foundation when the solution in the paper is extended to a soil foundation. Although the original paper which used a Winkler foundation was not expressly written for use with a soil foundation, the writer does not agree with Byrne's conclusions and believes that the method can often be advantageously used in such cases for reasons discussed subsequently.

First, note that new results are presented in the paper which apply to plates with and without an elastic foundation. It is only the application of those results pertaining to the elastic foundation and, moreover, only those cases in which the Winkler foundation is used to model a soil response, that is under discussion herein.

Second, this writer would rather believe that a principal reason why papers with a Winkler foundation appear widely in technical literature is the wide class of problems for which response analogies have been obtained. For example, Vlasov (18) has shown that a circular plate on a Winkler foundation can be used to model the response of a shallow spherical shell. Hetenyi (16) has shown that the response of a structure on a type of interconnected grillage beams can be modeled as on a Winkler foundation. Many other examples and basic assumptions can be found in Hetenyi's work and in technical literature. It should be noted that the Winkler foundation is used to model the response rather than the continuum. As was mentioned in the first paragraph of the paper, it was such a response analogy which principally motivated the use of an elastic foundation in this paper. Indeed, if Byrne has read the paper, it is difficult to see how "the writer is concerned that the author models a soil foundation," but then, Byrne may not have had an opportunity to present the latest results from his finite element program. In this regard it is significant that the example problem used by Byrne in his discussions hardly appears to be a representative example.

Nevertheless, it is natural to try to extend the results of the paper to a soil foundation. Certain limitations of the use of a Winkler foundation to model a soil continuum have been pointed out by many authors, e.g., by Hetenyi (16). The Boussinesq model, which for finite element applications to soil foundations was first obtained by Cheung and Zienciewicz (13), models the soil as a homogeneous and isotropic continuum and also considers only normal forces on the interface. The point that Byrne seems to overlook is that while the Boussinesq foundation models the foundation as ideal, the Winkler foundation can be considered as modeling an averaged response of a real foundation. Some limited test results (15,17) are available supporting the use of a Winkler model as a soil response model. Moreover, the Winkler foundation as shown by Byrne's results has one very forgiveable feature for many problems, i.e., it tends to produce conservative estimates of the plate deflections.

This writer is surprised to see words such as "real" and "correct" which appear in Byrne's discussion used to describe any model of a soil foundation. This writer would suggest that before any new idealized model is described in such terms that the analytical results should be supported by corresponding wide scale test results.

Appendix.—References

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VARIATIONAL FORMULATION OF DYNAMICS OF FLUID-SATURATED POROUS ELASTIC SOLIDS[a]

Discussion by Ismael Herrera[3] and Jacobo Bielak,[4] A. M. ASCE

The authors developed in their paper a variational principle for the dynamic analysis of saturated porous media. Specifically, they obtained a variational principle for Biot's (2) equations of motion. Their results are highly relevant as a tool for the application of the finite element method to this kind of problem.

However, the writers would like to point out that Ghaboussi and Wilson's variational principle was obtained along lines similar to those followed by Gurtin (8), which require transforming the equations of motion into a set of integro-differential equations. It is possible, however, to follow a more direct approach when dealing with transient problems, as Tonti has demonstrated for the heat equation (20). The system of partial differential equations may then be dealt with directly, and simpler variational principles may be obtained, because the functional to be varied involves fewer convolutions. Thus, for the problem defined by Eqs. 1, 2, and 3 of the paper under discussion, together with the boundary conditions

$$u_i(\underset{\sim}{x}, \tau) = \hat{u}_i(\underset{\sim}{x}, \tau) \quad \text{on} \quad S_1 \times [0, t];$$

$$\tau_{ij}(\underset{\sim}{x}, \tau)\, n_j(\underset{\sim}{x}) = \hat{T}_i(\underset{\sim}{x}, \tau) \quad \text{on} \quad S_2 \times [0, t];$$

$$w_i(\underset{\sim}{x}, \tau)\, n_i(\underset{\sim}{x}) = \hat{w}_i(\underset{\sim}{x}, \tau)\, n_i(\underset{\sim}{x}) \quad \text{on} \quad S_1 \times [0, t];$$

$$\pi(\underset{\sim}{x}, \tau) = \hat{\pi}(\underset{\sim}{x}, \tau) \quad \text{on} \quad S_2 \times [0, t] \quad \ldots\ldots\ldots\ldots \quad (41)$$

and the initial conditions $u_i(\underset{\sim}{x}, 0) = \overset{\circ}{u}_i(\underset{\sim}{x})$ on R

$\dot{u}_i(\underset{\sim}{x}, 0) = \dot{\overset{\circ}{u}}_i(\underset{\sim}{x})$ on R

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[a]August, 1972, by Jamshid Ghaboussi and Edward L. Wilson (Proc. Paper 9152).

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$$w_i(\underset{\sim}{x}, 0) = \bar{w}_i(\underset{\sim}{x}) \quad \text{on} \quad R$$

$$\dot{w}_i(\underset{\sim}{x}, 0) = \dot{\bar{w}}_i(\underset{\sim}{x}) \quad \text{on} \quad R \qquad (42)$$

the functional to be varied is given by:

$$\Omega(u, w) = \int_V \left(\rho\, \dot{u}_i * \dot{u}_i + \frac{1}{f} \rho_f\, \dot{w}_i * \dot{w}_i + 2\, \rho_f\, \dot{u}_i * \dot{w}_i + \frac{1}{k} w_i * \dot{w}_i + \frac{\partial u_i}{\partial x_j} * \tau_{ij} + \frac{\partial w_i}{\partial x_i} * \pi \right) dv - 2 \int_V (\rho b_i * u_i + \rho_f b_i * w_i)\, dv$$

$$- 2 \int_V \Big\{ [\rho\, \dot{u}_i(t) + \rho_f\, \dot{w}_i(t)]\, [\bar{u}_i - u_i(0)] + \left[\rho_f\, \dot{u}_i(t) + \frac{1}{f} \rho_f\, \dot{w}_i(t) + \frac{1}{2k} w_i(t) \right] [\bar{w}_i - w_i(0)]$$

$$+ [\rho\, u_i(t) + \rho_f\, w_i(t)]\, \dot{\bar{u}}_i + \rho_f \left[u_i(t) + \frac{1}{f} w_i(t) \right] \dot{\bar{w}}_i + \frac{1}{2k} \bar{w}_i\, w_i(t) \Big\} dv + 2 \int_{s_1} [T_i * (\hat{u}_i - u_i) + \pi * (\hat{w}_i - w_i)\, n_i]\, ds$$

$$- 2 \int_{s_2} (u_i * \hat{T}_i + \hat{\pi} * w_i\, n_i)\, ds \qquad (43)$$

in which, V = the region wherein the problem is defined and it is assumed that the boundary S of V satisfies $S = S_1 \cap S_2$ and $S_1 \cap S_2 = \phi$.

Several observations should be made regarding the functional given by Eq. 43. The functions, u_i, w_i, are assumed to be the only independent variables and they are not required to meet the initial conditions; consequently, when formulating a finite element method, a criterion for the choice of the "best" approximation of the discretized initial conditions will be given by the condition that the variation of Ω be zero. A more detailed derivation and a more thorough discussion of the variational principle implied by Eq. 43 will be given elsewhere.

It is also convenient to observe that boundary conditions given in Eq. 41 are a revised version of the authors' (Eq. 4); their last two equations have been modified. It can be shown that it is not possible to prescribe the whole vector, w_i, on the boundary, but that it is only its normal component that can be prescribed arbitrarily. On the other hand, the modification introduced in their last equation has the sole purpose of simplifying the notation.

Appendix.—References

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END EFFECTS IN TRUNCATED SEMI-INFINITE WEDGE[a]

Closure by Robert Wm. Little[4] and Tommie R. Thompson[5]

The writers thank Silverman for calling Brahtz work to their attention. As pointed out in the paper the defining equations for the eigenvalues are not new but the writers are unaware of applications of the eigenfunction analysis to wedge problems or an examination of the Saint-Venant region for this geometry.

BEAMS SUBJECTED TO FOLLOWER FORCE WITHIN THE SPAN[b]

Discussion by Karl K. Stevens[3]

The problem considered is that of the stability of columns with various end conditions subjected to a follower force applied at an arbitrary point within the span. This discussion is concerned not so much with the details of the technical content of the paper, as with its significance in a broader sense.

From time to time problems appear in literature which at first seem rather innocuous, but which, upon closer inspection, are found to contribute significantly to the understanding of system responses by pointing out phenomena not previously observed. Such is the case with the problem considered by the authors.

The most significant finding in the paper is the possibility of the loss of stability by what the authors term "higher flutter modes." As far as is known, this phenomenon has not been previously observed in systems similar to those treated in this paper. Moreover, the authors offer experimental evidence which substantiates this finding in a qualitative sense.

It is also shown that flutter is impossible for certain placements of the load. The loss of stability in these cases is always by divergence, which would indicate that the follower force is derivable from a potential function. It is well known that follower forces do have a potential when the component of the force which deviates from the corresponding conservative loading does no work. However,

[a]October, 1972, by Robert Wm. Little and Tommie R. Thompson (Proc. Paper 9251).

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