = \frac{\partial}{\partial \tau} G_{ijpq}(\mathbf{x}, t, \tau) \tag{2.3}$$

We say that $G_{ijpq}(\mathbf{x}, t, \tau)$ is initially symmetric when

$$C_{ijpq}(\mathbf{x}, t) = C_{pqij}(\mathbf{x}, t) \tag{2.4}$$

and initially positive definite if

$$C_{ijpq}(\mathbf{x}, t)a_{ij}a_{pq} > 0 \tag{2.5}$$

whenever the symmetric part of a_{ij} is nonzero.

Let $T > 0$; for every $t \in [0, T]$, $E^3 \times [t, t]$ is a three-dimensional Euclidean space immersed in the four-dimensional space-time. For every $t \in [0, T]$ we consider a compact region $R(t) \subset E^3 \times [t, t]$ which we assume to be the closure of a normal domain.

Let

$$V(t) = \bigcup_{\tau=0}^{\tau=t} R(\tau) \tag{2.6}$$

and $S^*(t)$ the boundary of $V(t)$ in the space-time. We assume that

(a) $V(t)$ is the closure of a normal domain (compact),

(b) $R(0) = R_0 \times [0, 0]$,

and adopt the following notation:

$$S_1(t) = R(0)$$

$$S_2(t) = \text{closure of } \{S^*(t) \cap [E^3 \times (0, t)]\}$$

$$S_3(t) = R(t)$$

Observe that

$$S^*(t) = S_1(t) \cup S_2(t) \cup S_3(t)$$

The unit outer normal vector to $S^*(t)$ will be denoted by $n_\alpha(\mathbf{x}, \tau)$ $(\alpha = 1, 2, 3, 4)$ at every point $(\mathbf{x}, \tau) \in S^*(t)$. It is assumed to be defined and continuous in $S_2(T)$. It must be understood that n_i $(i = 1, 2, 3)$ are the spatial components while n_4 is the time component.

The above assumptions imply that

$$n_i = 0 \qquad n_4 = -1 \qquad \text{on} \quad S_1(t) \tag{2.7}$$

$$n_i = 0 \qquad n_4 = 1 \qquad \text{on} \quad S_3(t) \tag{2.8}$$

We consider two types of domains $V(T)$. One type will be when $V(T)$ is a "hypercylinder" and the other one will be when $V(T)$ is a "conoid."

We say that $V(T)$ is a hypercylinder if

$$n_4(\mathbf{x}, t) = 0 \qquad \text{for every} \quad (\mathbf{x}, t) \in S_2(T) \tag{2.9}$$

In this case $S_2(T) = R_0 \times [0, T]$ and the spatial components n_i of n_α $(\alpha = 1, 2, 3, 4)$ constitute the outer unit normal vector to R_0.

In the case when $V(T)$ is a conoid we assume that $R(t)$ shrinks with time, i.e.,

$$R(t') \subset R(t) \qquad \text{whenever} \quad t' > t$$

We require even more than that. It is assumed that

$$n_4(\mathbf{x}, \tau) > 0 \qquad \text{whenever} \quad (\mathbf{x}, \tau) \in S_2(T) \tag{2.10}$$

Therefore, when $V(T)$ is a conoid there is a k $(0 < k < 1)$ such that

$$n_4(\mathbf{x}, \tau) > k \qquad \text{whenever} \quad (\mathbf{x}, \tau) \in S_2 \tag{2.11}$$

because n_4 is continuous on $S_2(T)$ which is compact.

Herrera and Gurtin[1] have established that the speed of propagation v of acceleration waves in any direction $\mathbf{e}$ satisfies the eigenvalue problem

$$(C_{ijkl} e_j e_l - \rho v^2 \, \delta_{ik}) a_k = 0 \tag{2.12}$$

where $\mathbf{a}$ is some vector $(\mathbf{a} \neq 0)$. The possible speeds, solutions of (2.12), will be denoted by v_1, v_2, v_3. It will be assumed that

$$0 < v_1 \leqslant v_2 \leqslant v_3 \tag{2.13}$$

3. Statement of the Problem

We consider the field equations of the linear theory of viscoelasticity

$$\tau_{ij,j}(\mathbf{x}, t) - \rho(\mathbf{x}) u_i^{(2)}(\mathbf{x}, t) + F_i(\mathbf{x}, t) = 0 \tag{3.1}$$

in the region $V(T)$ where u_i is the displacement field and F_i the body forces.

The stress relation is

$$\tau_{ij}(\mathbf{x}, t) = C_{ijpq}(\mathbf{x}, t) u_{p,q}(\mathbf{x}, t) - \int_{-\infty}^{t} \bar{G}_{ijpq}(\mathbf{x}, t, \tau) u_{p,q}(\mathbf{x}, \tau) \, d\tau \tag{3.2}$$

Given $\mathbf{u}(\mathbf{x}, t)$ defined in $V(T) \cup P$ and C^2 in $V(T)$, we give the name "initial velocity" to a function $\mathbf{S}(\mathbf{x})$ defined by

$$\mathbf{S}(\mathbf{x}) = \mathbf{u}^{(1)}(\mathbf{x}, 0+) \qquad \text{on} \quad R_0 \tag{3.3a}$$

and "past history" to the function $\pi(\mathbf{x}, t)$ which is the restriction to P of the function $\mathbf{u}(\mathbf{x}, t)$, i.e.,

$$\pi(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) \qquad \text{on} \quad P \tag{3.3b}$$

where the "past" P is the set

$$P = R_0 \times (-\infty, 0] \tag{3.4}$$

When $V(T)$ is a hypercylinder, given the functions $u_i^*(\mathbf{x}, t)$ defined on B_u and $T_i(\mathbf{x}, t)$ defined on B_τ we say that a function $\mathbf{u}(\mathbf{x}, t)$ defined on $V(T) \cup P$ meets the boundary conditions if

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*(\mathbf{x}, t) \qquad (\mathbf{x}, t) \in B_u \times [0, T] \tag{3.5}$$

$$\tau_{ij}(\mathbf{x}, t) n_j = T_i(\mathbf{x}, t) \qquad (\mathbf{x}, t) \in B_\tau \times [0, T] \tag{3.6}$$

In what follows we assume that

(a) $\rho(\mathbf{x})$ is continuous and positive in R_0.

(b) $\pi(\mathbf{x}, t) \in C^2$ and is such that

$$\int_{-\infty}^{0} \bar{G}_{ijpq}(\mathbf{x}, t, \tau) \pi_{p,q}(\mathbf{x}, \tau)\, d\tau$$

exists and belongs to C^2 on $V(T)$.

(c) $G_{ijpq}(\mathbf{x}, t, \tau) \in C^2$ in the closed region of E^5 where $(\mathbf{x}, t) \in V(T)$ and $0 \leqslant \tau \leqslant t$.

(d) $$G_{ijkl}(\mathbf{x}, t, \tau) = G_{jikl}(\mathbf{x}, t, \tau) = G_{ijlk}(\mathbf{x}, t, \tau) \tag{3.7}$$

whenever $(\mathbf{x}, t) \in V(T)$ and $0 \leqslant \tau \leqslant t$.

As is well known, in the linear theory of viscoelasticity the symmetries (3.7) are satisfied under very general conditions. Indeed, the relation

$$G_{ijkl} = G_{jikl}$$

follows from the symmetry of the stress tensor, while the relation

$$G_{ijkl} = G_{ijlk}$$

is a consequence of the principle of objectivity which implies that the stress tensor is a functional of the symmetric part of $u_{i,j}$ only.

Given $G_{ijpq}(\mathbf{x}, t, \tau)$ in the region where $(\mathbf{x}, t) \in P \cup V(T)$, $\tau \leqslant t$, $\rho(\mathbf{x})$ in R_0, $\pi(\mathbf{x}, t)$ in P, $\mathbf{S}(\mathbf{x})$ in R_0, $\mathbf{F}$ in $V(T)$, we consider two different types of problems.

A. The Mixed Problems of Dynamic Viscoelasticity

When $V(T)$ is a hypercylinder, given the functions u_i^* and T_i, we say that $\mathbf{u}(\mathbf{x}, t)$ defined in $V(T) \cup P$ is a solution of the "mixed problem of viscoelasticity" if:

(a) $\mathbf{u}(\mathbf{x}, t) \in C^2$ in $V(T)$.

(b) $\mathbf{u}, \mathbf{F}, G_{ijpq}$, and ρ satisfy the field equation (3.1) in $V(T)$. In these equations τ_{ij} is interpreted as a shorthand writing defined by (3.2).

(c) The past history and initial velocity of $\mathbf{u}$ are given by (3.3).

(d) $\mathbf{u}$ and τ_{ij} satisfy the boundary conditions (3.5) and (3.6), respectively.

B. Cauchy's Initial Value Problem for Dynamic Viscoelasticity

When $V(T)$ is a conoid we say that $\mathbf{u}(\mathbf{x}, t)$ defined in $P \cup V(T)$ is a solution of "Cauchy's initial value problem for viscoelasticity" if $\mathbf{u}$ satisfies (a), (b), and (c) of Section A.

We are now in a position to state the theorems.

THEOREM 1. *Assume that the relaxation tensor $G_{ijpq}(\mathbf{x}, t, \tau)$ is initially symmetric and initially positive definite. Then the mixed problem of viscoelasticity has at most one solution in $V(T) \cup P$.*

THEOREM 2. *Assume that the relaxation tensor $G_{ijpq}(\mathbf{x}, t, \tau)$ is initially symmetric and initially positive definite. Let the unit normal vector be given by*

$$\mathbf{n} = \lambda(e_1, e_2, e_3, c) \tag{3.8}$$

on $S_2(T)$, where $\mathbf{e}$ is a unit vector (normal to $R(t)$) and $\lambda > 0$ is a normalizing factor.

Then, if

$$c > v_3 \geqslant v_2 \geqslant v_1 > 0 \tag{3.9}$$

the Cauchy's initial value problem of dynamic viscoelasticity has at most one solution in $P \cup V(T)$.

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