Offprint from "Archive for Rational Mechanics and Analysis",
Volume 25, Number 3, 1967, P. 178–187

Springer-Verlag, Berlin · Heidelberg · New York

Riemann Representation Method in Viscoelasticity
II. Cauchy's Initial Value Problem

JOSÉ M. BARBERÁN & ISMAEL HERRERA

Communicated by W. NOLL

1. Introduction

In the first paper of this series [1], which will be called I from now on, we defined and constructed the Riemann function for the dynamic equations of linear visco-elasticity for one-dimensional motions. Then an integral representation theorem was given expressing the solution to problems with prescribed body forces in terms of the Riemann function.

It is well known [2–4] in the theory of partial differential equations that the solution to CAUCHY'S initial value problems can be expressed by means of integral representation formulas in terms of the Riemann function. A similar theory can be developed for partial differential equations with memory. In the first part of the present work this is done for the dynamic equations of viscoelasticity and the existence and uniqueness of the solution of CAUCHY'S initial value problem is shown in this manner. Then the method of the parametrix [3], here called quasi-Riemann function, is used to obtain an integral equation for the solution which is suitable for numerical treatment. From the practical point of view such a procedure has interest because it allows the construction of the solution of CAUCHY'S initial value problem directly, when the Riemann function is not known.

2. Notation

In this work the notation used in paper I will be adopted.

We shall be concerned with the dynamic equations of linear visco-elasticity for one-dimensional motions. If $u(x, t) \in C^2$ is the displacement field, they are

$$\frac{\partial}{\partial x}\left\{E(t)\frac{\partial u}{\partial x}(t)\right\}+\frac{\partial}{\partial x}\int_{-\infty}^{t} G(t,\tau)\frac{\partial u}{\partial x}(\tau)\,d\tau-\rho\frac{\partial^2 u}{\partial t^2}(t)=f(x,t) \qquad (2.1)$$

where

$E(x, t)>0$ is the initial value of the relaxation tensor, defined for every $(x, t)\in R_2$.

$G(x, t, \tau)$ is the relaxation tensor defined for every $(x, t)\in R_2$ and $\tau \leqq t$. It is convenient to extend the definition of $G(t, \tau)$ to be zero when $\tau > t$.

$\rho(x)>0$ is the density, defined on R_1.

$f(x, t)$ are the body forces.

We shall restrict attention mainly to solutions $u(x, t)$ of (2.1) for which there is a number T such that

$$u(x,t)=0 \qquad \text{whenever } t \leqq T$$

and we shall say that functions satisfying this condition possess "finite history". Functions defined on R_2 which vanish outside a bounded domain will be said to have "bounded support".

The speeds of propagation of wave fronts $c(x, t)$ satisfy

$$c^2 = \frac{E}{\rho}. \tag{2.2}$$

The positive and negative roots of this equation will be represented by c_+ and c_- respectively.

We systematically assume that

$$G(x,t,\tau) \in C^2 \qquad \text{when } (x,t) \in R_2, \ \tau \leqq t,$$

and

$$E(x,t) \in C^2 \qquad \text{on } R_2,$$
$$\rho(x) \in C^2 \qquad \text{on } R_1.$$

The domain of dependence $D_-(x_0, t_0)$ of the point (x_0, t_0) is the closed region of the x, t-plane whose boundary consists of the characteristics passing through (x_0, t_0) and whose points have the property that $t \leqq t_0$. The domain of influence $D_+(x_0, t_0)$ of (x_0, t_0) is defined in the same manner, except that its points satisfy $t \geqq t_0$.

The boundary of $D_+(x_0, t_0)$ is $S_+(x_0, t_0)$, whose equation is

$$t = t_+(x_0, t_0, x),$$

and the boundary of $D_-(x_0, t_0)$ is $S_-(x_0, t_0)$, whose equation is

$$t = t_-(x_0, t_0, x).$$

Associated with the two roots of (2.2) there are two families of characteristics. Those associated with c_+ will be called advancing and those associated with c_- will be called receding. For brevity we will write c for c_+ when dealing with advancing characteristics and c for c_- when dealing with receding characteristics. At every point of S_+ (except at x_0, t_0) we define the unit normal vector $\boldsymbol{n} = (n_x, n_t)$ outward to D_+. The vector $\boldsymbol{n}$ on S_- is defined similarly. Observe that

$$n_t = -c\, n_x. \tag{2.3}$$

The adjoint equation of (2.1) is defined by

$$\frac{\partial}{\partial x}\left(E \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x} \int_t^\infty G(\tau, t) \frac{\partial v}{\partial x}(\tau)\, d\tau - \rho \frac{\partial^2 v}{\partial t^2} = f^*. \tag{2.4}$$

It is convenient to adopt the notation

$$X=(x,t). \tag{2.5}$$

In the x, t-plane we shall consider line and surface integrals. The element of integration in both cases will be denoted by dX and the character of the integral will be specified by a subindex under the sign of integration. When dealing with functions of several variables, we write explicitly only those arguments which are not obvious from the context.

According to the definition of the quasi-Riemann function $Q(x, t, x_0, t_0)$ given in paper I, it satisfies the relations

$$\frac{\partial}{\partial x}\left(E\frac{\partial Q}{\partial x}\right)+\frac{\partial}{\partial x}\int_t^{t_-} G(\tau,t)\frac{\partial Q}{\partial x}(\tau)\,d\tau+\frac{\partial}{\partial x}\left\{\frac{G(t_-(x),t)\,Q(t_-(x))}{c(t_-(x))}\right\}-\rho\frac{\partial^2 Q}{\partial t^2}$$
$$=g(x,t,x_0,t_0);\qquad (x,t)\in D_-(x_0,t_0), \tag{2.6}$$

$$-2\rho(x_0)\,c_+(x_0,t_0)\,Q(x_0,t_0,x_0,t_0)=1, \tag{2.7.a}$$

$$2\rho c\frac{dQ}{ds}+\left(\frac{d\rho c}{ds}-\frac{G(t_-,t_-)}{c}n_x\right)Q=0;\qquad \text{on } S_-(x_0,t_0), \tag{2.7.b}$$

and

$$E(x_0,t)\left[\frac{\partial Q}{\partial x}(x_0,t_0,x_0,t_0)\right]=-\frac{G(x_0,t_0,t)}{E(x_0,t_0)}-$$
$$-\int_t^{t_0} G(x_0,\tau,t)\left[\frac{\partial Q}{\partial x}(x_0,\tau,x_0,t_0)\right]d\tau, \tag{2.7.c}$$

where

$$\frac{d}{ds}=n_t\frac{d}{dx}-n_x\frac{d}{dt}$$

and the square brackets stand for the jump discontinuity (the value on the right minus the value on the left). RIEMANN's function also satisfies (2.6) and (2.7), but with $g\equiv 0$.

We define the sets

$$D_{l+}(x_0,t_0)=\{(x,t)\,|\,(x,t)\in D_+(x_0,t_0),\ x\leqq x_0\}, \tag{2.8.a}$$

$$D_{r+}(x_0,t_0)=\{(x,t)\,|\,(x,t)\in D_+(x_0,t_0),\ x\geqq x_0\}, \tag{2.8.b}$$

and

$$\ddot{D}_r=\{(X,X')\,|\,X\in D_{r+}(X')\}, \tag{2.9.a}$$

$$\ddot{D}_l=\{(X,X')\,|\,X\in D_{l+}(X')\}. \tag{2.9.b}$$

3. Cauchy's Initial Value Problem

In this section we formulate the Cauchy initial value problem relevant to our work.

Let $[a, b]$ be a closed interval of the real axis and define the past by

$$\mathscr{P}=[a,b]\times(-\infty,0), \tag{3.1}$$

the "reduced domain of dependence" of (x_0, t_0) by

$$\mathscr{R}(x_0, t_0) = \{(x, t) \mid (x, t) \in D_-(x_0, t_0) \text{ and } t \geqq 0\}, \tag{3.2}$$

and the "domain of determinacy of $\mathscr{P}$" by

$$\Pi = \{(x, t) \mid \mathscr{R}(x, t) \subset [a, b] \times [0, \infty)\}. \tag{3.3}$$

Given any point $(x_0, t_0) \in \Pi$, define $\alpha(x_0, t_0)$ and $\beta(x_0, t_0)$ as two real numbers such that

$$[\alpha, \beta] = \{x \mid (x, 0) \in \mathscr{R}(x_0, t_0)\}, \tag{3.4}$$

and the "effective past of (x_0, t_0)" by

$$\hat{\mathscr{P}}(x_0, t_0) = [\alpha, \beta] \times (-\infty, 0). \tag{3.5}$$

We shall consider displacement field functions $u(x, t)$ defined on $\mathscr{P} \cup \Pi$ which are C^2 on Π and on $\mathscr{P}$ separately. The "past history" U is defined by

$$U(x, t) = u(x, t) \qquad \text{on } \mathscr{P}. \tag{3.6}$$

It is assumed to possess an extension to the closure of $\mathscr{P}$ which is C^2. The "initial value" u_0 is defined by

$$u_0(x) = u(x, 0+); \qquad x \in [a, b] \tag{3.7}$$

and the "initial velocity" by

$$\gamma(x) = \frac{\partial u}{\partial t}(x, 0+); \qquad x \in [a, b]. \tag{3.8}$$

With this notation we formulate the following Cauchy initial value problem.

"Given the functions $U(x, t) \in C^2$ of finite history on $\mathscr{P}$ (which can be extended to the closure of $\mathscr{P}$ as a C^2 function), $f(x, t) \in C$ on Π, $u_0(x) \in C^2$ and $\gamma(x) \in C^1$ on $[a, b]$, to find a function $u(x, t)$ defined on $\mathscr{P} \cup \Pi$ and C^2 on Π, which satisfies (2.1) on Π with f as body forces, and whose past history, initial value and initial velocity are U, u_0 and γ, respectively."

4. Integral Representation Theorem

Lemma 4.1. *Let $u(x, t)$ of finite history be a solution of Cauchy initial value problem as stated in the last section. Then if $Q(x, t, x_0, t_0)$ is a quasi-Riemann function of equation* (2.1) *with singularity at $(x_0, t_0) \in \Pi$, and $g(x, t, x_0, t_0)$ are the body forces associated with Q (as defined by* 2.6), *we have*

$$\begin{aligned} u(x_0, t_0) = & \int_{\mathscr{R}(x_0, t_0)} Q(\boldsymbol{X}, x_0, t_0) f(\boldsymbol{X})\, d\boldsymbol{X} - \int_{\mathscr{R}(x_0, t_0)} g(\boldsymbol{X}, x_0, t_0)\, u(\boldsymbol{X})\, d\boldsymbol{X} + \\ & + (\rho c_- Q u_0)(\beta, 0) - (\rho c_+ Q u_0)(\alpha, 0) - \\ & - \int_\alpha^\beta \rho \left(Q \gamma - u_0 \frac{\partial Q}{d t}\right)(x, 0)\, d x - \\ & - \int_{\mathscr{R}(x_0, t_0)} \left\{ Q(\boldsymbol{X}, x_0, t_0) \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t, \tau) \frac{\partial U}{\partial x}(\tau)\, d\tau \right\} d\boldsymbol{X}. \end{aligned} \tag{4.1}$$

Proof. Multiplying (2.1) by $Q(x, t, x_0, t_0)$ and (2.6) by $u(x, t)$ subtracting the resulting equations and integrating with respect to X over $\mathscr{R}(x_0, t_0)$, we obtain

$$\begin{aligned}\int_{\mathscr{R}} \Big\{ & Q(t) \frac{\partial}{\partial x} \int_{-\infty}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u(t) \frac{\partial}{\partial x} \int_{t}^{t_-} G(\tau, t) \frac{\partial Q}{\partial x}(\tau)\, d\tau - \\ & - u(t) \frac{\partial}{\partial x} \left(\frac{G(t_-, t)\, Q(t_-)}{c(t_-)} \right) \Big\}\, dX + \\ & + \int_{\mathscr{R}} \left\{ \frac{\partial}{\partial x} \left(Q E \frac{\partial u}{\partial x} - u E \frac{\partial Q}{\partial x} \right) - \rho \frac{\partial}{\partial t} \left(Q \frac{\partial u}{\partial t} - u \frac{\partial Q}{\partial t} \right) \right\} dX \\ = & \int_{\mathscr{R}} Q(X, x_0, t_0) f(X)\, dX - \int_{\mathscr{R}} g(X, x_0, t_0)\, u(X)\, dX .\end{aligned}$$

After some manipulation we get

$$\begin{aligned}\int_{\mathscr{R}} \Big\{ & Q(t) \frac{\partial}{\partial x} \int_{-\infty}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u(t) \frac{\partial}{\partial x} \int_{t}^{t_-} G(\tau, t) \frac{\partial Q}{\partial x}(\tau)\, d\tau - \\ & - u(t) \frac{\partial}{\partial x} \left(\frac{G(t_-, t)\, Q(t_-)}{c(t_-)} \right) \Big\}\, dX \\ = & - \int_{S_-} \frac{u(X)\, G(t_-, t_-)\, Q(X)}{c(t_-)}\, n_x\, dX - \\ & - \int_{0}^{t_0} u(x_0, t) \left\{ 2 \frac{G(t_-, t)\, Q(t_-)}{c_+(t_0)} - \int_{t}^{t_0} G(\tau, t) \left[\frac{\partial Q}{\partial x}(\tau) \right] d\tau \right\} dt + \\ & + \int_{\mathscr{R}(x_0, t_0)} \left\{ Q(X, x_0, t_0) \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t, \tau) \frac{\partial u}{\partial x}(\tau)\, d\tau \right\} dX .\end{aligned} \tag{4.2}$$

On the other hand, using the divergence theorem we get

$$\begin{aligned}\int_{\mathscr{R}} & \left\{ \frac{\partial}{\partial x} \left(Q E \frac{\partial u}{\partial x} - u E \frac{\partial Q}{\partial x} \right) - \rho \frac{\partial}{\partial t} \left(Q \frac{\partial u}{\partial t} - u \frac{\partial Q}{\partial t} \right) \right\} dX \\ = & - \int_{S_-} \left\{ \rho c Q \left(\frac{\partial u}{\partial x} n_t - \frac{\partial u}{\partial t} n_x \right) - \rho c u \left(\frac{\partial Q}{\partial x} n_t - \frac{\partial Q}{\partial t} n_x \right) \right\} dX + \\ & + \int_{0}^{t_0} u E \left[\frac{\partial Q}{\partial x} \right] dt + \int_{\alpha}^{\beta} \rho \left(Q \frac{\partial u}{\partial t} - u \frac{\partial Q}{\partial t} \right) dx \\ = & - \int_{S_-} \left\{ \rho c Q \frac{du}{ds} - \rho c u \frac{\partial Q}{\partial s} \right\} dX + \int_{0}^{t_0} u E \left[\frac{\partial Q}{\partial x} \right] dt + \int_{\alpha}^{\beta} \rho \left(Q \frac{\partial u}{\partial t} - u \frac{\partial Q}{\partial t} \right) dx\end{aligned} \tag{4.3}$$

where

$$\frac{d}{ds} = n_t \frac{d}{dx} - n_x \frac{d}{dt} .$$

Adding the expressions given by (4.2) and (4.3), using (2.6) and (2.7), we obtain (4.1).

Integral Representation Theorem 4.1. *Let $u(x, t)$ of finite history be a solution of* Cauchy's *initial value problem as stated in the last section. Then if $R(x, t, x_0, t_0)$*

is a Riemann function of equation (2.1) *with singularity at* $(x_0, t_0) \in \Pi$, *we have*

$$\begin{aligned} u(x_0, t_0) = & \int_{\mathscr{R}(x_0, t_0)} R(X, x_0, t_0) f(X)\, dX + \\ & + (\rho c_- R u_0)(\beta, 0) - (\rho c_+ R u_0)(\alpha, 0) - \\ & - \int_\alpha^\beta \rho \left(R\gamma - u_0 \frac{\partial R}{\partial t} \right)(x, 0)\, dx - \\ & - \int_{\mathscr{R}(x_0, t_0)} \left\{ R(X, x_0, t_0) \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t, \tau) \frac{\partial U}{\partial x}(\tau)\, d\tau \right\} dX . \end{aligned} \tag{4.4}$$

Proof. Set $g(X, x_0, t_0) = 0$ in equation (4.1) to get (4.4).

Uniqueness Theorem 4.2. *Let E, G,* and $\rho \in C^3$. *Assuming that a solution $u(x, t)$ of* CAUCHY'S *initial value problem exists in $\Pi \cup \mathscr{P}$, then $u(x, t)$ is the only solution and it is given by* (4.4).

Proof. Observe that by theorem 7.1 of paper I the Riemann function exists. Therefore we can apply the integral representation theorem 4.1 to get this result.

5. Existence

In this section we first prove some preliminary results and then an existence theorem for CAUCHY'S initial value problem.

Reciprocal Theorem 5.1.* *Let $u(x, t)$ be a solution to* CAUCHY'S *initial value problem and let $v(x, t)$ be an arbitrary function of class C^2 in R_2 whose support is bounded and contained in $\mathscr{P} \cup \Pi$. Let $f^*(x, t)$ be defined by* (2.4). *Then:*

$$\begin{aligned} \int_\Pi f(X) v(X)\, dX = & \int_\Pi f^*(X) u(X)\, dX + \\ & + \int_\Pi v \left\{ \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t, \tau) \frac{\partial U}{\partial x}(\tau)\, d\tau \right\} dX + \\ & + \int_a^b \rho \left(v\gamma - u_0 \frac{\partial v}{\partial t} \right)(x, 0)\, dx . \end{aligned} \tag{5.1}$$

Proof. Multiply (2.1) by v and (2.4) by u, subtract the resulting equations and integrate over Π to obtain

$$\begin{aligned} & \int_\Pi \frac{\partial}{\partial x} \left\{ v E \frac{\partial u}{\partial x} - u E \frac{\partial v}{\partial x} \right\} dX - \int_\Pi \rho \frac{\partial}{\partial t} \left\{ v \frac{\partial u}{\partial t} - u \frac{\partial v}{\partial t} \right\} dX + \\ & \quad + \int_\Pi \left\{ v \frac{\partial}{\partial x} \int_{-\infty}^{t} G(t, \tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u \frac{\partial}{\partial x} \int_t^\infty G(\tau, t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX \\ & = \int_\Pi \{ f(X) v(X) - f^*(X) u(X) \}\, dX . \end{aligned} \tag{5.2}$$

* *Note added in proof:* This theorem, like Theorem 8.1 of paper I, is obviously related to a reciprocal theorem given by V. VOLTERRA in 1909 (Sulle equazioni integro-differenziali della teoria dell'elasticita. Rendiconti della Reale Accademia dei Lincei **18**, 2, 295 (1909)).

Recall that the support of v is contained in $\mathscr{P} \cup \Pi$, so that

$$\int_{\Pi} \left\{ v \frac{\partial}{\partial x} \int_{-\infty}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u \frac{\partial}{\partial x} \int_{t}^{\infty} G(\tau,t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX$$

$$= \int_{\Pi} \left\{ v \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau \right\} dX + \int_{\Pi} \left\{ v \frac{\partial}{\partial x} \int_{0}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - \right.$$

$$\left. - u \frac{\partial}{\partial x} \int_{t}^{t-} G(\tau,t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX = \int_{\Pi} \left\{ v \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau \right\} dX$$

where the facts that

$$\int_{\Pi} \left\{ \frac{\partial v}{\partial x} \int_{0}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau \right\} dX = \int_{\Pi} \left\{ \frac{\partial u}{\partial x} \int_{t}^{t-} G(\tau,t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX$$

and

$$\int_{\Pi} \frac{\partial}{\partial x} \left\{ v \int_{0}^{t} G(t,\tau) \frac{\partial u}{\partial x}(\tau)\, d\tau - u \int_{t}^{t-} G(\tau,t) \frac{\partial v}{\partial x}(\tau)\, d\tau \right\} dX = 0$$

were used. Applying the divergence theorem to equation (5.2) and using again the fact that the support of v is contained in $\mathscr{P} \cup \Pi$, equation (5.1) follows.

Corollary 5.1. *Let $u(x,t)$ be a piecewise C^2 function on $\mathscr{P} \cup \Pi$, such that the only possible line of discontinuity for u and its first and second derivatives is the interval $[a,b]$ of the x-axis. Assume further that $u = U$ in $\mathscr{P}$. Let $f(x,t) \in C$ in Π be arbitrary. Associate with every $v(x,t) \in C^2$ in R_2 whose support is bounded and contained in $\mathscr{P} \cup \Pi$ a function $f^*(x,t)$ by means of* (2.6).

Then if for every v satisfying the above requirements, equation (5.1) *holds, the function $u(x,t)$ is a solution of* CAUCHY'S *initial value problem with body forces $f(x,t)$, initial value $u_0(x)$, initial velocity $\gamma(x)$, and $U(x,t)$ as past history.*

Proof. Define $F(x,t)$ as the value attained by the right-hand member of equation (2.1) when the given function u is substituted in it, and let

$$u_0'(x) = u(x, 0+) \quad \text{and} \quad \gamma'(x) = \frac{\partial u}{\partial t}(x, 0+).$$

Then u is a solution of CAUCHY's initial value problem with body forces F, initial value u_0', initial velocity γ', and past history U. Therefore we only need to prove $F = f$, $u_0' = u_0$ and $\gamma' = \gamma$.

By theorem 5.1 we have

$$\begin{aligned} \int_{\Pi} F(X)\, v(X)\, dX = {} & \int_{\Pi} f^*(X)\, u(X)\, dX + \\ & + \int_{\Pi} v(X) \left\{ \frac{\partial}{\partial x} \int_{-\infty}^{0} G(t,\tau) \frac{\partial U}{\partial x}(\tau)\, d\tau \right\} dX + \\ & + \int_{a}^{b} \rho \left(v \gamma' - u_0' \frac{\partial v}{\partial t} \right)(x, 0)\, dx . \end{aligned} \tag{5.3}$$

Subtracting (5.1) from (5.3) we get

$$\int_{\Pi} \{F(X)-f(X)\}\, v(X)\, dX = \int_a^b \rho \left\{ v(\gamma'-\gamma) - (u_0'-u_0)\frac{\partial v}{\partial t}\right\}(x,0)\, dx .$$

But this equation holds for every v if and only if $F=f$, $\gamma'=\gamma$, and $u_0'=u_0$.

Existence Theorem 5.2. *If E, G and $\rho \in C^3$, $f\in C^1$, $\gamma\in C^2$, u_0 and $U\in C^3$, then the function u, defined on $\mathscr{P}$ by $u(x,t)=U(x,t)$ and on Π by* (4.4), *is the solution of* CAUCHY'S *initial value problem with body forces f, initial value u_0, initial velocity γ, and past history U.*

Proof. It is convenient to show first that to prove this theorem it is enough to consider the case for which $u_0=\gamma=U=0$. To this end define an auxiliary function $u'(x,t)$ on $\mathscr{P}\cup\Pi$ by setting $u'(x,t)=U(x,t)$ on $\mathscr{P}$ and letting $u'(x,t)$ be on Π any function belonging to C^3 such that

$$u'(x,0+)=u_0(x), \qquad \frac{\partial u'}{\partial t}(x,0+)=\gamma(x).$$

Define $F(x,t)$ on Π as the value of the right-hand member of (2.1) when u is replaced by u'. It is clear that $u(x,t)$ will be a solution to the given Cauchy initial value problem if and only if $u-u'$ is a solution to a Cauchy initial value problem for which the initial displacement, initial velocity, and past history vanish, and for which the body forces are $f(x,t)-F(x,t)$. Since under the given hypotheses $f-F$ is C^1, it is therefore enough to prove the theorem for the case when $u_0=\gamma=U=0$.

In this case we have

$$u(x,t)=\int_{\mathscr{R}(x,t)} R(X,x,t)\, f(X)\, dX \qquad \text{on } \Pi .$$

The argument used in the proof of theorem 9.2 of paper I shows now that $u(x,t)\in C^2$ on Π. To show that

$$\int_{\Pi} f(X)\, v(X)\, dX = \int_{\Pi} f^*(X)\, u(X)\, dX$$

for every $v\in C^2$ whose support is bounded and is contained in $\mathscr{P}\cup\Pi$, observe that

$$\begin{aligned}\int_{\Pi} f^*(X)\, u(X)\, dX &= \int_{\Pi} f^*(X)\Big\{\int_{\mathscr{R}(x,t)} f(X')\, R(X',X)\, dX'\Big\}\, dX\\ &= \int_{\Pi}\int_{\Pi} f^*(X)\, f(X')\, R(X',X)\, dX'\, dX\\ &= \int_{\Pi} f(X')\Big\{\int_{\Pi} f^*(X)\, R^*(X,X')\, dX\Big\}\, dX'\\ &= \int_{\Pi} f(X')\, v(X')\, dX'\end{aligned}$$

where we have used the facts that

$$R(X',X)=0 \quad \text{when} \quad X'\in \Pi-\mathscr{R}(x,t)$$

and that

$$v(x,t)=\int_{D_+(x,t)} f^*(X')\,R^*(X',x,t)\,dX'=\int_{\Pi} f^*(X')\,R^*(X',x,t)\,dX' \qquad \text{on } \Pi$$

because the support of v is contained on $\mathscr{P}\cup\Pi$.

6. Direct Construction of the Solution

We now give a method to construct the solution of CAUCHY's initial value problem which does not assume previous knowledge of the RIEMANN's function.

Theorem 6.1. *Let $Q(X, X')$ be any quasi-Riemann function of* (2.1) *which as a function of (X, X') is C^2 on $\hat{D}_r$ and $\hat{D}_l$ separately. Assume that* CAUCHY's *initial value problem possesses a solution. Define*

$$u_1(x,t)=U(x,t) \qquad \text{on } \mathscr{P}, \tag{6.1.a}$$

$$\begin{aligned} u_1(x,t)=&\int_{\mathscr{R}(x,t)} Q(X',x,t)\,f(X')\,dX'+(\rho\,c_-\,Q\,u)(\beta(x,t),0)-\\ &-(\rho\,c_+\,Q\,u)(\alpha(x,t),0)-\int_{\alpha}^{\beta}\rho\left(Q\,\gamma-u_0\frac{\partial Q}{\partial t}\right)(x',0)\,dx'-\\ &-\int_{\mathscr{R}(x,t)}\left\{Q(X',x,t)\frac{\partial}{\partial x}\int_{-\infty}^{0}G(x',t',\tau)\frac{\partial U}{\partial x}(x',\tau)\,d\tau\right\}dX' \qquad \text{on } \Pi; \end{aligned} \tag{6.1.b}$$

$$u_{n+1}(x,t)=0 \qquad n=1,2,\ldots \text{ on } \mathscr{P}; \tag{6.1.c}$$

and

$$u_{n+1}(x,t)=-\int_{\mathscr{R}(x_0,t_0)} g(X',x,t)\,u_n(X')\,dX' \qquad n=1,2,\ldots \text{ on } \Pi. \tag{6.1.d}$$

Then the function

$$u_s(x,t)=\sum_{n=1}^{\infty}u_n(x,t) \qquad \text{on } \mathscr{P}\cup\Pi; \tag{6.2}$$

is the only solution of CAUCHY's *initial value problem.*

Proof. Let $\mathscr{D}$ be the set of continuous function defined on Π. The transformation

$$\begin{aligned} \tau(u)(x,t)=&\int_{\mathscr{R}(x,t)} Q(X',x,t)\,f(X')\,dX'-\\ &-\int_{\mathscr{R}(x,t)} g(X',x,t)\,u(X')\,dX'+(\rho\,c_-\,Q\,u_0)(\beta(x,t),0)-\\ &-(\rho\,c_+\,Q\,u_0)(\alpha(x,t),0)-\\ &-\int_{\mathscr{R}(x,t)}\left\{Q(X',x,t)\frac{\partial}{\partial x}\int_{-\infty}^{0}G(x',t',\tau)\frac{\partial U}{\partial x}(x',\tau)\,d\tau\right\}dX' \end{aligned}$$

is a transformation of $\mathscr{D}$ into itself.

We are going to define a metric on $\mathscr{D}$ with respect to which $\mathscr{D}$ will be a complete metric space. Then we will show that τ is a contraction with respect to this metric. Then the "principle of contraction mappings" [5] will assure us of the existence of one and only one fixed point of τ in $\mathscr{D}$. A particular approximating sequence to this fixed point is given by (6.2), so that $u_s(x, t)$ is the fixed point of τ. On the other hand, by virtue of Lemma 4.1 any solution of CAUCHY'S initial value problem is a fixed point of τ. Therefore, $u_s(x, t)$ is the only solution of CAUCHY'S initial value problem.

Given two functions u and $v \in \mathscr{D}$, define the distance between u and v by

$$\| v-u \| = \max_{0 \leqq t \leqq T} |v-u|_t \, e^{-Kt}$$

where

$$T = \max \{ t \mid (x, t) \in \Pi \},$$

$$|v-u|_t = \max \{ |u(x,\tau)| \mid 0 \leqq \tau \leqq t, (x,\tau) \in \Pi \},$$

and K is some positive number to be chosen later. With this metric $\mathscr{D}$ is a complete metric space. Observe

$$(\tau(u)-\tau(v))(x,t) = -\int_{\mathscr{R}(x,t)} g(X', x, t) \{u(X') - v(X')\} \, dX.$$

Now $g(X', x, t)$ is piecewise continuous on $\Pi \times \Pi$ which is compact in R_4. Therefore, there exists an $M > 0$ such that

$$|g(X', x, t)| < M \quad \text{whenever} \quad (X', X) \in \Pi \times \Pi.$$

Thus

$$|\tau(u)-\tau(v)|_t \leqq M \, m \int_0^t |u-v|_\tau \, d\tau \tag{6.3}$$

where $m > 0$ may be taken as the diameter of Π.

Multiplying (6.3) by e^{-Kt} we get

$$\begin{aligned} \| \tau(u)-\tau(v) \| &\leqq \max_{0 \leqq t \leqq T} \left\{ M \, m \, e^{-Kt} \int_0^t |u-v|_\tau \, e^{-K\tau} e^{K\tau} \, d\tau \right. \\ &\leqq M \, m \, \| u-v \| \left\{ \max_{0 \leqq t \leqq T} \left(e^{-Kt} \int_0^t e^{K\tau} \, d\tau \right) \right\} \\ &\leqq \frac{M \, m}{K} \| u-v \| \leqq \mu \, \| u-v \| \end{aligned}$$

where μ may be taken as a positive number less than one if K is taken sufficiently large. This completes the proof of the theorem.

References

[1] HERRERA, I., Riemann representation method in viscoelasticity, I. Arch. Rational Mech. Anal. **22**, 4 (1966).

[2] COURANT, R., & D. HILBERT, Methods of Mathematical Physics, vol. II. New York: Interscience Publ. 1962.

[3] GARABEDIAN, P., Partial Differential Equations. New York: John Wiley & Sons 1964.

[4] HELLWIG, G., Partial Differential Equations. Blaisdell Publishing Co. 1964.

[5] KOLMOGOROV, A. N., & S. V. FOMIN, Functional Analysis, vol. I: Metric and Normed Spaces, p. 43—46. Rochester (N. Y.): Graylock Press 1957.

Instituto de Geofísica de la Universidad
Nacional Autónoma de México

(Received December 2, 1966)

Druck der Universitätsdruckerei H. Stürtz AG., Würzburg