Theory of Multiple Leaky Aquifers

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Abstract. This paper is an extension of previous work by Herrera and Figueroa to multiple leaky aquifers. The equations governing the motion of such systems are transformed to obtain a simpler system suitable for numerical treatment. An essential feature of the method is the introduction of a lag time that occurs because the influence of any aquifer is not transmitted instantly to its neighbors. This approximation is suitable for treating cases for which the conditions on the aquifers vary slowly. The numerical treatment of the resulting system of equations is of the same order of complexity as that for a single aquifer.

INTRODUCTION

Leaky aquifers that lose or gain water from adjacent strata occur frequently in nature. *Hantush* [1967] cites among these systems the Dutch polder areas, Roswell and other areas in New Mexico, the southern areas of Minnesota, and many areas in Florida. Mexico also has areas of this type.

The treatment of this kind of problem has been incomplete so far. It is usually assumed in the theory of leaky aquifers [*Hantush*, 1960; *Hantush*, 1964; *De Wiest*, 1965] that the piezometric head in one of the aquifers is not altered by the discharge or recharge of the other one. Such an assumption is admissible in many situations, especially if the time of operation has not been too long, but for a longer operating time this hypothesis is not admissible because it leads to predictions that contradict the results observed in practice.

On the other hand, solutions for transient flow reported in the literature [*Hantush*, 1967] that take into account the interaction between acquifers do not include the storage capacity of the leaky media. It is well known [*Hantush*, 1960] that in many situations occurring in practice, this capacity cannot be neglected without incurring unacceptable errors.

In this paper I develop an extension of a theory presented previously by *Herrera and Figueroa* [1969] that is suitable for treating multiple leaky aquifers. In this theory interactions between the aquifers are taken into account as well as the storage capacity of the leaky layers. The derivation of the equations parallels the earlier work by Herrera and Figueroa; therefore previous reading of that paper could help in understanding this one.

The purpose of this paper is to transform the equations governing the behavior of a system of leaky aquifers to obtain equations more suitable for numerical handling of the problem. This is achieved by first transforming the equations into a system of partial differential equations with memory that do not contain the piezometric heads of the leaky layers. Then an approximation for the memory functions is introduced, and the problem is simplified. The resulting system is essentially uncoupled except for the fact that every equation contains the piezometric heads of the neighboring aquifers corresponding to a delayed time. Thus the numerical handling of the problem is quite simple.

If a step by step numerical method is used, then at a given step the values of the piezometric heads of the system corresponding to any previous time will be known. Thus, when carrying on the computations to extend the solution one step further, the piezometric heads at previous times may be taken as data, and the computations will be the same as if the system were uncoupled.

The derivation is done bearing in mind applications to hydraulics of wells, but the equations can be applied to similar problems in other fields.

FORMULATION OF THE PROBLEM

The problem consists of studying the behavior of a system of n elastic aquifers ($n \geq 2$),

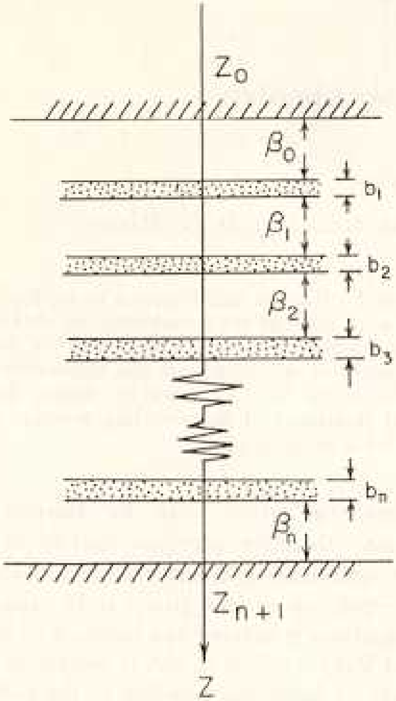

Fig. 1. Flow system.

separated from each other by semipervious layers. The storage capacity of these layers is taken into account. In previous works [*Hantush*, 1960; *Herrera and Figueroa*, 1969] this storage capacity has been taken into account for only a single aquifer, not for multiple aquifers [*Hantush*, 1967]. The discharge of the well or system of wells draining the aquifers is provided by the reduction of the storage in the aquifers and by leakage from the semipervious layers. The leakage is obtained from the reduction of the storage in the semipervious elastic beds or from other bodies of water over and/or underlying the semipervious strata limiting the system (Figure 1). The permeabilities in the leaky aquifers are very small compared with those in the main aquifers, so that the flow is vertical in the semipervious beds and horizontal in the main aquifers.

The goal will be to eliminate from the equations governing the behavior of the hydraulic system the drawdowns corresponding to the leaky layers. However, the resulting system will be a system of partial differential equations with memory. When the variations in the aquifers are slow (in application to hydraulics of wells, this corresponds to a long operating time), an approximation is obtained that reduces the problem to a system of essentially uncoupled equations. The coupling of the equations of the system is accomplished through terms containing the drawdowns in the neighboring aquifers retarded by some lag times. Therefore, when carrying numerical computations, every equation can be written as if it were uncoupled, because the values of the drawdowns at the neighboring aquifers in previous times are already known and can be introduced as data before doing the computations at a given step. The above reductions are obtained independently of the boundary conditions in the horizontal limits of the aquifer.

DIFFERENTIAL EQUATIONS GOVERNING THE MOTION

The general differential equation governing the laminar flow of water in a porous elastic media supposed to be isotropic but possibly inhomogeneous is

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)-q=S_s\frac{\partial h}{\partial t} \qquad (1)$$

where h represents the piezometric head and q the discharge per unit volume produced by a distribution of wells.

In leaky aquifers, according to *Hantush* [1960], the flow is vertical in the leaky layers and horizontal in the main aquifers. Because of this fact, the piezometric head is independent of the vertical coordinate in every one of the main aquifers. It is therefore convenient to identify the points of each main aquifer having the same horizontal coordinates and to consider the vertical coordinates as equal to z_j for the aquifer j. With this hypothesis, equation 1 can be approximated [*Hantush*, 1960] by

$$\begin{aligned}\frac{\partial}{\partial x}\left(T_i\frac{\partial h_i}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_i\frac{\partial h_i}{\partial y}\right)&\\ +\kappa_{i-1}\frac{\partial H_{i-1}}{\partial z}(x, y, z_i, t)&\\ -\kappa_i\frac{\partial H_i}{\partial z}(x, y, z_i, t)&\\ -Q_i=S_i\frac{\partial h_i}{\partial t}&\end{aligned} \qquad (2)$$

in the main aquifers and by

$$\frac{\partial^2 H_i}{\partial z^2} = \frac{1}{\nu_i'} \frac{\partial H_i}{\partial t} \tag{3}$$

in the semipervious strata, where h_i (x, y, t), H_i (x, y, z, t) are the piezometric heads in the main aquifers and in the semipervious layers, respectively.

The system of flow will be assumed to be in hydrostratic equilibrium initially. In that case, without loss of generality, it can be assumed that

$$h_i(x, y, 0) = H_i(x, y, z, 0) = 0$$

for every i.

Flow system. The flow system consists of n artesian aquifers ($n \geq 2$), each one over and underlying semipervious layers. The aquifers may be drained by one or several wells that are not necessarily steady. The boundary conditions in the horizontal limits of the aquifers are not specified.

Three different cases considered are defined by the conditions imposed on the semipervious layers limiting the system above and below (Figure 1).

Case 1. They are above and below, two other aquifers in which the piezometric heads remain constant.

Case 2. They are above and below two impervious layers.

Case 3. The first one rests above an impervious layer and the second one lies below an aquifer whose piezometric head remains constant.

According to the above description, leaving aside the boundary conditions on the horizontal limits of the aquifers, the drawdowns satisfy in each case the following system of equations:

At the main aquifers

$$\frac{\partial}{\partial x}\left(T_i \frac{\partial s_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_i \frac{\partial s_i}{\partial y}\right) + \kappa_{i-1} \frac{\partial \sigma_{i-1}}{\partial z}(x, y, z_i, t) - \kappa_i \frac{\partial \sigma_i}{\partial z}(x, y, z_i, t) + Q_i = S_i \frac{\partial s_i}{\partial t} \qquad i = 1, \cdots, n \tag{4}$$

In the semipervious layers

$$\frac{\partial^2 \sigma_i}{\partial z^2} = \frac{1}{\nu_i'} \frac{\partial \sigma_i}{\partial t} \tag{5}$$

In the planes separating the main aquifers from the semipervious layers

$$s_i(x, y, t) = \sigma_{i-1}(x, y, z_i, t) = \sigma_i(x, y, z_i, t) \qquad 1 \leq i \leq n \tag{6}$$

$$s_i(x, y, 0) = \sigma_i(x, y, z, 0) = 0 \text{ for every } i \text{ and for every } z \tag{7}$$

In addition,

For case 1

$$\sigma_0(x, y, z_0, t) = \sigma_n(x, y, z_{n+1}, t) = 0 \tag{8}$$

For case 2

$$\frac{\partial \sigma_0}{\partial z}(x, y, z_0, t) = \frac{\partial \sigma_n}{\partial z}(x, y, z_{n+1}, t) = 0 \tag{9}$$

For case 3

$$\sigma_0(x, y, z_0, t) = 0$$

$$\frac{\partial \sigma_n}{\partial z}(x, y, z_{n+1}, t) = 0 \tag{11}$$

Note that all the leaky layers satisfy the same conditions except the top and bottom ones, which satisfy special conditions. Therefore these layers have to be treated in a special manner. This special treatment leads to some complications in the notation. However, some effort can be spared when reading this paper by noting that those complications are due only to this fact.

Some remarks. When analyzing equations 1 to 11 we can see that the action any vertical fiber of any of the leaky strata can exert on neighboring fibers is only through the main aquifers. For example, if a point in the horizontal plan of coordinates (x_0, y_0) is fixed, then the motion in the fiber in the leaky layer made of the points

$$(x_0, y_0, z) \qquad z_j < z < z_{j-1}$$

where j is any integer such that $0 < j < n$ is completely determined by equations 5, 6, and 7 for every time $t > 0$, if one knows at that point the drawdowns s_j (x_0, y_0, t) and s_{j-1} (x_0, y_0, t) in the main aquifers that limit the fiber for every time $t > 0$. The problem defined by equations 5, 6, and 7 when s_j and s_{j-1} are regarded as data is one of the classical problems for the heat equation. This problem corresponds to determining the temperature in a bar covered by a thermal insulator when it is known that the temperature of the bar is 0 initially and that the temperatures in its end points vary in a known manner. In this case those temperatures are $s_j(x_0, y_0, t)$ and $s_{j-1}(x_0, y_0, t)$. In the three cases considered here, the conditions for the leaky layers limiting above and below the system of aquifers correspond also to classical problems of the heat equation.

On the other hand, observe equation 4, which governs the motion in the main aquifers. The influence of the leaky strata in the main aquifers is manifested through their water contributions, which in turn are characterized by the functions

$$\frac{\partial \sigma_i}{\partial z}(x, y, z_i, t) \quad \text{and} \quad \frac{\partial \sigma_{i-1}}{\partial z}(x, y, z_i, t)$$

Because of the observation we have made, these terms depend only on the functions $s_{i-1}(x, y, a)$, $s_i(x, y, t)$, and $s_{i+1}(x, y, t)$, but the dependence is not only on the present value of these functions but on the past values as well. In mathematical terms we say that $(\partial\sigma_{i-1}/\partial z)(x, y, z_i, t)$ and $(\partial\sigma_i/\partial z)(x, y, z_i, t)$ for $i = 1, \ldots, n$ are functionals of the functions $s_i(x, y, t)$ for $i = 1, \ldots, n$. Therefore it must be possible to eliminate the partial derivatives of σ_{i-1} and σ_i from (4) and to write instead something containing only the drawdown in the main aquifers s_i. Such terms must depend not only on the present value of s_i but also on all other past values. Hence the same must happen with the resulting system of equations. In this sense it will be a system of equations with memory. The aquifers remember what happened in the past. One may see that in fact the resulting system is a system of integro-differential equations in which the integral terms may be interpreted as a memory.

Transformation into a system of equations with memory. Define the functions $A_{i,\alpha}(z, t)$, where $z_i \le z \le z_{i+1}$; $t \ge 0$; $i = 0, 1, \cdots, n$; and $\alpha = i, i+1$; $i = 1, 2, \cdots, n-1$; $\alpha = 1$ $i = 0$; $\alpha = n$ if $i = n$. They are such that in the three cases considered, the equations

$$\frac{\partial^2 A_{i,\alpha}}{\partial z^2} = \frac{1}{\nu_i'}\frac{\partial A_{i,\alpha}}{\partial t} \qquad z_i \le z \le z_{i+1}$$
$$t > 0 \qquad i = 0, 1, \cdots, n$$

are satisfied.

On the other hand

$$A_{i,i+1}(z_{i+1}, t) = 1 \qquad i = 0, 1, 2, \cdots, n-1 \tag{13a}$$
$$A_{i,i}(z_i, t) = 1 \qquad i = 1, 2, \cdots, n$$
$$A_{i,i+1}(z_i, t) = A_{i,i}(z_{i+1}, t) = 0 \qquad i = 1, 2, \cdots, n-1$$

In addition in case 1

$$A_{0,1}(z_0, t) = A_{n,n}(z_{n+1}, t) = 0$$

In case 2

$$\frac{\partial A_{0,1}(z_0, t)}{\partial z} = \frac{\partial A_{n,n}(z_{n+1}, t)}{\partial z} = 0$$

In case 3

$$A_{0,1}(z_0, t) = \frac{\partial A_{n,n}}{\partial z}(z_{n+1}, t) = 0$$

Then for every one of the three cases we have

$$\sigma_i(x, y, z, t) = \sum_\alpha \left[\int_0^t \frac{\partial s_\alpha}{\partial t}(x, y, \xi) A_{i,\alpha}(z, t-\xi)\, d\xi\right]$$

as may be checked by direct substitution in equations 5 to 11.

Therefore

$$\frac{\partial \sigma_{i-1}}{\partial z}(x, y, z_i, t) = \sum_\alpha \left[\int_0^t \frac{\partial s_\alpha}{\partial t}(x, y, \xi) \frac{\partial A_{i-1,\alpha}}{\partial z}(z_i, t-\xi)\, d\xi\right] \tag{16a}$$

$$\frac{\partial \sigma_i}{\partial z}(x, y, z_i, t) = \sum_\alpha \left[\int_0^t \frac{\partial s_\alpha}{\partial t}(x, y, \xi) \frac{\partial A_{i,\alpha}}{\partial z}(z_i, t-\xi)\, d\xi\right]$$

where $i = 1, 2, \quad, n$.

Define

$$G_{i,\alpha}(t) = \kappa_{i-1} \frac{\partial A_{i-1,\alpha}}{\partial z}(z_i, t) - \kappa_i \frac{\partial A_{i,\alpha}}{\partial z}(z_i, t) \quad (17)$$

where $i = 1, 2, \cdots, n; \alpha = i - 1, i, i + 1$ if $i = 2, \cdots, n - 1$; $\alpha = 1, 2$, if $i = 1$; $\alpha = n - 1, n$, if $i = n$. Any nondefined $A_{i,\alpha}$ corresponding to these values of i and α that occurs in equation 17 must be interpreted to be identically 0.

By substituting equations 16 and 17 in equation 4, we obtain

$$\frac{\partial}{\partial x}\left(T_i \frac{\partial s_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_i \frac{\partial s_i}{\partial y}\right) + \sum_\alpha \left[\int_0^t \frac{\partial s_\alpha}{\partial t}(x, y, \xi) G_{i,\alpha}(t - \xi)\, d\xi\right] + Q_i = S_i \frac{\partial s_i}{\partial t} \qquad i = 1, 2, \cdots, n \quad (18)$$

This is the desired system of integral differential equations. Note that the integral terms depend on the histories of s_i up to the time t and therefore can be interpreted as a memory.

The functions $G_{i,\alpha}$. Using the classical methods for the heat equation [*Churchill*, 1941], it is easy to obtain the functions $G_{i,\alpha}$ corresponding to the different cases.

To express the results it is useful to introduce the functions

$$g_1(\tau) = 2 \sum_{m=1}^{\infty} e^{-m^2\tau} \quad (19a)$$

$$g_2(\tau) = 2 \sum_{m=0}^{\infty} e^{-(m+1/2)^2\tau} \quad (19b)$$

$$g_3(\tau) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m e^{-m^2\tau} \quad (19c)$$

Using them

$$G_{i,i}(t) = -\frac{\kappa_{i-1}}{\beta_{i-1}}\left[1 + g_1\left(\frac{\pi^2}{\beta_i^2}\nu_{i-1}'t\right)\right] - \frac{\kappa_i}{\beta_i}\left[1 + g_1\left(\frac{\pi^2\nu_i'}{\beta_i^2}t\right)\right] \qquad i = 2, \cdots, n-1 \quad (20a)$$

$$G_{i,i+1}(t) = \frac{\kappa_i}{\beta_i} g_3\left(\frac{\pi^2\nu_i' t}{\beta_i^2}\right) \qquad i = 1, \cdots, n \quad (20b)$$

$$G_{i,i-1}(t) = \frac{\kappa_{i-1}}{\beta_{i-1}} g_3\left(\frac{\pi^2\nu_{i-1}t}{\beta_{i-1}^2}\right) \qquad i = 2, \cdots, n \quad (20c)$$

In cases 1 and 3, $G_{1,1}(t)$ is given by formula 20a taking $i = 1$. In case 1, $G_{n,n}(t)$ is given by the same formula taking $i = n$.

In cases 2 and 3

$$G_{n,n}(t) = -\frac{\kappa_{n-1}}{\beta_{n-1}}\left[1 + g_1\left(\frac{\pi^2}{\beta_{n-1}^2}\nu_{n-1}'t\right)\right] - \frac{\kappa_n}{\beta_n} g_2\left(\frac{\pi^2}{\beta_n^2}\nu_n't\right) \quad (20d)$$

In case 2

$$G_{1,1}(t) = -\frac{\kappa_0}{\beta_0} g_2\left(\frac{\pi^2}{\beta_0^2}\nu_0't\right) - \frac{\kappa_1}{\beta_1}\left[1 + g_1\left(\frac{\pi^2}{\beta_2^2}\nu_2't\right)\right] \quad (20e)$$

As in a previous work [*Herrera and Figueroa*, 1969], we can make some observations regarding the functions $G_{i,i}(t)$. These functions may be written in the form

$$G_{i,i}(t) = -C_i - F_i(t) \quad (21)$$

where the C_i are constants and the $F_i(t)$ are infinite series of exponentials that tend to infinite at $t = 0$ and tend to 0 when t tends to infinite. They decay rapidly as t grows. It is easy to deduce the detailed shape of the functions F_i from the shape of the functions $g_1(\tau)$ and $g_2(\tau)$ illustrated in Figure 2.

On the other hand, the shape of the functions $G_{i,i+1}(t)$ and $G_{i,i-1}(t)$ is the same as that of $g_3(\tau)$, illustrated in Figure 3. Observe that

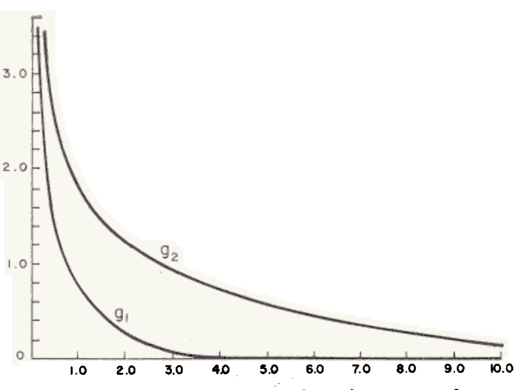

Fig. 2. Shape of memory functions g_1 and g_2.

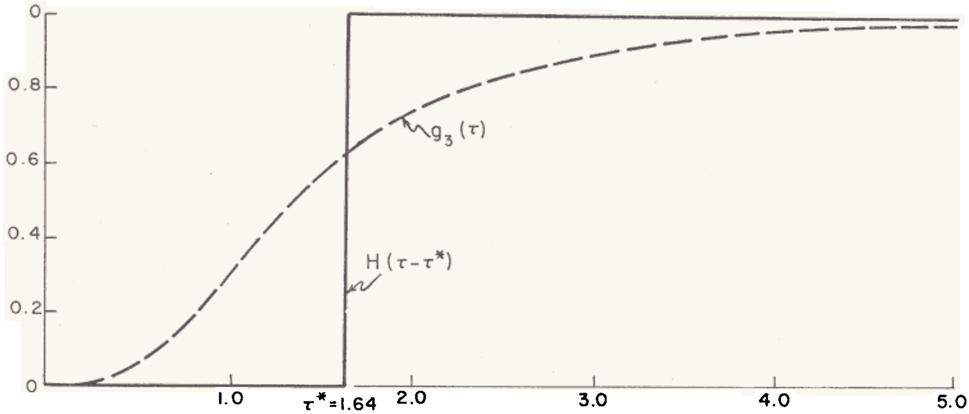

Fig. 3. Approximation of $g_3(\tau)$ by a step function.

$$g_3(0) = 0$$

$$\frac{dg_3}{d\tau}(0) = 0$$

These results may be shown using Cesaro summability criterion. It may be shown even more that

$$\frac{d^m g_3}{d\tau^m}(0) = 0 \quad \text{for any } m \tag{22}$$

It is important to observe that according to Figure 3 the shape of this function is similar to a step function.

Lag time approximation. Equations 18 were deduced directly from equations 4 to 11 without introducing any approximation.

Some simplifications are now made by introducing some approximations.

From Figure 2 we can see that most of the area under the graph of the function $F_i(t)$ is contained in a neighborhood of $t = 0$. Suppose that the changes of $\partial s_i/\partial t$ are small during a time interval equal to that in which most of the area contained under $G_{i,i}(t)$ lies. It is then possible to make the following approximations:

where

$$I_i = \int_0^\infty F_i(\xi)\, d\xi$$

Observe that this is equivalent to writing

$$F_i(t) \approx I_i\, \delta(t) \tag{24}$$

or to taking

$$G_{i,i}(t) \approx -C_i - I_i\, \delta(t) \tag{25}$$

These are precisely the approximations which were used in a previous work [*Herrera and Figueroa*, 1969].

However, notice that they are applicable only if $\partial s_i/\partial t$ varies slowly in comparison with a characteristic time of $F_i(t)$. Because of the way in which I_i was defined, the approximation 25 has the property of preserving the total water contribution of the leaky aquifers.

Since $g_3(\tau)$, as recalled previously, has a shape similar to a step function, it will be approximated in the following manner:

$$g_3(\tau) \approx H(\tau - \tau^*) \tag{26}$$

where $H(\tau)$ is the Heaviside unit step function

$$\int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi) G_{i,i}(t - \xi)\, d\xi = -C_i \int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi)\, d\xi - \int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi) F_i(t - \xi)\, d\xi$$

$$-C_i s_i(x, y, t) - \int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi) F_i(t - \xi)\, d\xi \approx -C_i s_i(x, y, t)$$

$$\frac{\partial s_i}{\partial t}(x, y, t) \int_0^t F_i(t - \xi)\, d\xi = -C_i s_i(x, y, t) - \frac{\partial s_i}{\partial t}(x, y, t) \int_0^t F_i(\xi)\, d\xi$$

$$\approx -C_i s_i(x, y, t) - I_i \frac{\partial s_i}{\partial t}(x, y, t) \tag{23}$$

and τ^* is chosen so that the total flow is preserved, i.e.,

$$\lim_{\tau\to\infty}\left[\int_0^\tau g_3(\xi)\,d\xi - \int_0^\tau H(\xi-\tau^*)\,d\xi\right] = 0 \qquad (27)$$

Equation 27 implies

$$\lim_{\tau\to\infty}\left[\int_0^\tau g_3(\xi)\,d\xi - (\tau-\tau^*)\right]$$
$$= \lim_{\tau\to\infty}\left[2\int_0^\tau \sum_{m=1}^{\infty}(-1)^m e^{-m^2\xi}\,d\xi + \int_0^\tau d\xi - (\tau-\tau^*)\right]$$
$$\tau^* + 2\sum_{m=1}^{\infty}(-1)^m\int_0^\infty e^{-m^2\xi}\,d\xi = 0$$

Therefore

$$\tau^* = 2\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^2} = \frac{\pi^2}{6} \qquad (28)$$

Substitution of this approximation in equations 20*b* and 20*c* leads to

$$G_{i,i+1}(t) = \frac{\kappa_i}{\beta_i}H(t - t^*_{i,i+1}) \qquad i = 1, \cdots, n-1 \qquad (29a)$$

$$G_{i,i-1}(t) = \frac{\kappa_{i-1}}{\beta_{i-1}}H(t - t^*_{i,i-1}) \qquad i = 2, \cdots, n \qquad (29b)$$

where

$$t^*_{i,i+1} = \frac{\beta_i^2}{6\nu_i'} \qquad i = 1, \cdots, n-1 \qquad (30a)$$

$$t^*_{i,i-1} = \frac{\beta_{i-1}^2}{6\nu_{i-1}'} \qquad i = 2, \cdots, n \qquad (30b)$$

Substitution of these approximations in the system of equations 18 leads to

$$\frac{\partial}{\partial x}\left(T_i\frac{\partial s_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_i\frac{\partial s_i}{\partial y}\right) + \frac{\kappa_{i-1}}{\beta_{i-1}}\int_0^t \frac{\partial s_{i-1}}{\partial t}(x, y, \xi)H(t - \xi - t^*_{i,i-1})\,d\xi$$
$$+ \frac{\kappa_i}{\beta_i}\int_0^t \frac{\partial s_{i+1}}{\partial t}(x, y, \xi)H(t - \xi - t^*_{i,i-1})\,d\xi - \int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi)[C_i + I_i\,\delta(t-\xi)]\,d\xi$$
$$+ Q_i = S_i\frac{\partial s_i}{\partial t} \qquad i = 1, 2, \cdots, n \qquad (31)$$

In these equations it must be understood that whenever any nondefined s_{i-1} or s_{i+1} occurs, the corresponding term must be dropped out.

Observe now that

$$\int_0^t \frac{\partial s_{i-1}}{\partial t}(x, y, \xi)H(t - \xi - t^*_{i,i-1})\,d\xi = \int_0^{t^*-t_{i-1}} \frac{\partial s_{i-1}}{\partial t}(x, y, \xi)\,d\xi$$
$$= s_{i-1}(t - t^*_{i,i-1}) \qquad (32a)$$

Similarly

$$\int_0^t \frac{\partial s_{i+1}}{\partial t}(x, y, \xi)H(t - \xi - t^*_{i,i+1})\,d\xi = s_{i+1}(t - t^*_{i,i+1}) \qquad (32b)$$

In addition

$$\int_0^t \frac{\partial s_i}{\partial t}(x, y, \xi)[C_i + I_i\,\delta(t+\xi)]\,d\xi$$
$$= C_i s_i(x, y, t) + I_i\frac{\partial s_i}{\partial t}(x, y, t) \qquad (32c)$$

When equations 32 and the definition

$$S_i^e = S_i + I_i \qquad (33)$$

are used, equation 31 becomes

$$\frac{\partial}{\partial x}\left(T_i\frac{\partial s_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_i\frac{\partial s_i}{\partial y}\right) + \frac{\kappa_{i-1}}{\beta_{i-1}}s_{i-1}(t - t^*_{i,i-1}) + \frac{\kappa_i}{\beta_i}s_{i+1}(t - t^*_{i,i+1}) - C_i s_i(t) + Q_i = S_i^e\frac{\partial s_i}{\partial t}$$
$$i = 1, \cdots, n \qquad (34)$$

where the values of C_i and I_i are given by

$$I_i = \tfrac{1}{3}(\Sigma_{i-1} + \Sigma_i)$$
$$i = 2, \quad , n - 1 \qquad (35a)$$

$$C_i = \frac{\kappa_{i-1}}{\beta_{i-1}} + \frac{\kappa_i}{\beta_i}$$
$$i = 2, \cdots, n - 1 \qquad (35b)$$

For $i = 1$ or $i = n$ we have for case 1

$$I_1 = \tfrac{1}{3}(\Sigma_0 + \Sigma_1) \qquad (35c)$$
$$C_1 = \frac{\kappa_0}{\beta_0} + \frac{\kappa_1}{\beta_1}$$

$$I_n = \tfrac{1}{3}(\Sigma_{n-1} + \Sigma_n) \qquad (35d)$$
$$C_n = \frac{\kappa_{n-1}}{\beta_{n-1}} + \frac{\kappa_n}{\beta_n}$$

For case 2

$$I_1 = \Sigma_0 + \tfrac{1}{3}\Sigma_1 \qquad C_1 = \frac{\kappa_1}{\beta_1} \qquad (35e)$$

$$I_n = \tfrac{1}{3}\Sigma_{n-1} + \Sigma_n \qquad C_n = \frac{\kappa_{n-1}}{\beta_{n-1}} \qquad (35f)$$

For case 3

$$I_1 = \tfrac{1}{3}(\Sigma_0 + \Sigma_1) \qquad C_1 = \frac{\kappa_0}{\beta_0} + \frac{\kappa_1}{\beta_1} \qquad (35g)$$

$$I_n = \tfrac{1}{3}\Sigma_{n-1} + \Sigma_n \qquad C_n = \frac{\kappa_{n-1}}{\beta_{n-1}} \qquad (35h)$$

To obtain the preceding formulas we have used the relations

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^2} = \frac{\pi^2}{2} \qquad (36)$$

In equation 34 it must be understood that the terms containing s_{i-1} or s_{i+1} must be dropped out whenever $i - 1 = 0$ or $i + 1 = n + 1$, respectively.

CONCLUSIONS

The system of equations 34 is very suitable to be treated numerically. In cases of practical interest $\tau^*_{i,i-1}$ and $\tau^*_{i,i+1}$ are large, larger than the time intervals used in the numerical integration. Therefore the system of equations 34 may be treated as an uncoupled system, solving every one of the n equations separately. Then the values of $s_{i-1}(t - t^*_{i,i-1})$ and $s_{i+1}(t - t^*_{i,i+1})$ at every step can be considered as data of the problem because they have been obtained in previous steps. A numerical program using the method presented here is being prepared for application in several cases of practical interest. The results will be reported later.

NOTATION

$b_1, b_2, \cdots, b_n,$	thickness of the main aquifers, L;
$\beta_0, \beta_1, \cdots, \beta_n,$	thickness of the leaky layers (the main aquifer j is limited by the leaky layers $j - 1$ and j (Figure 1)), L;
$h_1, \cdots, h_n,$	piezometric heads at the main aquifers, L;
$H_0, H_1, \cdots, H_n,$	piezometric heads at the semipervious layers, L;
$K_1, \cdots, K_n,$	hydraulic conductivities of the main aquifers, LT^{-1};
$\kappa_0, \kappa_1, \cdots, \kappa_n,$	hydraulic conductivities of the semipervious strata, LT^{-1};
$Q_1, \cdots, Q_n,$	contribution to the discharge of the wells per unit area, of every one of the main aquifers, L^3T^{-1};
$s_1, \cdots, s_n,$	drawdowns at the main aquifers at any point and at any time, L;
$\sigma_0, \sigma_1, \cdots, \sigma_n,$	drawdown at any point of the semipervious layer at any time, L;
$S_j = b_j S_s^{(j)}, j =$	$1, \cdots, n,$ storage coefficients of the main aquifer j;
$\Sigma_j = \beta_j \Sigma_s^{(j)},$	storage coefficient capacity of the semipervious layer j;
$S_s^{(j)},$	specific storage of the main aquifer j, L^{-1};
$\Sigma_s^{(j)},$	specific storage of the leaky layer j, L^{-1};
$t,$	time measured from some reference time, T;
$T_j = K_j b_j,$	transmissivity of the main aquifer j, L^2T^{-1};
$\tau_j = \kappa_j \beta_j,$	transmissivity of the leaky layer j, L^2T^{-1};
$z,$	vertical coordinate, L;
$z_0,$	vertical coordinate of the top of the flow system, L;
$z_j = z_0 - \sum_{i=0}^{j-1} \beta_i, j = 1, \cdots, n + 1,$	L;
$n,$	number of main aquifers;
$\nu_j = K_j/S_s^{(j)} = T_j/S_j,$	L^2T^{-1};
$\nu_j' = \kappa_j/\Sigma_s^{(j)} = \tau_j/\Sigma_j,$	L^2T^{-1}.

Acknowledgments. This work was sponsored by the Secretaría de Recursos Hidráulicos of México. I want to thank also Mr. Oscar Falcón for his assistance in several aspects of this work.

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(Manuscript received June 23, 1969; revised August 20, 1969.)