Integrodifferential Equations for Systems of Leaky Aquifers and Applications
1. The Nature of Approximate Theories

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The dynamics of leaky aquifers are governed by a system of integrodifferential equations derived in this paper. Alternative expressions for the memory functions are obtained, and it is shown that approximate theories of leaky aquifers correspond to several ways of approximating the memory functions.

In many situations the behavior of leaky aquifers can be understood only when they are viewed as part of a complex multiple-aquifer system [*Neuman and Witherspoon*, 1969*a*]. For many years the complexity involved in the treatment of such systems has given rise to the development of approximate solutions [*Jacob*, 1946; *Hantush*, 1956, 1959, 1960; *Hantush and Jacob*, 1954, 1955*a*, *b*, 1960]. It has only been recently that a numerical method of analysis has been developed [*Javandel and Witherspoon*, 1969], and only under the assumptions that flow is horizontal in the aquifers and vertical in the aquitards have some exact analytical solutions been obtained [*Neuman and Witherspoon*, 1969*a*].

The assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards have been extensively used, and *Neuman and Witherspoon* [1969*a*] have confirmed their validity for most cases of practical interest. Under these assumptions it has been shown [*Herrera and Figueroa*, 1969; *Herrera*, 1970] that the transient behavior of drawdown is governed by a system of integrodifferential equations. Some other fields, such as the theory of viscoelasticity, are also governed by integrodifferential equations, and for them the understanding of the corresponding systems of equations has played a fundamental role [*Truesdell and Noll*, 1965; *Gurtin and Herrera*, 1965; *Coleman et al.*, 1965; *Barberán and Herrera*, 1966*a*, *b*, 1967; *Herrera*, 1966]. We are convinced that the integrodifferential equations of leaky aquifer dynamics constitute the basis of a powerful method of analysis.

The interest in this system of equations is at least twofold. It can be used as a very flexible tool for preliminary analysis before a complex model is advanced. The system has this use because the memory functions appearing in it have universal shape; i.e., the shape of these functions does not depend on the particular problem considered. Therefore much information about a given situation can be derived before almost any computation has been made. As an example of the possibilities in this respect we mention that it has been shown [*Herrera*, 1970, equations 30] that the influence function of one aquifer on the next one has the shape of a unit step function (Figure 1) with a lag time t^*, given by

$$t^* = b'^2/6\alpha' \qquad (1)$$

Therefore one can conclude immediately that the system can be treated as uncoupled at times $< t^*$, as given by (1).

The system of integrodifferential equations can be used to develop improved computational methods. The possibilities in this respect have already been exhibited by the construction of approximate methods [*Herrera and Figueroa*, 1969; *Herrera*, 1970] that have been shown to apply in some transient conditions [*Neuman and

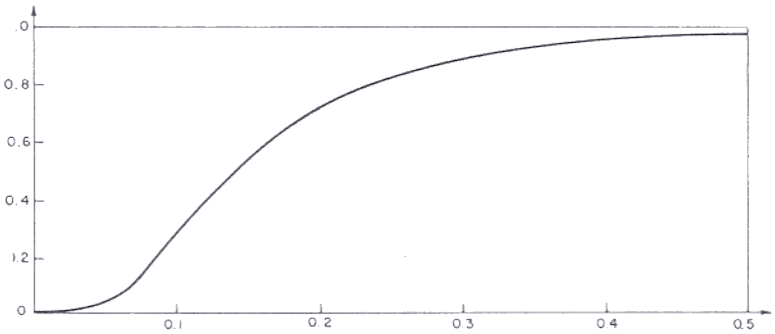


Fig. 1 The influence function h.

Witherspoon, 1970; *Herrera and Figueroa*, 1970]. However, it seems that the integrodifferential equations can be used to develop improved computational methods of more general applicability. This can be achieved through a better understanding of the role and structure of the memory functions, which may permit the construction of uniformly valid approximations.

Consequently, we have decided to devote a series of papers to discuss more thoroughly the system of integrodifferential equations and apply them to several problems of interest in aquifer dynamics.

In previous work [*Herrera and Figueroa*, 1969; *Herrera*, 1970] attention was focused on obtaining approximate methods of solution, and little attention was paid to the integrodifferential equations themselves. Thus in this paper the matter is attacked ab initio for the particular case of a two-aquifer system separated by an aquitard (Figure 2). First the integrodifferential equations are derived, and then they are applied to analyze the nature of some approximations of leaky aquifer dynamics.

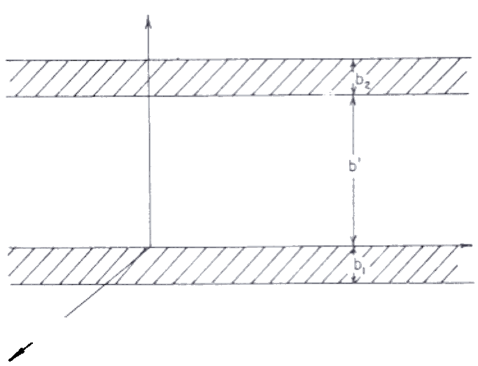


Fig. 2. The aquifer system.

We consider first the *Hantush and Jacob* [1955*b*] solution obtained on the assumption that the rate of leakage into the pumped aquifer is proportional to the potential drop across the leaky aquitard. Then we deal with *Hantush*'s [1960] solution for short periods of time. Finally, Hantush's solution for large values of time is analyzed.

In all these three cases it is shown that the corresponding solutions are obtained by approximating the memory functions in suitable ways. Therefore we can summarize our results by stating that the method on which approximate theories of flow in leaky aquifers have been based consists essentially of making suitable approximations of the memory functions of the exact integrodifferential equations governing the aquifer behavior. The importance of this result can be seen by the fact that the method has been used only for very specific problems. Thus, for example, Hantush's solutions were obtained for the problem of a single well in which radial symmetry is essential. The above result shows that the approximations that he used can be applied with greater generality by introducing them into the integrodifferential equations regardless of the boundary conditions. It is obvious that, by doing so, greater flexibility is achieved.

Among the specific results we obtain in this paper we consider of special interest the memory function implied by Hantush's solution for short periods of time. This function is essentially the reciprocal of the square root of time. In spite of its remarkable simplicity, by means of it, an approximation uniformly valid for all values of time can easily be derived.

A very important contribution to establish the range of applicability of current theories of flow in leaky aquifers was made by *Neuman and Witherspoon* [1969*b*]. They compared the approximate solutions with some exact solutions. The explicit expressions obtained here for the memory functions used in approximate theories of leaky aquifers will help to improve the understanding of their results. Indeed, in a further paper we will present a systematic analysis of the error involved.

In this paper, alternative expressions for the memory functions f and g not previously derived are established.

INTEGRODIFFERENTIAL EQUATIONS OF AQUIFER MECHANICS

In this section the set of integrodifferential equations, equivalent to the partial differential equations governing the transient behavior of multiple-aquifer systems, will be obtained for the particular case of a two-aquifer system (Figure 2) separated by a semipervious layer (aquitard or aquiclude).

According to *Hantush* [1960] and *Neuman and Witherspoon* [1969*a*] the problem can be formulated as follows:

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} + \frac{K'}{T_1}\left(\frac{\partial s'}{\partial z}\right)_{z=0} = \frac{1}{\alpha_1}\frac{\partial s_1}{\partial t} \qquad (2a)$$

$$\frac{\partial^2 s'}{\partial z^2} = \frac{1}{\alpha'}\frac{\partial s'}{\partial t} \qquad (2b)$$

$$\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} - \frac{K'}{T_2}\left(\frac{\partial s'}{\partial z}\right)_{z=b'} = \frac{1}{\alpha_2}\frac{\partial s_2}{\partial t} \qquad (2c)$$

In addition,

$$s'(x, y, 0, t) = s_1(x, y, t) \qquad (3a)$$

$$s'(x, y, b', t) = s_2(x, y, t) \qquad (3b)$$

$$s_1(x, y, 0) = 0 \qquad (3c)$$

$$s_2(x, y, 0) = 0 \qquad (3d)$$

$$s'(x, y, z, 0) = 0 \qquad (3e)$$

Here no contribution to the drawdown by distributed wells has been considered, but it can easily be incorporated if necessary. In any particular problem, appropriate boundary conditions have to be added to (2) and (3). Some of the results that we obtain will be changed if the initial conditions 3*c*, *d*, *e* are modified, but it is not difficult to modify the analysis to account for those changes.

As has been mentioned in previous papers [*Herrera and Figueroa*, 1969; *Herrera*, 1970], the set of equations 2*b* and 3 constitutes a well-posed problem for the heat equation, whose solution is given by Duhamel's integral [*Courant and Hilbert*, 1962]:

$$s'(x, y, z, t) = \int_0^t \frac{\partial s_1}{\partial t}(x, y, t-\tau)u(z, \tau)\, d\tau + \int_0^t \frac{\partial s_2}{\partial t}(x, y, t-\tau)v(z, \tau)\, d\tau \qquad (4)$$

where the auxiliary function $u(z, t)$ satisfies

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\alpha'}\frac{\partial u}{\partial t} \qquad 0 < z < b' \qquad 0 < t \qquad (5a)$$

$$\begin{aligned} u(0, t) &= 1 & t &> 0 \\ u(b', t) &= 0 & t &> 0 \\ u(z, 0) &= 0 & 0 &< z < b' \end{aligned} \qquad (5b)$$

The function $v(z, t)$ satisfies the same set of equations except that the boundary conditions 5*b* have to be replaced by

$$\begin{aligned} v(0, t) &= 0 & t &> 0 \\ v(b', t) &= 1 & t &> 0 \\ v(z, 0) &= 0 & 0 &< z < b' \end{aligned} \qquad (5c)$$

If we write

$$u(z, t) = \omega(\zeta, t') \qquad (6)$$

where ζ and t' are given by

$$\zeta = z/b' \qquad (7a)$$

$$t' = \alpha' t/b'^2 \qquad (7b)$$

then ω satisfies

$$\frac{\partial^2 \omega}{\partial \zeta^2} = \frac{\partial \omega}{\partial t'} \qquad 0 < \zeta < 1 \qquad 0 < t' \qquad (8a)$$

$$\omega(0, t') = 1 \qquad t' > 0 \qquad (8b)$$

$$\omega(1, t') = 0 \qquad t' > 0 \qquad (8c)$$

$$\omega(\zeta, 0) = 0 \qquad 0 < \zeta < 1 \qquad (8d)$$

To apply the method of separation of variables to these equations, the inhomogeneous boundary condition 8*b* has to be removed. It can be removed by introducing

$$\omega_1(\zeta, t') = \omega(\zeta, t') - 1 + \zeta \tag{9}$$

which satisfies (8*a*) and

$$\omega_1(0, t') = 0 \tag{10a}$$

$$\omega_1(1, t') = 0 \tag{10b}$$

$$\omega_1(\zeta, 0) = \zeta - 1 \tag{10c}$$

Using the method of separation of variables [*Churchill*, 1941], we obtain

$$\omega(\zeta, t') = 1 - \zeta - 2 \sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2 t'}}{n\pi} \sin n\pi\zeta \tag{11}$$

On the other hand, it is easily seen that

$$v(z, t) = u(b' - z, t) \tag{12}$$

so that in view of (6) and (7) we have

$$v(z, t) = \omega(1 - \zeta, t') \tag{13}$$

Finally, from (4), (6), (12), and the definition of ζ and t' (7) it follows that

$$\frac{\partial s'}{\partial z}(x, y, 0, t) = \frac{1}{b'}\left(\int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) \frac{\partial \omega}{\partial \zeta}(0, \alpha'\tau/b'^2)\, d\tau - \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) \frac{\partial \omega}{\partial \zeta}(1, \alpha'\tau/b'^2)\, d\tau\right) \tag{14}$$

But by virtue of (11) we have

$$\frac{\partial \omega}{\partial \zeta}(0, \tau) = -1 - 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2\tau} \tag{15a}$$

$$\frac{\partial \omega}{\partial \zeta}(1, \tau) = -1 - 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2\tau} \tag{15b}$$

and, in view of (6) and (7),

$$\frac{\partial u}{\partial z}(z, t) = \frac{1}{b'} \frac{\partial \omega}{\partial \zeta}(\zeta, t) \tag{15c}$$

Consequently, if we define f and h by

$$f(t') = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t'} \tag{16a}$$

$$h(t') = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t'} \tag{16b}$$

then

$$\frac{\partial s'}{\partial z}(x, y, 0, t) = \frac{1}{b'}\left(\int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) f(\alpha'\tau/b'^2)\, d\tau - \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) h(\alpha'\tau/b'^2)\, d\tau\right) \tag{17}$$

When this expression for $\partial s'/\partial z$ is substituted in (2*a*), we obtain

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} - C_1 \int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) f(\alpha'\tau/b'^2)\, d\tau + C_1 \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) h(\alpha'\tau/b'^2)\, d\tau = \frac{1}{\alpha_1} \frac{\partial s_1}{\partial t} \tag{18a}$$

In a similar manner from (2*c*) it follows that

$$\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} - C_2 \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) f(\alpha'\tau/b'^2)\, d\tau + C_2 \int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) h(\alpha'\tau/b'^2)\, d\tau = \frac{1}{\alpha_2} \frac{\partial s_2}{\partial t} \tag{18b}$$

In these equations, $C_i = K'/T_i b'$, $i = 1, 2$. Equations 18*a*, *b*, together with (4), i.e.,

$$s'(x, y, z, t) = \int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) u(z, \tau)\, d\tau + \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) v(z, \tau)\, d\tau \tag{18c}$$

constitute a complete set of equations that, when they are complemented by appropriate conditions on the horizontal boundaries, determine the behavior of the drawdown in the system of aquifers. When they are applied, it must be recalled that according to (6), (7), and (13) we have

$$u(z, t) = 1 - z/b' - 2 \sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2\alpha' t/b'^2}}{n\pi} \sin n\pi z/b' \tag{19a}$$

$$v(z, t) = z/b' + 2 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-n^2\pi^2\alpha' t/b'^2}}{n\pi} \sin n\pi z/b' \tag{19b}$$

An alternative form of these equations is achieved by defining

$$g(t') = 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \tag{20a}$$

,e.,

$$f(t') = 1 + g(t') \tag{20b}$$

When this expression is substituted in (18a, b), the integration from 0 to t of the derivatives of s_1 and s_2 can be performed, (3c, d) being taken into account, and we obtain

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} - C_1 s_1 - C_1 \int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) g(\alpha'\tau/b'^2)\, d\tau + C_1 \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) h(\alpha'\tau/b'^2)\, d\tau = \frac{1}{\alpha_1}\frac{\partial s_1}{\partial t}$$

$$\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} \quad C_2 s_2 - C_2 \int_0^t \frac{\partial s_2}{\partial t}(x, y, t - \tau) g(\alpha'\tau/b'^2)\, d\tau + C_2 \int_0^t \frac{\partial s_1}{\partial t}(x, y, t - \tau) h(\alpha'\tau/b'^2)\, d\tau = \frac{1}{\alpha_2}\frac{\partial s_2}{\partial t} \tag{21b}$$

The shape of functions f, g, h, and ω is illustrated in Figures 1, 3, and 4.

The variables ζ and t', introduced previously, are dimensionless. They, together with

$$\alpha_{ai} = \frac{b' S_s'}{b_i S_{si}} = \frac{S'}{S_i} \qquad i = 1, 2$$

$$\xi_i = x(K'/K_i b_i b')^{1/2} \qquad i = 1, 2 \tag{22b}$$

$$\eta_i = y(K'/K_i b_i b')^{1/2} \qquad i = 1, 2 \tag{22c}$$

or

$$R_i = r(K'/K_i b_i b')^{1/2} = (\xi_i^2 + \eta_i^2)^{1/2} \qquad i = 1, 2$$

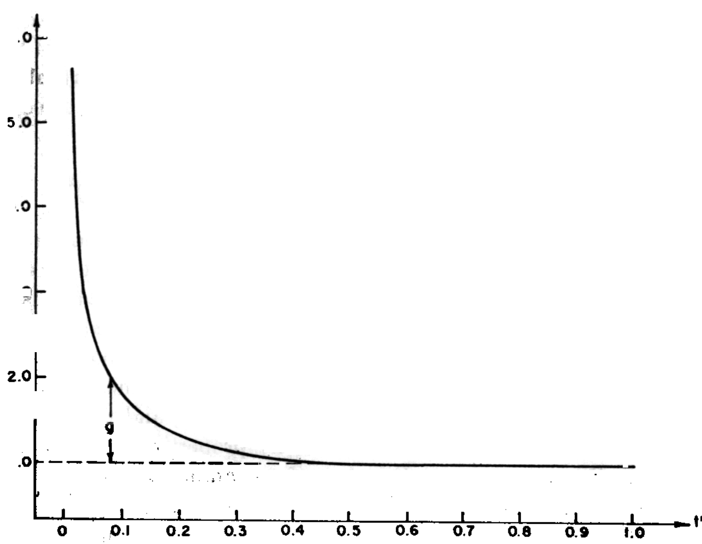
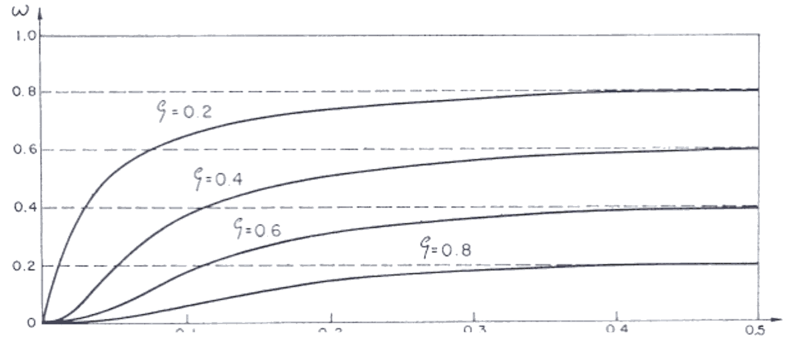


Fig. 3. The memory functions f and g.



for axially symmetric problems, constitute a whole set of dimensionless variables. When (18c) and (21) are transformed into these variables, they become

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - s_1 \quad \frac{\partial s_1}{\partial t'}(\xi_1, \eta_1, t' - \tau) g(\tau)\, d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(\xi_1, \eta_1, t' - \tau) h(\tau)\, d\tau = \frac{1}{\alpha_{a1}} \frac{\partial s_1}{\partial t'} \tag{23a}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2} + \frac{\partial^2 s_2}{\partial \eta_2^2} - s_2 \quad \frac{\partial s_2}{\partial t'}(\xi_2, \eta_2, \quad \tau) g(\tau)\, d\tau + \int_0^{t'} \frac{\partial s_1}{\partial t'}(\xi_2, \eta_2, t' - \tau) h(\tau)\, d\tau \tag{23b}$$

$$s'(\xi_i, \eta_i, \zeta, t') = \int_0^{t'} \frac{\partial s_1}{\partial t'}(\xi_i, \eta_i, t' \quad \tau)\omega(\zeta, \tau)\, d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(\xi_i, \eta_i, t' - \tau)\omega(1 - \zeta, \tau)\, d\tau \tag{23c}$$

The dimensionless variables R_i and t' correspond to the variables r/B_{ij} and t_D'', used previously by *Neuman and Witherspoon* [1968, 1969a, b]. These authors recall that some of their results can be expressed more simply by using the dimensionless time t', but, to make easier the comparison of their results with previous ones, they did not use it extensively. We have found that the variables t' and α_{ai} are quite suitable to analyze the behavior of the aquifer system itself irrespectively of the problem considered.

For the discussion of the well problem it has been customary to consider the dimensionless variables

$$t_{Di} = \alpha_i t/r^2 \tag{24a}$$

and

$$\beta_i = \frac{r}{4b_i}(K'S_s'/K_iS_{si})^{1/2} \tag{24b}$$

The advantages of t' and α_{ai} when the properties of the aquifer system itself are being discussed become apparent by recalling that $\alpha_{ai} = S'/S_i$ is the ratio of the storage coefficient of the aquitard to that of the aquifer, an important property of the aquifer system that is not directly given in terms of R_i, t_{Di}, and β_i. Indeed,

$$\alpha_{ai} = (4\beta_i/R_i)^2 \tag{25}$$

On the other hand, the memory functions f, g, and h have universal shape, but the time is scaled by the factor α'/b'^2, so that what is relevant as far as the memory is concerned is the dimensionless time $t' = (\alpha'/b'^2)t$. Note that the scaling factor α'/b'^2 determines the rate at which the memory functions act, and this factor is in turn determined by properties of the aquitard alone. For clays, for example, this factor is very small, and the rate at which the memory acts is very small unless b' is also very small.

All these facts are obscured if other parameters are used. Thus, for example, the dimensionless time t_{Di} depends on the distance r to the well, an irrelevant parameter if interest is focused on the aquifer system itself.

ALTERNATIVE EXPRESSIONS FOR THE MEMORY FUNCTIONS

In the section on the integrodifferential equations of aquifer mechanics we obtained the functions $u(z, t)$ and $v(z, t)$ by using the method of separation of variables. In this section we will use the method of images to obtain the same functions, getting in this manner alternative expressions for the memory functions.

First we will get the function $u(z, t)$ for the case when $b' = \infty$. This function will be denoted by $u_0(z, t)$. In this case, (5a) and (5b) become

$$\frac{\partial^2 u_0}{\partial z^2} = \frac{1}{\alpha'} \frac{\partial u_0}{\partial t} \qquad 0 < z \qquad 0 < t \tag{26a}$$

$$u_0(0, t) = 1 \qquad t > 0 \tag{26b}$$

$$u_0(z, 0) = 0 \qquad z > 0 \tag{26c}$$

We express u_0 in terms of an auxiliary function w_0 that satisfies

$$u_0(z, t) = 1 + w_0(z, t) \tag{27}$$

so that w_0 meets (26a), and

$$w_0(0, t) = 0 \tag{28a}$$

$$w_0(z, 0) = -1 \tag{28b}$$

Using the fundamental solution of the heat equation together with the method of images yields

$$w_0(z, t) = -\frac{1}{(\pi\alpha' t)^{1/2}} \int_0^z e^{-\lambda^2/4\alpha' t} \, d\lambda \tag{29}$$

Therefore

$$u_0(z, t) = 1 - \frac{1}{(\pi\alpha' t)^{1/2}} \int_0^z e^{-\lambda^2/4\alpha' t} \, d\lambda \qquad z > 0 \tag{30a}$$

and

$$\frac{\partial u_0}{\partial z}(0, t) = -\frac{1}{(\pi\alpha' t)^{1/2}} \tag{30b}$$

From (2a) and (4) it follows that, when the aquitard is of infinite thickness, s_1 satisfies

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} - \frac{K'}{T_1} \int_0^t \frac{\partial s_1}{\partial t}(t - \tau) f_0(\tau) \, d\tau = \frac{1}{\alpha_1} \frac{\partial s_1}{\partial t} \tag{31}$$

where

$$f_{[illegible]}([illegible]) = \frac{1}{([illegible])^{1/2}} \tag{32}$$

Using the method of images makes it easy to construct the function $u(z, t)$ that satisfies (5a, b). It is

$$u(z, t) = u_0(z, t) + \sum_{n=1}^{\infty} [u_0(2nb' + z, t) - u_0(2nb' - z, t)] \tag{33}$$

Observe that all the arguments of u_0 appearing in (33) are positive whenever $0 < z < b'$. This condition had to be satisfied because u_0 is defined by (30a) only for nonnegative values of z.

From (33) it follows that

$$\frac{\partial u}{\partial z}(0, t) = \frac{\partial u_0}{\partial z}(0, t) + 2 \sum_{n=1}^{\infty} \frac{\partial u_0}{\partial z}(2nb', t) \tag{34}$$

Thus, by virtue of (30a),

$$\frac{\partial u}{\partial z}(0, t) = \frac{-1}{(\pi\alpha' t)^{1/2}} \left(1 + 2 \sum_{n=1}^{\infty} e^{-n^2 b'^2/\alpha' t}\right) \tag{35}$$

But

$$\frac{\partial u}{\partial z}(0, t) = \frac{+1}{b'} \frac{\partial \omega}{\partial \zeta}(0, t') = \frac{-1}{b'} f(t') = \frac{-1}{b'} [1 + g(t')] \tag{36}$$

We see that

$$f(t') = 1 + g(t') = \frac{1}{(\pi t')^{1/2}} \left(1 + \sum_{n=1}^{\infty} e^{-n^2/t'}\right) \tag{37}$$

This equation yields, by the way, the relation

$$1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t'} = \frac{1}{(\pi t')^{1/2}} \left(1 + 2 \sum_{n=1}^{\infty} e^{-n^2/t'}\right) \tag{38}$$

Equation 37 is the alternative expression of the memory function that we were looking for.

Observe that the first term of the right-hand member in (37) is the exact memory function for the case of an aquitard of infinite thickness, because

$$C_1 \frac{}{(\pi t')^{1/2}} = \frac{K'}{T_1 b'} \cdot \frac{b'}{(\pi \alpha' t)^{1/2}} = \frac{K'}{T_1} f_0(t) \qquad (39)$$

Here (32) has been used.

Aquifer with Arbitrary Memory

We consider now the case for which the drawdown in one of the aquifers can be neglected, so that we set

$$s_2(x, y, t) \equiv 0 \qquad (40)$$

In the section on the integrodifferential equations of aquifer mechanics we considered several alternative forms of the integrodifferential equations governing aquifer systems. In this section the most suitable set of equations is that constituted by (21) and (18c). In view of (40) they reduce to

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - C_1 \int_0^t \frac{\partial s}{\partial t}(x, \quad, t - \tau) f(\alpha' \tau / b'^2)\, d\tau = \frac{1}{\alpha_1} \frac{\partial s}{\partial t} \qquad (41a)$$

$$s'(x, y, z, t) = \int_0^t \frac{\partial s}{\partial t}(x, y, t - \tau) u(z, \tau)\, d\tau \qquad (41b)$$

where for simplicity we have written s instead of s_1.

We will show that the approximate theories of flow in leaky aquifers that have been used in the past correspond to alternative ways of approximating the memory function f. To this end we consider the problem of a completely penetrating well discharging at a constant rate Q. In this case the solution is axially symmetrical, so that (41a) becomes

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} - C_1 \int_0^t \frac{\partial s}{\partial t}(x, y, t - \tau) f(\alpha' \tau / b'^2)\, d\tau \quad \frac{1}{\alpha_1} \frac{\partial s}{\partial t} \qquad (42a)$$

The boundary conditions are

$$s(\infty, t) = 0$$

$$\lim_{r \to 0} r \frac{\partial s}{\partial r}(r, t) = -\frac{Q}{2\pi T}$$

and the initial condition 3c is recalled

$$s(x, y, 0) = 0 \qquad (42d)$$

By means of taking the Laplace and Hankel transforms successively and using the boundary and initial conditions, (42a) is transformed into

$$\mathcal{L}\,[\mathcal{H}(s)](a, p) = \frac{Q}{2\pi T p} \frac{}{a^2 + A(p)} \qquad (43)$$

where $\mathcal{L}[\mathcal{H}(s)]$ stands for the Laplace transform of the Hankel transform of s, and

$$A(p) = p[(S'/T_1)\bar{f}(b'p/\alpha') + (1/\alpha_1)] \qquad (44)$$

Taking the inverse Hankel transform of (43), we see that the Laplace transform of s is given by

$$\mathcal{L}(s)(a, p) = \frac{Q}{4\pi T p} \cdot \int_0^\infty \frac{dy}{y} \exp -\left(y + \frac{Ar^2}{4y}\right) \qquad (45)$$

Equation 45 will be used later to derive the different forms that $A(p)$ has taken in several of the approximations of leaky aquifers.

Jacob's Approximation

The approach of *Jacob* [1946], on which the r/B solution [*Hantush and Jacob*, 1955*b*] is based, corresponds to the partial differential equation

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - C_1 s = \frac{1}{\alpha_1} \frac{\partial s}{\partial t} \qquad (46)$$

which is derived from the assumption that the rate of leakage is proportional to the potential drop across the aquitard. In view of the initial condition 3c this equation can also be written in the form

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} - C_1 \int_0^t \frac{\partial s}{\partial t}(t - \tau)\, d\tau = \frac{1}{\alpha_1} \frac{\partial s}{\partial t} \qquad (47)$$

Comparing (47) with (41a) shows clearly that in Jacob's partial differential equation the memory function g has been neglected; i.e.,

$$f(t) = 1 + g(t) \approx 1 \qquad (48)$$

Thus the function $g(t)$ has been neglected. Obviously, this function has the physical interpretation of being the contribution of the storage capacity of the aquitard to the leakage into the aquifer.

HANTUSH APPROXIMATION FOR SMALL VALUES OF TIME

When drawdown in the unpumped aquifer remains 0, the well problem considered in the section on the aquifer with arbitrary memory becomes equivalent to *Hantush*'s [1960] case 1. The Laplace transform (equations 42 and 43 of Hantush's paper) of his asymptotic solution for small values of time reduces to

$$\mathfrak{L}(s)(r, p) = \frac{Q}{4\pi T_1} \int_0^\infty \exp\{-y - (r^2/4y)p[1/\alpha_1 + (S'/T_1)(b'^2p/\alpha')^{-1/2}]\}/yp\, dy \tag{49}$$

when the aquifer is limited by only one aquitard. Therefore, comparing this equation with (45), we see that

$$A(p) = p[1/\alpha_1 + (S'/T_1)(b'^2p/\alpha')^{-1/2}] \tag{50}$$

and consequently by (44)

$$\bar{f}(b'p/\alpha') = (b'^2p/\alpha')^{-1/2} \tag{51}$$

Thus

$$\bar{f}(p) = p^{-1/2} \tag{52}$$

The inverse Laplace transform [*Sneddon*, 1951] of this equation yields

$$f(t') = (\pi t')^{-1/2} \tag{53}$$

Thus Hantush's solution for short periods of time is a solution of (31); i.e., as is well known [*Hantush*, 1964, p. 337], it is the exact solution of our problem when the aquitard is of infinite thickness. We conclude that the memory function f has been approximated by taking the first term in (37), which corresponds to neglecting all the reflections on the vertical boundaries of the aquitard.

The approximation

$$f(t') \approx 1/(\pi t')^{1/2} \tag{54}$$

deserves special attention. Its usefulness in leaky aquifer dynamics has already been demonstrated; it yielded the Hantush approximation for small values of time. Its simplicity is remarkable and offers great advantage for its handling. From it, it is not difficult to construct a uniformly valid approximation for all values of time, as will be shown elsewhere.

Before leaving this matter, we recall that *Hantush*'s [1960] case 2 also satisfies (41*a*) with a suitable memory function f [*Herrera and Figueroa*, 1969], which also has an alternative expression of the type derived in the section on an aquifer with arbitrary memory. The approximation used by Hantush for that case is also given by (54). The coincidence of both cases at this order of approximation is not surprising because, as was mentioned before, reflections on the horizontal boundaries of the aquitard are neglected, and consequently the boundary condition on the top of the aquitard is irrelevant.

HANTUSH APPROXIMATION FOR LARGE VALUES OF TIME

The Laplace transform of the Hantush asymptotic solution for large values of time (equations 47 and 48 of *Hantush* [1960]) reduces to

$$\mathfrak{L}(s)(r, p) = \frac{Q}{[illegible]} \cdot \int_0^\infty \exp\{-[y + (r^2/4y)p(K'/T_1b'p + 1/\alpha_1 + S'/3T_1)]\}/y\, dy \tag{55}$$

when the aquifer is limited by only one aquitard. Therefore, comparing this equation with (45), we see that

$$A(p) = p(1/\alpha_1 + K'/T_1b'p + S'/3T_1) \tag{56}$$

and in view of (44),

$$\frac{S'}{T_1}\bar{f}\frac{b'^2p}{\alpha'} = \frac{S'}{3T_1} + \frac{K'}{T_1b'p} \tag{57}$$

Thus

$$\bar{f}(p) = \tfrac{1}{3} + 1/p \tag{58}$$

Taking the inverse Laplace transform of (58) yields

$$f(t') = \quad + \delta(t')/3 \tag{59}$$

.e.,

$$g(t') = \delta(t')/3 \tag{60}$$

This approximation has already been used to construct approximate methods suitable for numerical computation [*Herrera and Figueroa*, 1969; *Herrera*, 1970].

CONCLUSIONS

It has been shown that approximate theories of leaky aquifers are based on several approximations of the memory functions in the integrodifferential equations of the dynamics of leaky aquifers. By means of analysis of these functions in the time domain it is possible to achieve a better understanding of the nature of these approximations as well as to foresee the possibility of their application to new situations. These types of results constitute important steps toward making the integrodifferential equations a powerful method of analysis of the very complex situations arising in the study of actual leaky aquifer systems. They can also be used to construct improved, simplified methods for numerical computation.

NOTATION

b_i, thickness of the ith aquifer, L;
b', thickness of the aquitard, L;
$C_i = K'/T_i b'$, $i = 1, 2$, L^{-2};
$f(t')$, memory function, equal to $1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$;
$f_0(t)$, memory function for aquitard of infinite thickness, equal to $(\pi\alpha' t)^{-1/2}$;
$g(t') = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$;
$h(t')$, influence function, equal to $1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t'}$;
K_i, permeability of the ith aquifer, L/T;
K', permeability of the aquitard, L/T;
Q, pumping rate from aquifer 1, L^3/T;
r_i, radial distance to the pumping well, L;
$R_i = r_i(K'/K_i b_i b')^{1/2}$;
S_{si}, specific storage of the ith aquifer, L^{-1};
S_s', specific storage of the aquitard, L^{-1};
S_i, storage coefficient of the ith aquifer, equal to $S_{si} b_i$;
S', storage coefficient of the aquitard, equal to $S_s' b'$;
s_i, drawdown in the ith aquifer, L;
s', drawdown in the aquitard, L;
T_i, transmissibility of the ith aquifer, equal to $K_i b_i$, L^2/T;
t, time, T;
t', dimensionless time, equal to $\alpha' t/b'^2$;
x, y, z, coordinates, L;
$\alpha_i = K_i/S_{si} = T_i/S_i$, L^2T^{-1};
$\alpha' = K'/S_{si}'$, L^2T^{-1};
$\alpha_{ai} = S'/S_i$, $i = 1, 2$;
$\delta(t')$, Dirac's delta function;
$\eta_i = y(K'/K_i b_i b')^{1/2}$;
$\zeta = z/b'$;
$\xi_i = x(K'/K_i b_i b')^{1/2}$.

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REFERENCES

Barberán, J., and I. Herrera, Uniqueness theorems and the speed of propagation of signals in viscoelastic materials, *Arch. Ration. Mech. Anal.*, *23*(3), 173–190, 1966*a*.

Barberán, J., and I. Herrera, The speed of propagation of signals in viscoelastic materials, in *Modern Developments in the Mechanics of Continua*, edited by S. Eskinazi, pp. 175–182, Academic, 1966*b*.

Barberán, J., and I. Herrera, Riemann representation method in viscoelasticity, 2, Cauchy's initial value problem, *Arch. Ration. Mech. Anal.*, *25*(3), 178–187, 1967.

Churchill, R. V., *Fourier Series and Boundary Value Problems*, pp. 102–111, McGraw Hill, New York, 1941.

Coleman, B. D., M. E. Gurtin, I. Herrera, and C. Truesdell, *Wave Propagation in Dissipative Materials*, Springer-Verlag, New York, 1965.

Courant, R., and D. Hilbert, *Methods of Mathematical Physics*, vol. 2, pp. 512–515, Interscience, New York, 1962.

Gurtin, M. E., and I. Herrera, A correspondence principle for viscoelastic wave propagation, *Quart. Appl. Math.*, *22*(4), 360–364, 1965.

Hantush, M. S., Analysis of data from pumping tests in leaky aquifers, *Eos Trans. AGU*, *37*(6), 702–714, 1956.

Hantush, M. S., Nonsteady flow to flowing wells in leaky aquifers, *J. Geophys. Res.*, *64*(8), 1043–1052, 1959.

Hantush, M. S., Modification of the theory of leaky aquifers, *J. Geophys. Res.*, *65*(11), 3713–3725, 1960.

Hantush, M. S., Hydraulics of wells, in *Advances in Hydroscience*, vol. 1, edited by V. T. Chow, pp. 281–432, Academic, New York, 1964.

Hantush, M. S., and C. E. Jacob, Plane potential flow of ground water with linear leakage, *Eos Trans. AGU*, *35*(6), 917–936, 1954.

Hantush, M. S., and C. E. Jacob, Nonsteady Green's functions for an infinite strip of leaky aquifer, *Eos Trans. AGU*, *36*(1), 101–112, 1955*a*.

Hantush, M. S., and C. E. Jacob, Non-steady radial flow in an infinite leaky aquifer, *Eos Trans. AGU*, *36*(1), 95–100, 1955*b*.

Hantush, M. S., and C. E. Jacob, Flow to an eccentric well in a leaky circular aquifer, *J. Geophys. Res.*, *65*(10), 3425–3432, 1960.

Herrera, I., Riemann representation method in viscoelasticity, 1, Characterization and construction of the Riemann function, Solution of problems with prescribed body forces, *Arch. Ration. Mech. Anal.*, *22*(4), 270–291, 1966.

Herrera, I., Theory of multiple leaky aquifers, *Water Resour. Res.*, *6*(1), 185–193, 1970.

Herrera, I., and G. E. Figueroa, A correspondence principle for the theory of leaky aquifers, *Water Resour. Res.*, *5*(4), 900–904, 1969.

Herrera, I., and G. E. Figueroa, Reply, *Water Resour. Res.*, *6*(3), 1011, 1970.

Jacob, C. E., Radial flow in a leaky artesian aquifer, *Eos Trans. AGU*, *27*(2), 198–208, 1946.

Javandel, I., and P. A. Witherspoon, A method of analyzing transient fluid flow in multilayered aquifers, *Water Resour. Res.*, *5*(4), 856–869, 1969.

Neuman, S. P., and P. A. Witherspoon, Theory of flow in aquicludes adjacent to slightly leaky aquifers, *Water Resour. Res.*, *4*(1), 103–112, 1968.

Neuman, S. P., and P. A. Witherspoon, Theory of flow in a confined two-aquifer system, *Water Resour. Res.*, *5*(4), 803–816, 1969*a*.

Neuman, S. P., and P. A. Witherspoon, Applicability of current theories of flow in leaky aquifers, *Water Resour. Res.*, *5*(4), 817–829, 1969*b*.

Neuman, S. P., and P. A. Witherspoon, Comments on 'A correspondence principle for the theory of leaky aquifers' by Ismael Herrera and Germán E. Figueroa V., *Water Resour. Res.*, *6*(3), 1009, 1970.

Sneddon, I. N., *Fourier Transforms*, p. 526, McGraw Hill, New York, 1951.

Truesdell, C., and W. Noll, Non-linear fields theories of mechanics, in *Encyclopedia of Physics*, vol. 3, part 3, pp. 382–385, Springer-Verlag, New York, 1965.

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