VOL. 6, NO. 3 WATER RESOURCES RESEARCH JUNE 1970

Reply

ISMAEL HERRERA

Instituto de Geofísica, Universidad Nacional Autónoma de México, México, D. F., México

GERMÁN E. FIGUEROA V.

Comisión Hidrológica de la Cuenca del Valle de México, S.R.H., México, D. F., México

We are especially pleased to read the comments on our paper [*Herrera and Figueroa,* 1969] by Neuman and Witherspoon because now that an exact solution for a particular case is available [*Neuman and Witherspoon,* 1969*a*], it is possible to discuss more thoroughly the range of applicability of the approximation we propose.

Neuman and Witherspoon assert in their comments that the applicability of our theory is limited to the steady state or at best, a quasi-steady state. In support of their statement they say that the 'solution previously obtained by *Hantush* [1960] for large values of time is restricted largely to the steady state.'

This statement is very interesting because it implies that the solution for large values of time that *Hantush* [1960] obtained using his modified theory of leaky aquifers is no better than the r/B solution, which also predicts the steady state correctly. However, this assertion seems to contradict what *Neumàn and Witherspoon* [1969*b*, p. 818] wrote when discussing the applicability of current theories of flow in leaky aquifers: 'We shall demonstrate later that Hantush's solutions are quite good over a broader time span than he has indicated.'

In their comments they refer to Figures 2, 3, and 4 of the paper by *Neuman and Witherspoon* [1969*b*] and state that their equation 5, which is the same as our equation 26, corresponds exclusively to the horizontal. However, those figures indicate only the range of applicability that Hantush expected for his theory. As mentioned before, Neuman and Witherspoon have found that the range of applicability of Hantush's results is broader than he has anticipated, so it is necessary to find out which is the actual range of applicability.

Hantush's approximation for large values of time is the same as the r/B solution, except that the time is divided by the factor

$$1 + \frac{S'}{3S} = 1 + \frac{16}{3}\frac{\beta^2}{(r/B)^2} \qquad (1)$$

Since Neuman and Witherspoon in Figures 2, 3, 4, and 5 have used logarithmic scales, we need only make a translation to the right of every one of the dotted curves appearing in their Figure 5 by amounts determined by equation 1 above.

We have done that and obtained Figures 1, 2, and 3, which illustrate the comparison between Hantush's solution for large values of time and the exact solution given by Neuman and Witherspoon in their Figures 2, 3, and 4.

It is clear from the figures that for the values of the parameters corresponding to Figures 1 and 2 the approximation gives results that are in agreement with the exact solution over the whole time range for most cases.

It is important to recall that the approximation for large values of time is satisfactory for all cases illustrated in Figures 1, 2, and 3 on a time range whose lower limit is smaller than or equal to the upper bound of *Hantush*'s [1960] definition of small values of time.

Therefore the ranges where Hantush's solution for small and large values of time is acceptable overlap each other and thus, when taken together, cover the whole time range. It may be seen in Figures 1, 2, and 3 that the approximation for large values of time covers in all cases a region of nonsteady flow that begins before

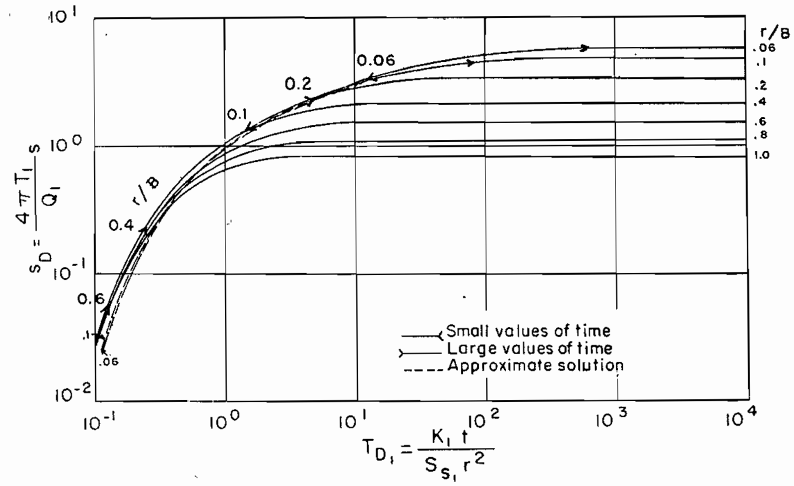

Fig. 1. Comparison of exact solution with Hantush's approximation for large values of time, when $\beta = 0.01$.

the end of the region corresponding to Hantush's small values of time. Therefore we conclude that there are no grounds for Neuman and Witherspoon's statement that the example we have treated with the method we propose is restricted to the steady state. On the contrary, Neuman and Witherspoon's results corroborate that the method is suitable for application to numerical studies of regional evolution of piezometric heads.

This result is very important because our method has been extended to include very general systems of multilayered aquifers [*Herrera*, 1970].

We proceed now to discuss the theoretical arguments presented by Neuman and Witherspoon in their comments. To simplify the discussion, assume that K_2 vanishes. Then

$$F(t) = \frac{K_1}{Tb_1} g\left(\frac{\pi^2 \nu_1}{b^2} t\right) \qquad (2)$$

where for case 1

$$g(\tau) = 2 \sum_{n=1}^{\infty} e^{-n^2 \tau} \qquad (3a)$$

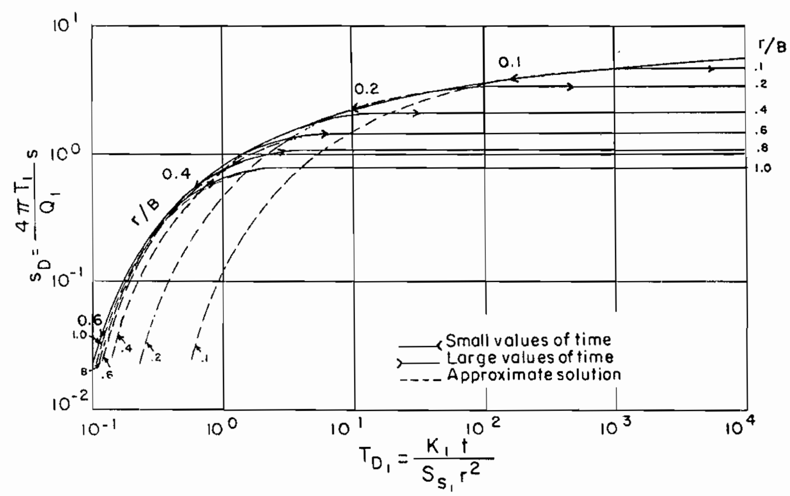

Fig. 2. Comparison of exact solution with Hantush's approximation for large values of time, when $\beta = 0.1$.

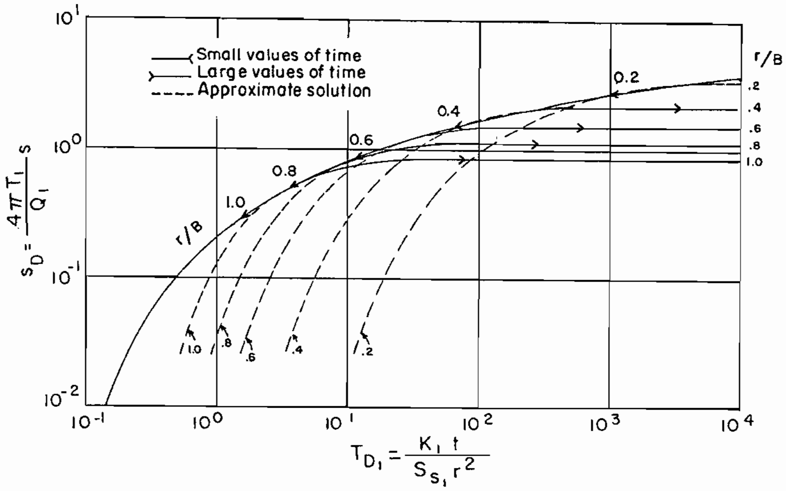

Fig. 3. Comparison of exact solution with Hantush's approximation for large values of time, when $\beta = 1.0$.

and for case 2

$$g(\tau) = 2 \sum_{n=0}^{\infty} \exp\left[-(n + \tfrac{1}{2})^2 \tau\right] \qquad (3b)$$

Case 3 is the same as case 1 and will not be considered.

We have used the approximation

$$\int_0^t \frac{\partial s}{\partial t}(\tau) F(t - \tau)\, d\tau \approx \frac{\partial s}{\partial t}(t) \int_0^{\infty} F(\tau)\, d\tau \qquad (4)$$

Thus the error δ introduced is

$$\delta = \left| \int_0^t \frac{\partial s}{\partial t}(\tau) F(t - \tau)\, d\tau - \frac{\partial s}{\partial t}(t) \int_0^{\infty} F(\tau)\, d\tau \right| \qquad (5)$$

Because of the identity

$$\int_0^t \frac{\partial s}{\partial t}(\tau) F(t - \tau)\, d\tau = \int_0^t \frac{\partial s}{\partial t}(t - \tau) F(\tau)\, d\tau \qquad (6)$$

we can write

$$\delta \le \int_0^t \left| \frac{\partial s}{\partial t}(t - \tau) - \frac{\partial s}{\partial t}(t) \right| F(\tau)\, d\tau + \left| \frac{\partial s}{\partial t}(t) \right| \int_t^{\infty} F(\tau)\, d\tau \qquad (7)$$

where we used the fact that F is positive.

Let t_ϵ be such that $0 < t_\epsilon < t$ and M an upper bound of $|(\partial s/\partial t)(\tau)|$ in the range $0 \le \tau \le t$. Then

$$\delta \le \int_0^{t_\epsilon} \left| \frac{\partial s}{\partial t}(t - \tau) - \frac{\partial s}{\partial t}(t) \right| F(\tau)\, d\tau + 2M \int_{t_\epsilon}^{t} F(\tau)\, d\tau + M \int_t^{\infty} F(\tau)$$
$$\le \int_0^{t_\epsilon} \left| \frac{\partial s}{\partial t}(t - \tau) - \frac{\partial s}{\partial t}(t) \right| F(\tau)\, d\tau + 2M \int_{t_\epsilon}^{\infty} F(\tau)\, d\tau \qquad (8)$$

Given any small number $\epsilon > 0$ we can choose t_ϵ so that

$$\int_{t_\epsilon}^{\infty} F(\tau)\, d\tau = \epsilon \int_0^{\infty} F(\tau)\, d\tau = \epsilon I \qquad (9)$$

because F decays exponentially as t grows to infinite. Now let Δ be such that

$$\left| \frac{\partial s}{\partial t}(t - \tau) - \frac{\partial s}{\partial t}(t) \right| \le \Delta \qquad t - t_\epsilon \le \tau \le t \qquad (10)$$

Then the error

$$\delta \le \Delta \int_0^{t_\epsilon} F(\tau)\, d\tau + 2\epsilon M \int_0^{\infty} F(\tau)\, d\tau \le \Delta I + 2\epsilon I M \qquad (11)$$

Since I and M are fixed numbers, the error is controlled by ϵ and Δ. If ϵ is chosen to reduce its contribution to the error given by (11) to a desired size, then we obtain in that manner some t_ϵ that in turn will determine the value of Δ by means of (10).

Therefore $(\partial s/\partial t)$ (τ) must vary little in the period $t - t_\epsilon$ to t, but not in the entire period 0 to t as stated by Neuman and Witherspoon in their comments. The length of the time interval $t - t_\epsilon$ to t in general may be much smaller than the length of the time interval 0 to t. Thus the early variation of s is irrelevant.

Furthermore, t_ϵ is uniquely determined once F is given and therefore depends on the characteristics of the aquitard only; i.e., t_ϵ is independent of the characteristics of the aquifer.

The number t_ϵ is a parameter characterizing the time required by the memory to forget and has no connection with the time required for the aquifer to reach the steady state.

The definition of t_ϵ is given by equation 9 and can be easily computed. For example, for $\epsilon = .03$ using (2) and (3*a*) we obtain

$$t_{.03} = \frac{3S_1b_1}{\pi^2 K_1} \qquad (12)$$

The approximation we have used in our work [*Herrera and Figueroa*, 1969] could be anticipated on more general grounds because of the shape of the functions g illustrated in Figure 4. These functions are very large when their argument is small and very small when their argument is large. This situation implies that the memory in our integrodifferential equation is fading, in a sense similar to the definition of fading memory for viscoelastic materials [*Truesdell and Noll*, 1965]. In brief, this concept implies that changes occurring in the distant past have less influence than those occurring in the recent past.

The above discussion shows that on theoretical grounds the validity of the approximation we have developed [*Herrera and Figueroa*, 1969] is well founded. On the other hand, Figures 1, 2, and 3 show that its range of applicability is within the region of practical interest.

The work we have done so far [*Herrera and Figueroa*, 1969; *Herrera*, 1970] has been devoted to the use of equation 14 in this paper and similar integrodifferential equations to obtain approximations for long times of operation. However, those equations can be used to obtain much better approximations, which could be used over the whole time span. For example, the approximation used in our paper to obtain (17) from (14) is equivalent to approximating the memory function $F(t)$ by a delta function, i.e.,

$$F(t) \approx I\ \delta(t) \qquad (13)$$

which is a suitable approximation because of equation 2 and the shape of the functions g shown in Figure 4.

The approximation (13) has the following physical interpretation. When a change occurs in the piezometric head of the aquifer, some water is supplied by consolidation of the aquitard. The time distribution of this water supply is given by $F(t)$, and (13) corresponds to the assumption that all the supply is given immediately after the occurrence of the change in the piezometric head of the aquifer. Obviously a better approximation would be

$$F(t) \approx A\ \delta(t) + BH(t_0 - t) \qquad t_0 > 0 \qquad (14)$$

where H is Heaviside unit step function and A, B, and t_0 can be chosen so that the time distribution of the water supply given by the right-hand member of (14) be as close as possible to the time distribution given by $F(t)$. However, A, B, and t_0 must satisfy the restriction

$$A + Bt_0 = I \qquad (15)$$

to assure that the total volume of water yielded by the aquitard is correctly given by the approximation (14).

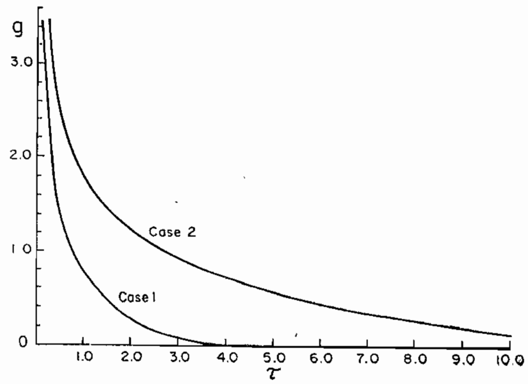

Fig. 4. The shape of functions g.

If this were done, equation 17 of our paper [*Herrera and Figueroa*, 1969] would be replaced by

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - [C + B]s(t) + Bs(t - t_0) = \left(\frac{1}{\nu} + A\right)\frac{\partial s}{\partial t} \qquad (16)$$

which is almost as convenient for numerical application to studies of regional evolution as equation 17.

Finally, we would like to point out some misprints in our paper. Both negative signs in (11) should be replaced by a plus sign. In equations 8, 9, 11, 14, 15, and 25, $\partial\tau$ should be replaced by $d\tau$.

REFERENCES

Hantush, M. S., Modification of the theory of leaky aquifers, *J. Geophys. Res., 65*(11), 3713–3725, 1960.

Herrera, I., and G. E. Figueroa, A correspondence principle for the theory of leaky aquifers, *Water Resour. Res., 5*(4), 900–904, 1969.

Herrera, I., Theory of multiple leaky aquifers, *Water Resour. Res., 6*(1), 185–193, 1970.

Neuman, S. P., and P. A. Witherspoon, Theory of flow in a two aquifer system, *Water Resour. Res., 5*(4), 803–816, 1969*a*.

Neuman, S. P., and P. A. Witherspoon, Applicability of current theories of flow in leaky aquifers, *Water Resour. Res., 5*(4), 817–829, 1969*b*.
bility of current theories of flow in leaky aqui-
Truesdell, C., and W. Noll, Non-linear field theories of mechanics, in *Encyclopedia of Physics*, vol. 3/3, pp. 101–106, Springer-Verlag, Berlin, 1965.

(Manuscript received October 31, 1969.)