A Correspondence Principle for the Theory of Leaky Aquifers

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Abstract. A correspondence principle for leaky aquifers in which the storage in the confining layers is taken into consideration associates a nonleaky aquifer with every leaky aquifer. It is valid for long periods of operation. Using this principle, the study of the leaky aquifers is reduced to the study of the corresponding nonleaky aquifers. When the quality of the approximation is tested by applying it to an isolated well, complete agreement with results obtained by other methods is found. The principle is suitable for application to numeric studies of regional evolution of piezometric heads.

INTRODUCTION

Leaky aquifers, introduced formerly by *Jacob* [1946] and studied extensively later on by *Hantush* [1964] have been the subject of great interest from the point of view of ground water and recently from the point of view of soil mechanics as well [*Figueroa*, 1968]. However, the application of existing theories to regional problems is difficult, leading to rather complicated situations that are not supported in practice by the quality and quantity of the existing field measurements. Under such circumstances it seems desirable to develop simplified theories permitting the treatment of such problems with rigour yet in agreement with the quality of the available data.

This paper has been prepared with that goal in mind. Starting from equations governing the behavior of leaky aquifers confined above or below semipervious elastic clay or silt that yields a significant amount of water from storage, the problem is reduced to that of a nonleaky aquifer by means of some mathematical transformations and simplifications.

A more precise statement of the correspondence principle is given later in this paper.

When the quality of the approximation was tested by applying it to a problem solved previously by *Hantush* [1960], complete agreement with his results was found. The approximation is suitable in the treatment of problems in which the flow changes slowly, as in long-period operations.

As a byproduct of the work, the system of equations governing leaky aquifers is equivalent to a single equation with memory. The possibility of using that equation to obtain more refined approximations or for direct computation has not been explored so far and remains an open question.

ANALYSIS

Statement of the problem. The problem is to study the behavior of elastic aquifers confined by semipervious elastic strata; storage within the semiconfining layers is taken into account by *Hantush* [1960]. The discharge of the well or system of wells is supplied by the reduction of storage in the aquifer and by leakage from the semipervious layers. The leakage is obtained from the reduction of storage in the semipervious elastic beds or from other bodies of water overlying and/or underlying the semipervious beds or from both sources. The permeabilities in the semiconfining layers are very small compared with that in the main aquifer, so that the flow is vertical through the semiconfining beds and horizontal in the main aquifer. The goal will be to reduce the system of equations governing the drawdown to a single equation containing the drawdown in the aquifer only. However, this equation turns out to be a partial differential equation with memory. For large values of time an approximation is obtained that reduces the problem of a leaky aquifer to that of a completely confined one by means of a suitable transformation. The above reductions are obtained regardless of the lateral boundary conditions. For the particular case of an aquifer unlimited in the horizontal direction, the approximation so obtained is precisely the same as that obtained previously by *Hantush* [1960].

Flow system. The flow system is an artesian aquifer over- and underlain by semipervious layers. The aquifer may be drained by one or several steady wells. The aquifer may be limited in the horizontal directions by boundaries of any type. Therefore radial symmetry is not assumed. The semipervious layers are over- and underlying two other aquifers in which the hydraulic heads remain uniform (case 1); over- and overlying two impermeable layers (case 2); or one layer is overlying an impermeable bed and the other is underlying an aquifer in which the hydraulic head remains uniform (case 3) (Figure 1).

According to *Hantush* [1960] the flow in this system is vertical in the semipervious layers and horizontal in the main aquifer. Therefore the hydraulic head is independent of the vertical coordinate in the main aquifer, and one may identify all points having the same horizontal coordinates and consider their vertical coordinate as equal to 0. Leaving aside suitable boundary conditions in the horizontal direction and assuming that the drawdown vanishes initially, the following system of equations is satisfied at points not occupied by wells.

Case 1

$$\frac{\partial^2 s_1}{\partial z^2} = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \qquad \text{(a)}$$

$$s_1(x, y, z, 0) = 0 \qquad \text{(b)}$$

$$s_1(x, y, b_1, t) = 0 \qquad \text{(c)}$$

$$s_1(x, y, 0, t) = s(x, y, t) \qquad \text{(d)}$$

Upper semipervious layer (1)

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{K_1}{T}\frac{\partial}{\partial z} s_1(x, y, 0, t) - \frac{K_2}{T}\frac{\partial}{\partial z} s_2(x, y, 0, t) = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad \text{(a)}$$

$$s(x, y, 0) = 0 \qquad \text{(b)}$$

Main aquifer (2)

$$\frac{\partial^2 s_2}{\partial z^2} = \frac{1}{\nu_2}\frac{\partial s_2}{\partial t} \qquad \text{(a)}$$

$$s_2(x, y, z, 0) = 0 \qquad \text{(b)}$$

$$s_2(x, y, 0, t) = s(x, y, t) \qquad \text{(c)}$$

$$s_2(x, y, -b_2, t) = 0 \qquad \text{(d)}$$

Lower semipervious layer (3)

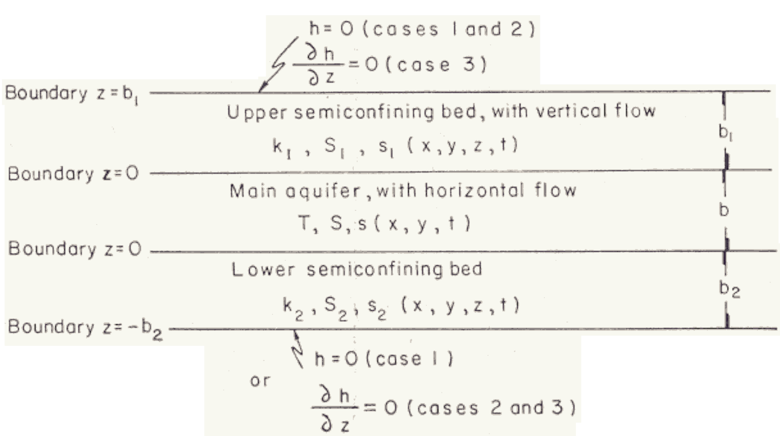

Fig. 1. Sketch of the leaky system.

Case 2 Same as case 1, the conditions (1c) and (3d) being replaced respectively by

$$\frac{\partial}{\partial z} s_1(x, y, b_1, t) = 0 \qquad (4a)$$

$$\frac{\partial}{\partial z} s_2(x, y, -b_2, t) = 0 \qquad (4b)$$

Case 3 Same as case 1, the condition (3d) being replaced by

$$\frac{\partial}{\partial z} s_2(x, y, \quad b_2, t) = 0 \qquad (5)$$

Reduction to a single equation with memory. Assume that $A_1\ (z, t)$ and $A_2\ (z, t)$ are two functions that satisfy

$$\frac{\partial^2 A_1}{\partial z^2} = \frac{1}{\nu_1} \frac{\partial A_1}{\partial t}, \ 0 \leq z \leq b_1, \ t > 0 \qquad (6a)$$

$$\frac{\partial^2 A_2}{\partial z^2} = \frac{1}{\nu_2} \frac{\partial A_2}{\partial t}, \ -b_2 \leq z \leq 0, \ t > 0 \qquad (6b)$$

$$A_1(0, t) = A_2(0, t) = 1 \qquad (6c)$$

$$A_1(z, 0) = A_2(z, 0) = 0 \qquad (6d)$$

and the following conditions

$$A_1(b_1, t) = 0 \quad \text{in case 1 and 3} \qquad (7a)$$

$$\frac{\partial A_1}{\partial z}(b_1, t) = 0 \quad \text{in case 2} \qquad (7b)$$

$$A_2(-b_2, t) = 0 \quad \text{in case 1} \qquad (7c)$$

$$\frac{\partial A_2}{\partial z}(-b_2, t) = 0 \quad \text{in case 2 and 3} \qquad (7d)$$

Then in any case

$$s_1(x, y, z, t) = \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) A_1(z, \quad \tau)\, \partial\tau \qquad (8a)$$

$$s_2(x, y, z, t) = \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) A_2(z, \quad \tau)\, \partial\tau \qquad (8b)$$

as may be checked by direct substitution in equations 1, 3, 4, and 5, using 6 and 7.

Therefore

$$\frac{\partial s_1}{\partial z}(x, y, 0, t) = \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) \frac{\partial}{\partial z} A_1(0, t - \tau)\, \partial\tau$$

$$\frac{\partial s_2}{\partial z}(x, y, 0, t) = \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) \frac{\partial}{\partial z} A_2(0, t \quad \tau)\, \partial\tau$$

Define

$$G(t) = \frac{K_1}{T} \frac{\partial}{\partial z} A_1(0, t) - \frac{K_2}{T} \frac{\partial}{\partial z} A_2(0, t) \qquad (10)$$

Then substitution of (9) and (10) in equation 2a yields

$$\frac{\partial^2 s}{\partial x^2} - \frac{\partial^2 s}{\partial y^2} - \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) \cdot G(t - \tau)\, \partial\tau = \frac{1}{\nu} \frac{\partial s}{\partial t} \qquad (11)$$

This is the desired partial differential equation with memory for s.

The function G. The standard methods for the heat equation [Churchill, 1941] yield easily the function G for the different cases.

Case 1

$$G(t) = -\frac{K_1}{Tb_1}\left[1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi^2\nu_1}{b_1^2} t\right)\right] - \frac{K_2}{Tb_2}\left[1 + 2 \sum_{n=1}^{\infty} \exp\left(\quad \frac{n^2\pi^2\nu_2}{b_2^2} t\right)\right]$$

Case 2

$$G(t) = -\frac{2K_1}{Tb_1} \sum_{n=0}^{\infty} \exp\left[-\frac{(n + \frac{1}{2})^2\pi^2\nu'}{b_1^2}\right] - \frac{2K_2}{Tb_2} \sum_{n=0}^{\infty} \exp\left[-\frac{(n + \frac{1}{2})^2\pi^2\nu_2}{b_2^2} t\right]$$

Case 3

$$G(t) = -\frac{K_1}{Tb_1}\left[1 + 2 \sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi^2\nu_1}{b_1^2} t\right)\right] - \frac{2K_2}{Tb_2} \sum_{n=0}^{\infty} \exp\left[-\frac{(n + \frac{1}{2})^2\pi^2\nu_2}{b_2^2} t\right]$$

Observe that in any of these cases, it is possible to write

$$G(t) = -C - F(t) \qquad (13)$$

where C is a constant (possibly zero) and $F(t)$ is an infinite series of exponentials.

Equation 11 may be reduced to

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - Cs - \int_0^t \frac{\partial}{\partial t} s(x, y, \tau) F(t - \tau)\, \partial\tau = \frac{1}{\nu} \frac{\partial s}{\partial t} \quad (14)$$

by virtue of (13).

The correspondence principle. Equation 14 is implied by the system of equations 1, 2, 3, 4, and 5. It has not introduced any approximation in its derivation. We will assume now that the system of wells has been operated for a long time, so that s and $\partial s/\partial t$ vary slowly. Then

$$\int_0^t \frac{\partial}{\partial t} s(x, y, \tau) F(t - \tau)\, \partial\tau \approx \frac{\partial}{\partial t} s(x, y, t) \int_0^t F(t - \tau)\, \partial\tau = \frac{\partial}{\partial t} s(x, y, t) \int_0^t F(\tau)\, \partial\tau \approx \left[\int_r F(\tau)\, \partial\tau \right] \quad (15)$$

Writing

$$\int_0^\infty F(\tau)\, \partial\tau = I \quad (16)$$

we obtain

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = \left(\frac{1}{\nu} + I\right) \frac{\partial s}{\partial t} + Cs \quad (17)$$

Write

$$\nu_c = \frac{\nu}{1 + I\nu} \quad (18a)$$

and

$$s_c = s e^{C\nu_c t} \quad (18b)$$

so that equation 17 becomes

$$\frac{\partial^2 s_c}{\partial x^2} + \frac{\partial^2 s_c}{\partial y^2} = \frac{1}{\nu_c} \frac{\partial s_c}{\partial t} \quad (19)$$

This is the equation corresponding to a nonleaky aquifer.

Therefore when dealing with a problem in the modified theory of leaky aquifers, one need only solve a corresponding problem for a completely confined aquifer given by (19). Then

$$s = s_c e^{-C\nu_c t} \quad (20)$$

where ν_c is given by (18a) and I and C are given as follows:

Case 1

$$I = \frac{1}{3T}\left(\frac{K_1 b_1}{\nu_1} + \frac{K_2 b_2}{\nu_2}\right) = \frac{1}{3T}(S_1 + S_2)$$

$$C = \frac{1}{T}\left(\frac{K_1}{b_1} + \frac{K_2}{b_2}\right) \quad (21b)$$

$$\nu_c = \frac{3T}{3S + S_1 + S_2}$$

Case 2

$$I = \frac{1}{T}(S_1 + S_2)$$

$$C = 0$$

$$\nu_c = \frac{T}{S + S_1 + S_2}$$

Case 3

$$I = \frac{1}{T}\left(\frac{S_1}{3} + S_2\right)$$

$$C = \frac{K_1}{T b_1}$$

$$\nu_c = \frac{3T}{3S + S_1 + 3S_2}$$

where the equalities

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^2} = \frac{\pi^2}{2} \quad (24)$$

have been used.

Thus the above results can be resumed in the following correspondence principle: To every aquifer in the modified theory of leaky aquifers corresponds an aquifer in the theory of nonleaky aquifers. This correspondence is such that for slowly varying conditions of operation the behavior of the leaky aquifer can be predicted by means of a simple change of variables, provided the behavior of the nonleaky aquifer is known.

An isolated well. As an illustration of the method, consider the problem of an isolated well in a leaky aquifer unlimited in all horizontal directions.

First consider the problem corresponding to the instantaneous extraction of a finite volume Q from the well. The solution to this problem for the corresponding nonleaky aquifer is given by the fundamental solution of the heat equation. This is

$$+\frac{Q}{4\pi T t}\exp\left(-\frac{r^2}{4\nu_e t}\right)$$

where $r = (x^2 + y^2)^{1/2}$.

Thus according to equation 20 the solution for the leaky aquifer is given by

$$+\frac{Q}{4\pi T t}\exp\left(-C\nu_e t-\frac{r^2}{4\nu_e t}\right)$$

Therefore the drawdown for the case of a steady extraction of a volume Q per unit time is given by the integral of the above expression, i.e.

$$s = +\frac{Q}{4\pi T}\int_0^t \frac{\partial\tau}{\tau}\exp\left(-C\nu_e\tau-\frac{r^2}{4\nu_e\tau}\right) \quad (25)$$

which by a change of variable is seen to be

$$s = +\frac{Q}{4\pi T}\int_{(r/4\nu_e t)}^{\infty}\exp\left(-y-\frac{Cr^2}{4y}\right)/y\;dy$$

$$= +\frac{Q}{4\pi T}W\left(\frac{r^2}{4\nu_e t},\,C^{1/2}r\right) \quad (26)$$

Simplification of numeric and analog methods. The correspondence principle established previously can be used to simplify the numeric and analog methods used in the modified theory of leaky aquifers for long periods of operation, i.e., large values of t.

Indeed equation 19 can be used instead of the original system of equations 1, 2, 3, 4, and 5. Since equation 19 is the equation for nonleaky aquifers, the standard techniques for them can be used to save time and effort.

NOTATION

		Dimension
b, b_1, b_2,	thickness of stratified layers,	L;
K,	hydraulic conductivity of main aquifer,	LT^{-1};
K_1, K_2,	vertical hydraulic conductivity of semipervious layers,	LT^{-1};
Q,	constant well discharge,	L^3T^{-1};
r,	horizontal radial distance measured from center of the well to any observation point in the field,	L;
s,	the drawdown of the piezometric surface at any time and at any point in the main aquifer,	L;
s_1, s_2,	drawdowns of piezometric surfaces at any time and at any point in the semipervious layers,	L;
$S = bS_s$, $S_1 = b_1S_s'$, $S_2 = b_2S_s''$,	storage coefficients of the main and of the semipervious layers, respectively,	—
S_s, S_s', S_s'',	specific storage of the main and semipervious layers, respectively (the volume of water a unit volume of the aquifer releases from storage under a unit head decline);	L^{-1};
t,	time measured from some reference time,	T;
$T = Kb$,	coefficient of transmissivity of the main aquifer,	L^2T^{-1}
$W(u, v) = \int_u^\infty dy/y \exp(-y - v^2/4y)$,	the well function for leaky aquifers,	L;
z,	vertical coordinate,	
$\nu = K/S_s = T/S$,		L^2T^{-1};
$\nu_1 = K_1/S_s' = K_1b_1/S_1$,		L^2T^{-1};
$\nu_2 = K_2/S_s'' = K_2b_2/S_2$,		L^2T^{-1}.

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