COMPUTATIONS USING A SIMPLIFIED THEORY OF MULTIPLE LEAKY AQUIFERS

ISMAEL HERRERA* AND
LEOPOLDO RODARTE**

RESUMEN

Se hacen algunas aplicaciones del sistema de ecuaciones diferenciales desarrollado por Herrera y Figueroa (1969), Herrera (1970) para acuíferos semiconfinados múltiples. Las soluciones así obtenidas se compararon con algunas soluciones exactas disponibles, encontrándose una coincidencia muy satisfactoria entre ambas para todos los tiempos en los cuales la interacción entre los acuíferos es significativa.

ABSTRACT

Computations are carried out using the simplified differential system developed by Herrera and Figueroa (1969), Herrera (1970) for multiple leaky aquifers. Very good agreement with the exact solutions was found, in all the ranges of time for which interaction between the aquifers is relevant.

* *Instituto de Geofísica, UNAM.*
** *Instituto de Geofísica, UNAM.*

NOTATIONS

b_i	Thickness of ith aquifer, L;
b'	Thickness of the aquitard, L;
C_i	$K'/T_i b'$; $i = 1, 2$; L^{-2};
$f(t')$	$1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t'}$, memory function
$F(r,t,C_1,C_2)$	Auxiliary function given by equation (4.2)
$h(t')$	$1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t'}$, influence function;
K_i	Permeability of ith aquifer, L/T ;
K'	Permeability of the aquitard, L/T ;
Q	Pumping rate from aquifer 1, L^3/T ;
r	Radial distance to the pumping well, L ;
S_{si};	Specific storage of ith aquifer L^{-1} ;
S'_S;	Specific storage of aquitard L^{-1} ;
S_i;	Storage coefficient of ith aquifer $(S_{si} b_i)$;
S';	Storage coefficient of aquitard $(S'_s b')$;
s_i;	Drawdown in ith aquifer, L;
s';	Drawdown in aquitard, L ;
T_i	Transmissibility of ith aquifer $(K_i b_i)$, L^2/T ;
t	Time, T ;
t'	Dimensionless time $(\alpha' t/b'^2)$;

$$W(\xi, \eta) = \int_{\xi}^{\infty} \frac{dy}{y} \exp\left(-y - \frac{\eta^2}{4y}\right).$$ The well function for leaky aquifers.

$$w(r,t,C_1) = \frac{1}{t} \exp \left\{ - \left[\frac{r^2}{4\alpha_{c_1} t} + C_1 \alpha_{c_1} t \right] \right\}$$

x, y, z	coordinates, L;
	$\alpha_{ci} = \dfrac{3T_i}{3S_i + S'}$
α_i	$K_i/S_{si} = T_i/S_i$, $L^2\ T^{-1}$;
α'	K'/S'_{si}, $L^2\ T^{-1}$
$\delta(t')$	Dirac's delta function
$\omega(\varphi, t')$	Auxiliary function given by (2. 4. c)

1. INTRODUCTION

In previous papers (Herrera and Figueroa, 1969; Herrera, 1970) a method of uncoupling the system of partial differential equations governing the flow in multiple systems of leaky aquifers, was developed. Essentially what was done, was to transform the system of differential equations into a system of integro-differential equations, in which the integrals involved can be interpreted as the memory of the system. Then, by suitable approximations, an uncoupled system of differential equations was obtained.

It is clear, that, when they are applicable, the above results can be used to simplify the numerical methods available (Javandel and Witherspoon, 1968 a, 1968 b, and 1969) because in that manner one of the independant variables of the problem is eliminated and consequently, the size of the matrices involved is reduced. However, these results are based on the approximation of the memory function f by a Dirac's delta function and the influence function h, by a Heaviside unit step function; consequently, the range of applicability of the method is not unlimited.

It has been shown previously, that the approximation of the memory function by the delta function is equivalent to Hantush's approximation for long times and a brief discussion of its range of applicability is already available (Neuman and Whiterspoon, 1970; Herrera and Figueroa, 1970). In this paper, we present some of the results we have obtained using the approximate system of differential equations, for a two aquifer system. Our conclusions can be summarized as follows: For all the cases considered, the results obtained using the equations of Herrera and Figueroa (1969), and Herrera (1970), agree very well with those obtained using the exact equations, in a range of times whose lower limit is much smaller than the time at which the influence of the unpumped aquifer on the pumped aquifer, begins to be noticeable. It seems therefore, that the approximation of the influence function h, by a unit step function is quite suitable. We conclude that the method of Herrera and Figueroa (1969), Herrera (1970) can safely be applied in a range of time which is not limited by above, and whose lower limit is the same as that for the range of applicability of the approximation of the memory function by a delta function. This limit in turn, is the same as that for Hantush's long time approximation (Herrera and Figueroa, 1969), which has been shown (Herrera and Figueroa, 1970) to be much smaller than Hantush had anticipated.

Although, the simplified system of differential equations can be used safely in the range indicated above, it would be desirable to develop a simplified method of more general applicability. The above results indicate that to achieve this, attention can be restricted to improve the approximation of the memory function and work to this end is under way (Herrera and Rodarte, 1973; Rodarte, 1973).

2. INTEGRO-DIFFERENTIAL EQUATIONS GOVERNING THE MOTION

For the case of the two aquifer system illustrated in figure 1, the exact integro-differential equations of motion (Herrera, 1970), for problems with axial symmetry can be written as:

$$\frac{\partial^2_{Si}}{\partial r^2} + \frac{1}{r} \frac{\partial s_i}{\partial r} + \sum_{j=1}^{2} \int_0^t \frac{\partial s_j}{\partial t} (x, y, \tau) \, G_{ij} \, (t-\tau) \, d\tau$$

$$= \frac{1}{\alpha_i} \frac{\partial s_i}{\partial t} \quad ; \quad i = 1, 2 \tag{2.1.a}$$

and [Herrera and Rodarte, 1973]:

$$s'(r,z,t) = \int_0^t \frac{\partial s_i}{\partial t} (r, z, t-\tau) \, u(z, \tau) \, d\tau +$$

$$\int_0^t \frac{\partial s_2}{\partial t} (r, z, t-\tau) \, \nu(z, \tau) \, d\tau \tag{2.1.b}$$

where

$$u(z,t) = \omega(\zeta, t') \tag{2.2.a}$$

$$\omega(\zeta, t') = 1 - \zeta - 2 \sum_{n=1}^{\infty} \frac{e^{-n^2 \pi^2 t'}}{n\pi} \sin n\pi\zeta \tag{2.2.b}$$

$$\nu(z, t) = u(b' - z, t) \tag{2.2.c}$$

$$t' = \alpha' t / b'^2 \tag{2.2.d}$$

and

$$\zeta = z / b' \tag{2.2.e}$$

On the other hand

$$G_{ij}(t) = - C_i f(t'); \quad \text{if } i = j \tag{2.3.a}$$

$$G_{ij}(t) = C_i h(t'); \quad \text{if } i \neq j \tag{2.3.b}$$

where

$$f(t') = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t'} \qquad (2.3.c)$$

$$h(t') = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t'} \qquad (2.3.d)$$

$$C_i = K'/T_i b' \qquad (2.3.c)$$

Herrera and Figueroa (1969), Herrera (1970), Herrera and Rodarte (1973), have approximated the functions f and h by:

$$f(t') \approx 1 + \delta(t')/3 \qquad (2.4.a)$$

$$h(t') \approx H\ (t' - 1/6) \qquad (2.4.b)$$

where δ and H are respectively Dirac's delta function and Heaviside unit step function.

When these approximations are used equations (2.1) become

$$\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r}\frac{\partial s_i}{\partial r} - C_i s_i + C_i s_j (r, t-t^*) = \frac{1}{\alpha_{ic}}\frac{\partial s_i}{\partial t}$$

$$i = 1, 2 \ ; \quad j \neq i \qquad (2.5)$$

where

$$t^* = \frac{(b')^2}{6\,\alpha'} \tag{2.6}$$

To carry numerical computations, by use of methods such as the finite element method, equations (2.5) are much easier to handle than the original system of partial differential equations, because they are uncoupled.

3. THE WELL PROBLEM

We consider the problem of an isolated steady well when only aquifer 1 is being pumped. In this case, equations (2.5) must be suplemented by the boundary and initial conditions:

$$s_1\,(r, 0) = 0 \quad ; \quad s_2\,(r, 0) = 0 \tag{3.1.a}$$

$$\lim_{r\to 0} r\,\frac{\partial s_1}{\partial r} = -\frac{Q_1}{2\pi T_1}\ ; \ \lim_{r\to 0} r\,\frac{\partial s_2}{\partial r} = 0 \tag{3.1.b}$$

$$s_1\,(\infty, t) = 0 \ ; \ s_2\,(\infty, t) = 0 \tag{3.1.c}$$

4. SOLUTION OF THE WELL PROBLEM

To check the accuracy of the method, we have compared the solutions of the system (2.5) and (3.1) whit those of the exact system (Neuman and Witherspoon, 1969 a, 1969 b).

We have obtained analytical solutions. In Appendix, it is shown that they are:

$$s_1\,(r,t) = \frac{Q}{4\pi T_1}\left\{W\left(\frac{r^2}{4\alpha_{c\,1}\,t}, C_1^{1/2}\, r\right)\right.$$

$$\sum_{n=1}^{n=N_1} \frac{C_1^n C_2^n}{n!\,(n-1)!} \int_{2nt*}^{t} \frac{\partial^{2n-1} F}{\partial C_1^n \partial C_2^{n-1}} (r, \tau - 2nt*)\, d\tau \Bigg\} \quad (4.1.a)$$

$$s_2 (r,t) = \frac{Q}{4\pi T_1} \sum_{n=0}^{n=N_2} \frac{C_1^n C_2^{n+1}}{(n!)^2} \int_{(2n+1)t*}^{t} \frac{\partial^{2n}}{\partial C_1^n \partial C_2^n} F (r, \tau - [2n+1]t*)\, d\tau$$

(4.1.b)

where N_1 and N_2 are chosen as the largest integrers that make the lower limits in the integrals smaller than the upper limits. On the other hand

$$F (r, t, C_1, C_2) = \frac{\alpha_{c_1} \alpha_{c_2}}{\alpha_{c_1} - \alpha_{c_2}}\, e^{-\frac{\alpha_{c_1} \alpha_{c_2}}{\alpha_{c_1} - \alpha_{c_2}} (C_2 - C_1) t}$$

$$\int_{\frac{r^2}{4\alpha_{c_1} t}}^{\frac{r^2}{4\alpha_{c_2} t}} \frac{e^{-\left[y + \frac{C_1 \alpha_{c_1} - C_2 \alpha_{c_2}}{4(\alpha_{c_1} - \alpha_{c_2}) y} \right]}}{y}\, dy \quad (4.2.)$$

if $\alpha_{c_1} > \alpha_{c_2}$. If $\alpha_{c_1} < \alpha_{c_2}$, F is also given by (4.2) except that the subscripts 1 and 2 must be interchanged. If $\alpha_{c_1} = \alpha_{c_2}$ we do not define F and special formulas have to be developed for these cases. An important case is that for which both aquifers have the same properties, so that $\alpha_{c_1} = \alpha_{c_2}$ and $C_1 = C_2$.

5. SOLUTION FOR THE CASE WHEN THE AQUIFERS HAVE IDENTICAL PROPERTIES

In the Appendix it is shown that when the aquifers have identical properties

$$s_1(r, t) = \frac{Q}{4\pi T_1}\left\{W\left(\frac{r^2}{4\alpha_{c_1} t}, C_1^{1/2} r\right) + \sum_{n=1}^{n=N_1} \frac{C_1^{2n}}{(2n)!} \int_{2nt^*}^{t} \frac{\partial^{2n} w}{\partial C_1^{2n}}\right.$$

$$\left.(r, \tau\text{-}2nt^*)\, d\tau\right\} \qquad (5.1.a)$$

$$s_2(r, t) = \frac{Q}{4\pi T_1} \sum_{n=0}^{n=N_2} \frac{C_1^{2n+1}}{(2n+1)!} \int_{(2n+1)t^*}^{t} \frac{\partial^{2n+1} w}{\partial C_1^{2n+1}}$$

$$(r, \tau\text{-}[2n+1]t^*)\, d\tau \qquad (5.1.b)$$

where N_1 and N_2 are chosen in the way explained in section 3. On the other hand

$$w(r, t, c_1) = \frac{1}{t}\, e^{-\left[\frac{r^2}{4\alpha_{c_1} t} + C_1 \alpha_{c_1} t\right]} \qquad (5.2)$$

EVALUATION OF SOLUTIONS

We have evaluated the solutions in the way explained in the Appendix for the case of identical properties of the aquifers, and have compared our results with those obtained by Neuman and Witherspoon (1969 a, b). The results are shown in the figures 2, 3, 4 and 5.

To facilite the comparison we have included in the figures the solutions for the case when the drawdown at the unpumped aquifer is taken equal to zero. The point at which this solution and that for which the aquifers have identical properties, deviate from each other, corresponds to the time at which tne interaction between the aquifers begins to be noticeable and therefore, after this time that interaction must be taken into account when computing the drawdown in the pumped aquifer. Observing the figures, it becomes clear that for the cases considered, the exact and approximate solutions coincide in the pumped aquifer, for a range of time whose lower limit is much smaller than the time at which the interaction between the aquifers starts to be noticeable.

On the other hand, at the unpumped aquifer there is a noticeable difference between the exact and the approximate solutions, only at the early stages, when the drawdown in this aquifer is very small in comparison with that at the pumped aquifer. The situation in the aquitard lies between that at the pumped and the unpumped aquifers.

APPENDIX

Applying Hankel and Laplace transforms succesively to equations (2.1.a), we get

$$\bar{\bar{s}}_1 = \frac{D(a^2 + A_2)}{(a^2 + A_1)(a^2 + A_2) - B_1 B_2} \qquad \text{(A.1.a)}$$

$$\bar{\bar{s}}_2 = \frac{DB_2}{(a^2 + A_1)(a^2 + A_2) - B_1 B_2} \qquad \text{(A.1.b)}$$

where

$$A_i(p) = p\left(\frac{1}{\alpha_i} - \bar{G}_{ii}\right) ; i = 1, 2; \qquad \text{(A.2.a)}$$

$$B_i(p) = p\,\bar{G}_{ij} \;;\; i = 1, 2 \;; j \neq i. \qquad \text{(A.2.b)}$$

$$D = \frac{Q_1}{2\pi T_1 p} \qquad \text{(A.2.c)}$$

Observe that

$$\frac{1}{(a^2 + A_1)(a^2 + A_2) - B_1 B_2} = \sum_{n=0}^{\infty} \frac{B_1^n B_2^n}{(a^2 + A_1)^{n+1} \, (a^2 + A_2)^{n+1}} \qquad \text{(A.3)}$$

Thus

$$\bar{\bar{s}}_1 = D \sum_{n=0}^{\infty} \frac{(B_1 B_2)^n}{(a^2 + A_1)^{n+1} (a^2 + A_2)^n} \tag{A.4.a}$$

$$\bar{\bar{s}}_2 = D \sum_{n=0}^{\infty} \frac{B_1^n B_2^{\,n+1}}{(a^2 + A_1)^{n+1} \quad (a^2 + A_2)^{n+1}} \tag{A.4.b}$$

so that

$$\bar{\bar{s}}_1 = \frac{D}{a^2 + A_1} - D \sum_{n=1}^{\infty} \frac{(B_1 B_2)^n}{n!\,(n+1)!} \frac{\partial^{2n-1}}{(\partial A_1)^n (\partial A_2)^{n-1}}$$

$$\frac{1}{(a^2 + A_1)(a^2 + A_2)} \tag{A.5.a}$$

$$\bar{\bar{s}}_2 = D \sum_{n=0}^{\infty} \frac{B_1^n B_2^n +}{(n!)^2} \frac{\partial^{2n}}{(\partial A_1)^n (\partial A_2)^n} \frac{1}{(a^2 + A_1)(a^2 + A_2)} \tag{A.5.b}$$

when the approximations (2.4) are introduced, equations (A.2) become

$$A_i(p) = \frac{p}{\alpha_{c_i}} + C_i \tag{A.6.a}$$

$$B_i(p) = C_i\, e^{-pt*} \tag{A. 6.b}$$

$$D = \frac{Q_1}{2\pi T_1 p} \tag{A.6.c}$$

Observe that (A.6.a) imply that

$$\frac{\partial A_i}{\partial C_i} = 1 \tag{A.6.d}$$

and consequently the partial derivatives in (A.5) can be taken with respect to C_1 and C_2.

Taking inverse Hankel and Laplace transforms, we get:

$$\mathcal{L}^{-1}\left\{H^{-1}\left(\frac{1}{a^2+A_1}\cdot\frac{1}{a^2+A_2}\right)\right\}=\frac{1}{2}\ F\,(r,t,C_1,C_2) \qquad (A.7.a)$$

$$\mathcal{L}^{-1}\left\{B_i\right\}=C_i\ \delta\,(t-t^*) \qquad (A.7.b)$$

where $F(r,t,C_1,C_2)$ is given by (4.2).

Taking the inverse Hankel and Laplace transforms of (A.5), we get (4.1) in view of the above equations.

To obtain (4.1), we have to assume $\alpha_{c1} \neq \alpha_{c2}$. This assumption leaves out the case when both aquifers have identical properties. For that case equations (A.4) become

$$\bar{\bar{s}}_1 = D\sum_{n=1}^{\infty}\frac{B^{2n}}{(a^2+A_1)^{2n+1}} \qquad (A.8.a)$$

$$\bar{\bar{s}}_2 = D\sum_{n=0}^{\infty}\frac{B_1^{2n+1}}{(a^2+A_1)^{2n+2}} \qquad (A.8.b)$$

and

$$\bar{\bar{s}}_1 = \frac{D}{a^2+A_1} + D\sum_{n=1}^{\infty}\frac{B_1^{2n}}{(2n)!}\ \frac{\partial^{2n}}{\partial A_1^{2n}}\left(\frac{1}{a^2+A_1}\right) \qquad (A.9.a)$$

$$\bar{\bar{s}}_2 = -D\sum_{n=0}^{\infty}\frac{B_1^{2n+1}}{(2n+1)!}\ \frac{\partial^{2n+1}}{\partial A_1^{2n+1}}\left(\frac{1}{a^2+A_1}\right) \qquad (A.9.b)$$

Taking inverse Hankel and Laplace transforms of equations (A.9), we get (5.1) after making use of the fact that

$$\mathcal{L}^{-1}\left\{H^{-1}\left(\frac{1}{a^2 + A_1}\right)\right\} = \frac{1}{2}\, w\,(r, t, C_1) \qquad \text{(A.10)}$$

where w is given by (5.2).

The evaluation of the integrals in equations (5.1) can be simplified by repeated use of the recurrence relations:

$$\int_0^t \frac{\partial^n w}{\partial C_1^n}(r,\tau)\, d\tau = \frac{1}{C_1 \alpha_{c_1}}\left[\frac{r^2}{4\alpha_{c_1}} \int_0^t \frac{\partial^{n-2} w}{\partial C_1^{n-2}}(r,\tau)\, d\tau \right.$$

$$\left. + (n-1)\int_0^t \frac{\partial^{n-1} w}{\partial C_1^{n-1}}(r,\tau)\, d\tau - \frac{\partial^n w}{\partial C_1^n}(r, t)\right] \; ; n \geqslant 2 \qquad \text{(A.11)}$$

Using (A.11), it is possible to express all the integrals appearing in (5.1) in terms of the well function w and of

$$\int_0^t e^{-\left(\frac{r^2}{4\alpha_{c_1}\tau} + C_1 \alpha_{c_1} \tau\right)}\, d\tau$$

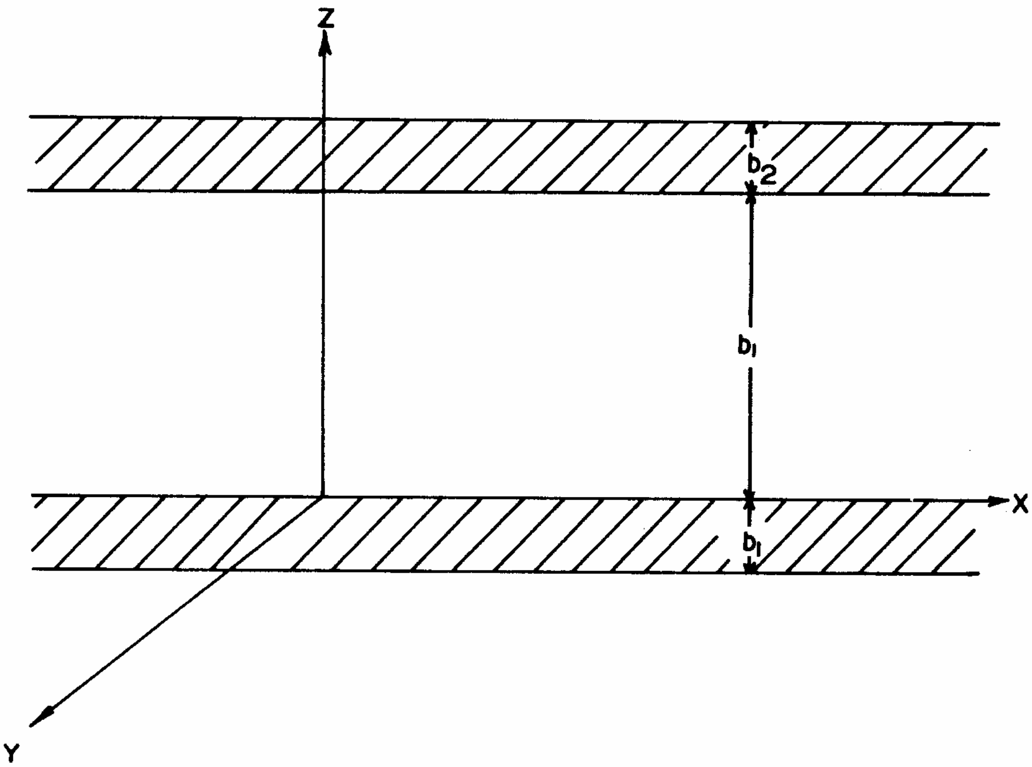

Figure 1. The aquifer system.

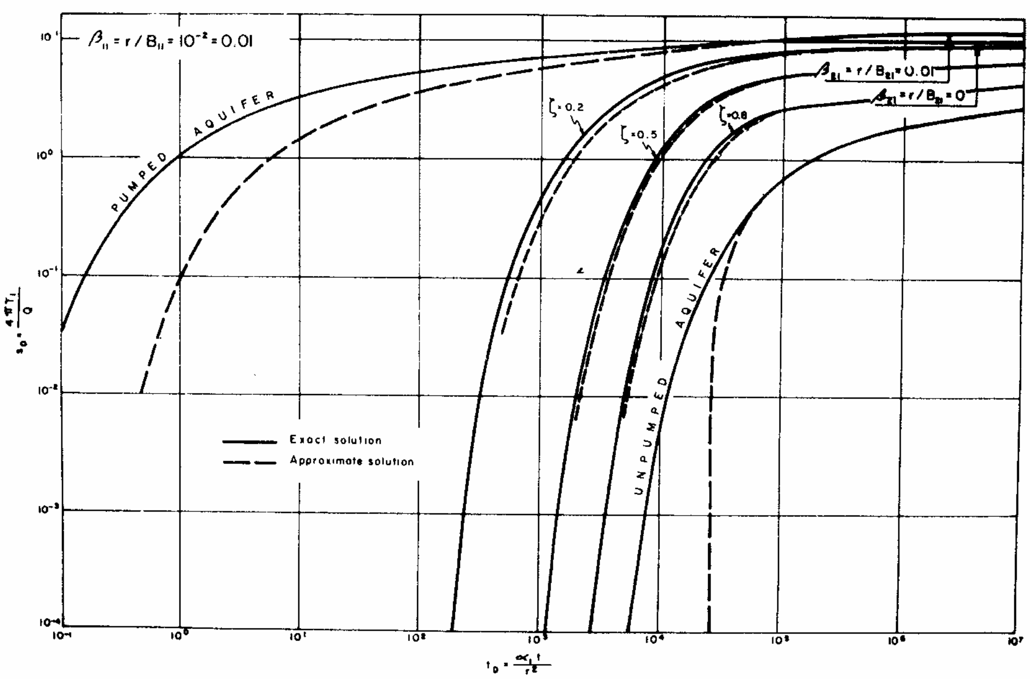

Figure 2. Comparison between exact and approximate solutions ($\beta_{11} = {}^r/_{B_{11}} = 10^{-2}$).

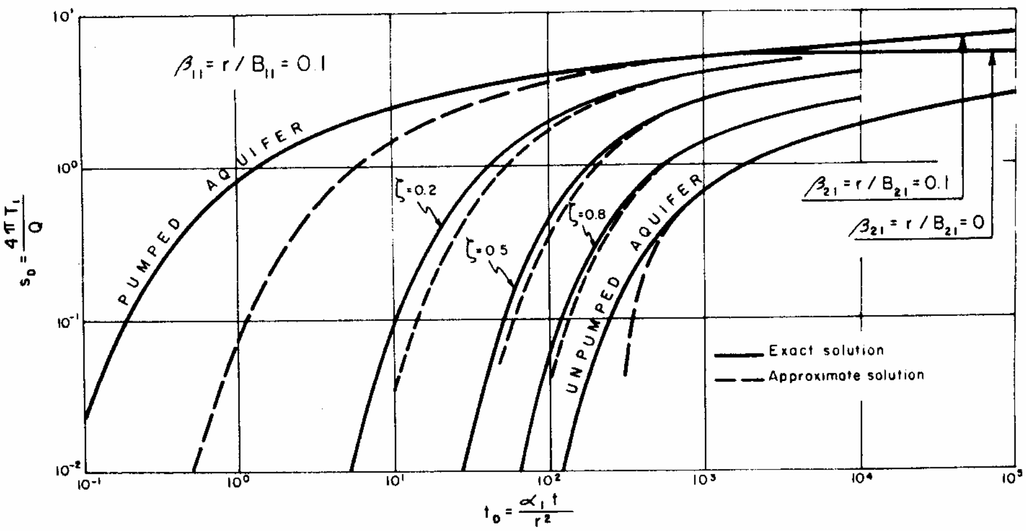

Figure 3. Comparison between exact and approximate solutions ($\beta_{11} = {}^r/_B = 10^{-1}$).

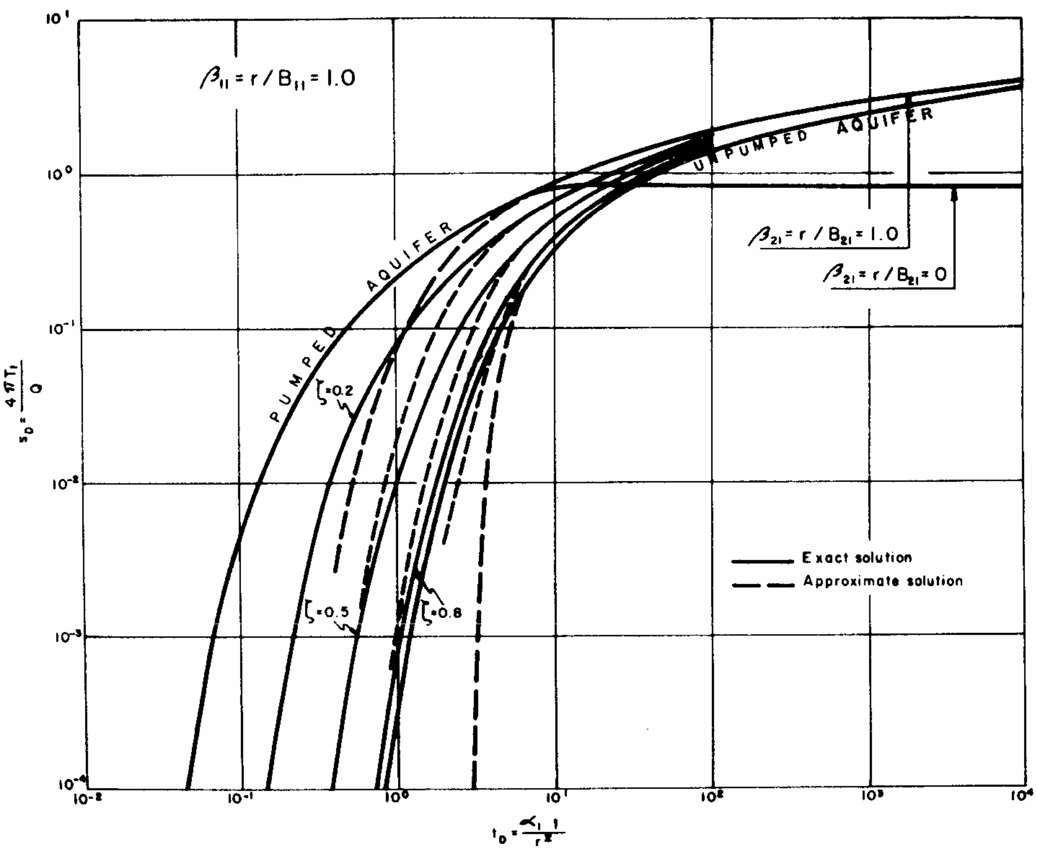

Figure 4. Comparison between exact and approximate solutions ($\beta_{11} = {}^r/_B = 1.0$).

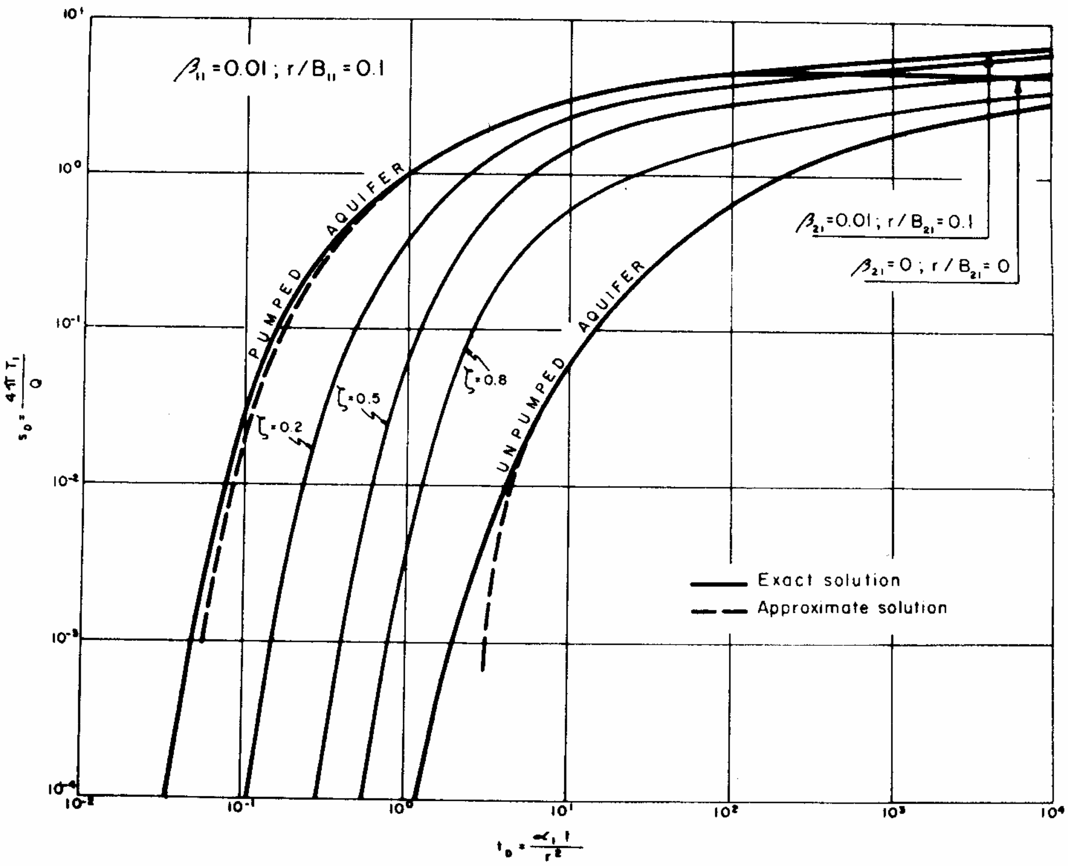

Figure 5. Comparison between exact and approximate solutions ($\beta_{11} = 10^{-2}$; $r/B = 10^{-1}$).

ACKNOWLEDGMENTS

This work has been sponsored by the Secretaría de Recursos Hidráulicos, México.

BIBLIOGRAPHY

HERRERA, I., Theory of Multiple Leaky Aquifers, *Water Resour. Res.* 6 (1), 1970.

HERRERA, I., and G.E. FIGUEROA, A Correspondence Principle for the Theory of Multiple Leaky Aquifers, *Water Resour. Res.* 5 (4), 1969.

HERRERA, I., and G.E. FIGUEROA, Reply to Comments by Shlomo P. Neuman and Paul A. Witherspoon, *Water Resour. Res.* 6 (3), pp. 1011-1015, 1970.

HERRERA, I., and L. RODARTE, Integro-Differential Equations for Leaky Aquifers and applications Part 1, *Water Resour. Res.* 1973 (to appear).

JAVANDEL, I., and P.A. WITHERSPOON, Analysis of Transient Fluid Flow in Multilayered Systems, Contribution 124, *Water Resources Center,* University of California, Berkeley, 1968 a.

JAVENDEL I., and P.A. WITHERSPOON, Application of the finite element method to transient flow in porous media, *Soc. Petrol. Engrs.* J., 8 (3), 1968 b.

JAVANDEL, I., and P.A. WITHERSPOON, A method of analyzing transient fluid flow in multilayered aquifers, *Water Resour. Res.,* 5 (4), 1969.

NEUMAN, S.P., and P.A. WITHERSPOON, Transient Flow of ground water to wells in multiple-aquifer systems, *Geotech. Eng. Rept.* 69-1 University of California, Berkeley. January 1969a.

NEUMAN, S.P., and P.A. WITHERSPOON, Theory of flow in a confined two aquifer system. *Water Resour. Res.* 5 (4), 1969 b.

NEUMAN, S.P., and P.A. WITHERSPOON, Comments on "A Correspondence Principle for the Theory of Leaky Aquifers" by Ismael Herrera and Germán E. Figueroa, *Water Resour. Res.* 5 (4), 1970.

RODARTE, L., Issledovanie sistemy vodonos nykh plastov, granichashikh so slavo pronitsaemymi plastami (Ph. D *Dissertation,* Institute for Civil Engineers, Kuibuishev Moscu, URSS., to be presented).