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A Simplified Version of Gurtin's Variational Principles

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1. Introduction

Variational principles have been used extensively in mechanics. Recently they have played an important role in the formulation of the finite element method [1, 2] which is currently subject to vigorous development.

The basic concepts applicable to the formulation of variational principles for *static* boundary value problems have been known for some time. In *dynamics* the development has been less satisfactory. Though Hamilton-type variational principles are well known, they correspond to transient problems which are not well-posed.

In this paper we are concerned with principles applicable to initial-value problems. Complementary variational principles for these problems can be constructed [3], but they raise the order of the operators considered. Variational principles which do not have that inconvenient feature were first formulated by GURTIN [4, 5]. By use of GURTIN's approach, numerous applications have been made to problems in mechanics and related fields. Thus, to list but a few, variational principles for initial-value problems in elastodynamics and viscoelasticity have been derived by GURTIN [5] and LEITMAN [6], respectively; NICKELL & SACKMAN [7] did corresponding work for thermoelasticity. Heat flow problems have been considered by GURTIN [4], EMERY & CARSON [8] and WILSON & NICKELL [9]. Extensive applications to ground water hydrology have been made by NEUMAN & WITHERSPOON [10] and JAVANDEL & WITHERSPOON [11], and additional work has been carried out by PRODHAN & SARMA [12]. Applications in soil mechanics have been carried out by BREBBIA [13] and GHABOUSSI & WILSON [14].

In GURTIN's method, given an initial value problem, the inverse of the time operator is applied to obtain integro-differential equations which contain the initial conditions implicitly and for which variational principles can be derived with the use of convolutions. Recently SANDHU & PISTER [15] and TONTI [16] have shown that GURTIN's formulation can be set within the framework of the general theory of variational methods [17]. What is essential is the use of the convolution as inner product, and TONTI [16] has given an example for an initial-value problem for which the transformation into an integro-differential equation is not required.

In this paper we present a general formulation of initial-value problems in terms of functional valued operators. This formulation is quite suitable for treating initial and boundary conditions systematically; using it, one obtains variational principles for which the admissible functions are required to satisfy neither the initial nor the boundary conditions. We apply that formulation to derive variational principles for a large class of initial-value problems, which are simpler than those that Gurtin's method yields. In order to make clearer the relation between these variational principles and Hamilton's principles, the latter are obtained using functional valued operators. Then we establish a systematic method of simplifying Gurtin's variational principles. This result has special interest, because Gurtin-type principles have been obtained for many problems and here it is shown how to transform them into a simpler form.

In Section 2, we recall mathematical results and notations that will be used in the sequel. Section 3 is devoted to the recasting of some known facts about the theory of variational principles for linear operators [17, 18] in terms of functional-valued operators. The time dependent problems to be considered in this study are formulated in Section 4. Sections 5 and 6 are devoted to obtaining variational principles of Hamilton-type and for initial-value problems, respectively. The connection of these principles with Gurtin's principles is established in Section 7. To illustrate the use of these results, in Section 8 we obtain a simpler variational principle for elastodynamics and exhibit its connection with one obtained by GURTIN [5].

2. Mathematical Preliminaries

Let D and K be two linear spaces over the field of real numbers R^1. By a linear operator or simply an operator, we mean a linear mapping

$$L: D \to K. \tag{2.1}$$

The set

$$\mathscr{R}(L) = \{Lu \mid u \in D\} \tag{2.2}$$

is called the range of L. Let L_1 and L_2 be two linear mappings of D into K, and $a \in R^1$. Then the operators $L_1 + L_2$ and aL_1 of D into K, are defined for every $u \in D$ by [19]:

$$(L_1 + L_2)u = L_1 u + L_2 u \tag{2.3a}$$

and

$$(aL_1)u = a(L_1 u). \tag{2.3b}$$

The set of all linear operators of D into K is itself a linear space with respect to these operations. In the special case when $K = R^1$, an operator is called a linear functional; thus, the set of all linear functionals on D is a linear space that will be denoted by D^*.

On the other hand, a bilinear functional is a mapping

$$\{,\}: D \times K \to R^1 \tag{2.4}$$

such that for every $u, v \in D$; $\bar{u}, \bar{v} \in K$ and $a, b \in R^1$, we have

$$\{au + bv, \bar{u}\} = a\{u, \bar{u}\} + b\{v, \bar{u}\}, \tag{2.5a}$$

$$\{u, a\bar{u} + b\bar{v}\} = a\{u, \bar{u}\} + b\{u, \bar{v}\}. \tag{2.5b}$$

A bilinear functional is called an inner product if

$$\{u, \bar{u}\}=0 \quad \text{for fixed } u\in D \quad \text{and every } \quad \bar{u}\in K \Rightarrow u=0 \tag{2.6a}$$

and

$$\{u, \bar{u}\}=0 \quad \text{for fixed } \bar{u}\in K \quad \text{and every } \quad u\in D \Rightarrow \bar{u}=0. \tag{2.6b}$$

Define the bilinear functional

$$\langle\,|\,\rangle: D\times D^* \to R^1 \tag{2.7a}$$

by the equation

$$\langle u\,|\,T\rangle = T(u) \tag{2.7b}$$

for every $u\in D$ and every $T\in D^*$. It can be easily shown that this bilinear functional is an inner product on $D\times D^*$. In this paper, functional valued operators play an important role; specifically, we will restrict attention to functional valued operators for which

$$K=D^*. \tag{2.8}$$

Observe that to define a functional valued operator L^*, it is sufficient to know $\langle u|L^* v\rangle$ for every $u, v\in D$. Thus, we can associate with every functional valued operator L another operator L^*, which will be called its adjoint, and is defined by the condition that

$$\langle u\,|\,L^* v\rangle = \langle v\,|\,Lu\rangle \tag{2.9}$$

hold for every $u, v\in D$. In particular, the operator L is called self-adjoint if

$$L^*=L. \tag{2.10}$$

Given a linear mapping

$$\Sigma: D\to D, \tag{2.11a}$$

we define another mapping

$$S: D^*\to D^* \tag{2.11b}$$

for every $w\in D^*$ by

$$\langle v\,|\,Sw\rangle = \langle \Sigma v\,|\,w\rangle \tag{2.11c}$$

which holds for every $v\in D$. Therefore, if the product SL of two operators is defined in the usual manner as the operator obtained by the application of L followed by S, it is seen that SL is a functional valued operator whenever L is a functional valued operator; it satisfies

$$\langle u\,|\,SLv\rangle = \langle \Sigma u\,|\,Lv\rangle \tag{2.12}$$

for every $u, v\in D$.

3. Variational Principles for Linear Operators

In this paper we are concerned with linear operators only; for them, the results from the theory of variational principles that we will need take a very simple form. It is advantageous to formulate these results in terms of functional valued operators.

Let L be a functional valued operator and consider the equation

$$Lu=f, \tag{3.1}$$

where we assume that a solution exists; *i.e.*, we assume $f \in \mathscr{R}(L) \subset D^*$. Define

$$\Omega(u) = \langle u \mid Lu \rangle - 2\langle u \mid f \rangle \tag{3.2}$$

for every $u \in D$. Then, for fixed $u, v \in D$, $\Omega(u+\lambda v)$ is a function of the real number λ. The derivative

$$\frac{d\Omega}{d\lambda}(u) = \langle v \mid Lu \rangle + \langle u \mid Lv \rangle - 2\langle v \mid f \rangle \tag{3.3}$$

is a functional $\delta\Omega_u$ whose domain is D; $\delta\Omega_u$ is called the variation of Ω at u. The main result we are going to use is contained in the following

Theorem 3.1. *Let L be a self-adjoint functional valued operator. Then*

$$\tfrac{1}{2}\delta\Omega_u = Lu - f \tag{3.4a}$$

for every $u \in D$. *In particular*

$$\delta\Omega_u = 0 \tag{3.4b}$$

if and only if u satisfies (3.1).

Proof. If L is self-adjoint, equation (3.3) becomes

$$\delta\Omega_u(v) = 2\langle v \mid Lu - f \rangle \tag{3.5}$$

for every $u, v \in D$. For fixed u, this implies (3.4a). Consequently, (3.4b) holds if and only if (3.1) holds.

The case when L is not self-adjoint can easily be handled, at least in principle. Indeed, let P be a linear mapping

$$P: D^* \to D^*. \tag{3.6}$$

The mapping P is called non-singular if, for every $w \in D^*$, we have

$$Pw = 0 \Rightarrow w = 0. \tag{3.7}$$

It then becomes a simple matter to establish the following

Corollary 3.1. *Let P be non-singular and such that PL is self-adjoint. Define*

$$\Omega = \langle u \mid PLu \rangle - 2\langle u \mid Pf \rangle. \tag{3.8}$$

Then

$$\delta\Omega_u = 0 \tag{3.9}$$

if and only if u satisfies (3.1).

Proof. By Theorem 3.1, Eq. (3.9) is satisfied if and only if

$$PLu = Pf; \tag{3.10}$$

but P is non-singular and consequently (3.10) is equivalent to (3.1).

4. Formulation of Time Dependent Problems

Let R be the closure of an open region in R^n (the n-dimensional Euclidean space) with boundary B, and let the subsets B_1 and B_2 of B denote a decomposition of B, *i.e.*:

$$B_1 \cap B_2 = \phi; \quad B_1 \cup B_2 = B. \tag{4.1}$$

Henceforth we shall be concerned with functions $u, v, \ldots$ defined on $R \times [0, t_1]$, with values in R^m. Each point of the set $R \times [0, t_1]$ will be denoted by (x, t) where $x \in R$ and $t \in [0, t_1]$; the components $x_1, \ldots, x_n$ of x will be called the spatial variables, while t will be called the time. C^M will be the set of such functions possessing continuous derivatives up to order M on $R \times [0, t_1]$. Partial derivatives with respect to time will often be denoted by dots; thus, $\dot{u}$ and $\ddot{u}$ will be the same as $\partial u/\partial t$ and $\partial^2 u/\partial t^2$, respectively.

Given $u, v \in R^m$, we define

$$u v = \sum_{j=1}^{m} u_j v_j \tag{4.2}$$

where u_j and v_j are the Cartesian components of u and v, respectively.

For functions $u, v \in C^0$, we define the inner product:

$$\{u, v\} = \int_0^{t_1} \int_R u v \, dx \, dt. \tag{4.3}$$

Then, with every $u \in C^0$ we associate a linear functional $\bar{u}$ whose domain is C^0 and whose value for each $v \in C^0$ is given by

$$\bar{u}(v) = \{u, v\}. \tag{4.4}$$

Consider the second order differential operator:*

$$\mathscr{L} u \equiv \frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial u}{\partial x_j}\right) + F u \tag{4.5}$$

whose domain D is taken to be C^2. Here and in what follows i and j run from 1 to n and summation over the range of repeated indices is understood. F and A_{ij}, for each i and j, are functions defined on $R \times [0, t_1]$ whose values are $m \times m$ matrices; they are assumed to be time independent, F is required to be continuous and symmetric, A_{ij} is assumed to possess continuous first order partial derivatives and to be such that for every u, v

$$A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = A_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{4.6}$$

On the boundary B of R, we define

$$\mathscr{L}_B u \equiv A_{ij} \frac{\partial u}{\partial x_j} n_i; \quad x \in B \times [0, t_1], \tag{4.7}$$

where n_i $(i = 1, \ldots, n)$ are the components of the outward unit normal vector to B.

The domain D of all the functionals and of the functional valued operators to be considered will be taken to be the set C^2. Thus, given any two continuous

* Felippa & Clough [20] and Oden [2] have considered a general class of higher order operators. The analysis presented here can be extended along similar lines to that kind of operators.

functions

$$u_{B1}: B_1 \to R^m \tag{4.8a}$$

and

$$N: B_2 \to R^m \tag{4.8b}$$

we associate with them the linear functionals $\beta_1[u_{B1}]$ and $\beta_2[N]$. The values of these functionals, for each $v \in D = C^2$, are given by

$$\beta_1[u_{B1}](v) = \int_0^{t_1} \int_{B_1} u_{B1} \mathscr{L}_B v \, dx \, dt \tag{4.9a}$$

and

$$\beta_2[N](v) = -\int_0^{t_1} \int_{B_2} N v \, dx \, dt. \tag{4.9b}$$

The operator $\hat{\mathscr{P}}$ is defined as a functional valued operator such that, for every $u \in C^2$, we have

$$\hat{\mathscr{P}} u = \overline{(\mathscr{L} u)} + \beta_1[u] + \beta_2[\mathscr{L}_B u], \tag{4.10}$$

where $\overline{(\mathscr{L} u)}$ is the restriction to C^2 of the functional associated to $\mathscr{L} u$ by means of (4.3) and (4.4).

It is easy to prove that

$$\overline{(\mathscr{L} u)} = 0 \Rightarrow \mathscr{L} u \equiv 0, \quad \text{on} \quad R \times [0, t_1] \tag{4.11a}$$

and

$$\beta_2[\mathscr{L}_B u] = 0 \Rightarrow \mathscr{L}_B u \equiv 0, \quad \text{on} \quad B_2 \times [0, t_1]. \tag{4.11b}$$

The vanishing of the functional $\beta_1[u]$, however, does not generally imply the vanishing of u on $B_1 \times [0, t_1]$. Accordingly, we introduce the following

Definition 4.1. The differential form $\mathscr{L}_B$, given by (4.7), is called non-singular if

$$\beta_1[u] = 0 \Rightarrow u \equiv 0, \quad \text{on} \quad B_1 \times [0, t_1]. \tag{4.12}$$

We introduce now three functional valued operators,

$$\Delta_{2P} u \equiv \overline{\frac{\partial^2 u}{\partial t^2}} - u(t_1)\,\delta_1' + u(0)\,\delta_0', \tag{4.13a}$$

$$\Delta_{2I} u \equiv \overline{\frac{\partial^2 u}{\partial t^2}} + u(0)\,\delta_0' + \dot{u}(0)\,\delta_0, \tag{4.13b}$$

and

$$\Delta_{1I} u \equiv \overline{\frac{\partial u}{\partial t}} + u(0)\,\delta_0. \tag{4.13c}$$

In these definitions δ_0, δ_0', δ_1 and δ_1' are taken as the delta functions and their first derivatives, with support on $t=0$ and $t=t_1$ respectively [21]. To be more precise, for every $v \in C^2$, the values of these linear functionals are

$$\delta_0(v) = \int_R v(x, 0)\, dx, \tag{4.14a}$$

$$\delta_0'(v) = -\int_R \dot{v}(x, 0)\, dx, \tag{4.14b}$$

$$\delta_1(v)=\int_R v(x,t_1)\,dx, \tag{4.14c}$$

and

$$\delta_1'(v)=-\int_R \dot{v}(x,t_1)\,dx. \tag{4.14d}$$

We recall that

$$\int_0^{t_1}\int_R \{u\mathscr{L}v-v\mathscr{L}u\}\,dx\,dt=\int_0^{t_1}\int_B \{u\mathscr{L}_B v-v\mathscr{L}_B u\}\,dx\,dt, \tag{4.15a}$$

$$\int_0^{t_1}\int_R \{v\ddot{u}-u\ddot{v}\}\,dx\,dt=\int_R \{\dot{u}(t_1)v(t_1)-\dot{u}(0)v(0)-u(t_1)\dot{v}(t_1)+u(0)\dot{v}(0)\}\,dx, \tag{4.15b}$$

and

$$\int_0^{t_1}\int_R \{v\dot{u}+u\dot{v}\}\,dx\,dt=\int_R \{u(t_1)v(t_1)-u(0)v(0)\}\,dx. \tag{4.15c}$$

The first of these relations follows from the definitions (4.5) and (4.7) of $\mathscr{L}$ and $\mathscr{L}_B$, applying the divergence theorem and using the symmetry of F, as well as equation (4.6). Equations (4.15b and c) are obvious.

In view of equations (4.15), it is clear that $\hat{\mathscr{P}}$ and Δ_{2P} are symmetric; on the other hand, the adjoints of Δ_{2I} and Δ_{1I} are given by:

$$\Delta_{2I}^{*}u=\overline{\frac{\partial^2 u}{\partial t^2}}-u(x,t_1)\,\delta_1'-\dot{u}(x,t_1)\,\delta_1, \tag{4.16a}$$

$$\Delta_{1I}^{*}u=-\overline{\frac{\partial u}{\partial t}}+u(x,t_1)\,\delta_1. \tag{4.16b}$$

Consider the functional valued operators

$$\mathscr{L}_{2P}=\rho\Delta_{2P}-\hat{\mathscr{P}}, \tag{4.17a}$$

$$\mathscr{L}_{2I}=\rho\Delta_{2I}-\hat{\mathscr{P}}, \tag{4.17b}$$

$$\mathscr{L}_{1I}=\rho\Delta_{1I}-\hat{\mathscr{P}}, \tag{4.17c}$$

where ρ is a function defined on $R\times[0,t_1]$ whose values are non-singular, $m\times m$ symmetric matrices. We assume ρ is independent of t. The time dependent problems with which we shall be concerned consist in finding a solution $u\in C^2$ to any one of the equations

$$\mathscr{L}_{2P}u=f_{2P}, \tag{4.18a}$$

$$\mathscr{L}_{2I}u=f_{2I}, \tag{4.18b}$$

or

$$\mathscr{L}_{1I}u=f_{1I}. \tag{4.18c}$$

We do not discuss the existence of solutions and consequently assume that a solution exists; *i.e.* that $f_{2P}\in\mathscr{R}(\mathscr{L}_{2P})$, $f_{2I}\in\mathscr{R}(\mathscr{L}_{2I})$ and $f_{1I}\in\mathscr{R}(\mathscr{L}_{1I})$. This assumption implies that

$$f_{2P}=\hat{f}_R-\rho u_1(x)\,\delta_1'+\rho u_0(x)\,\delta_0', \tag{4.19a}$$

$$f_{2I}=\hat{f}_R+\rho u_0(x)\,\delta_0'+\rho\dot{u}_0(x)\,\delta_0 \tag{4.19b}$$

and

$$f_{1I} = \hat{f}_R + \rho u_0(x)\delta_0, \tag{4.19c}$$

where

$$\hat{f}_R = \bar{f}_R - \beta_1[u_{B1}] - \beta_2[N]. \tag{4.20}$$

Here $\bar{f}_R$ stands for the functional associated with f_R by means of equations (4.3) and (4.4); u_{B1} and N are functions defined on B_1 and B_2, respectively, whereas u_0, $\dot{u}_0$ and u_1 are functions defined on R. As mentioned previously, it is assumed that f_R, u_{B1}, N, u_0, $\dot{u}_0$ and u_1 are such that f_{2P}, f_{2I} and f_{1I} belong to the range of the respective linear operators.

Let three time-dependent problems be defined as follows:

Problem i) A function u is a solution of this problem if and only if $u \in C^2$ and

$$\rho \frac{\partial^2 u}{\partial t^2} - \mathscr{L} u = f_R \quad \text{on} \quad R \times [0, t_1], \tag{4.21}$$

$$u = u_{B1} \quad \text{on} \quad B_1 \times [0, t_1], \tag{4.22a}$$

$$\mathscr{L}_B u = N \quad \text{on} \quad B_2 \times [0, t_1], \tag{4.22b}$$

$$u(x, 0) = u_0(x), \quad x \in R, \tag{4.23a}$$

$$u(x, t_1) = u_1(x), \quad x \in R. \tag{4.23b}$$

Problem ii) A function u is a solution of this problem if and only if $u \in C^2$ and it meets equations (4.21), (4.22), (4.23a), together with

$$\dot{u}(x, 0) = \dot{u}_0(x); \quad x \in R. \tag{4.24}$$

Problem iii) A function u is a solution of this problem if and only if $u \in C^2$ and it meets equations (4.22), (4.23a) together with

$$\rho \frac{\partial u}{\partial t} - \mathscr{L} u = f_R \quad \text{on} \quad R \times [0, t_1]. \tag{4.25}$$

The equivalence between these problems and those defined by equations (4.18) is given by

Theorem 4.1. *Let $\mathscr{L}_B$ be non-singular and $u \in C^2$. Then:*

a) *u is a solution of problem* i) *if and only if u satisfies Eq.* (4.18a).

b) *u is a solution of problem* ii) *if and only if u satisfies Eq.* (4.18b).

c) *u is a solution of problem* iii) *if and only if u satisfies Eq.* (4.18c).

Proof. We prove part a) of the theorem only. The proofs of the other two parts are similar.

Assume equations (4.21–4.23) are satisfied by $u \in C^2$. Then straightforward application of definitions (4.17a), (4.19a) and (4.20) of $\mathscr{L}_{2P}$, f_{2P} and $\hat{f}_R$ shows that (4.18a) is met by u.

Assume now that (4.18a) is met by $u \in C^2$. Take $v \in C^2$ and such that the closure of its support is contained in an open subset of $R \times [0, t_1]$. Then v together with all its derivatives vanish on the boundary of $R\times[0,t_1]$. Thus, we have

$$0=\langle v\,|\,\mathscr{L}_{2P}u-f_{2P}\rangle=\int_0^{t_1}\int_R\left\{\rho\frac{\partial^2 u}{\partial t^2}-\mathscr{L}u-f_R\right\}v\,dx\,dt. \tag{4.26}$$

Since this is true for any such v, it follows that (4.21) is satisfied at points (x,t) of the interior of $R\times[0,t_1]$. The continuity of both members of (4.21) implies that this equation is met also on the boundary of $R\times[0,t_1]$.

Now take $v\in C^2$ so that the closure of its support is disjoint to $B\times[0,t_1]$. Since the validity of (4.21) has already been established, for any such v we get

$$\begin{aligned}0=\langle v\,|\,\mathscr{L}_{2P}u-f_{2P}\rangle&=\langle v\,|\,\rho[u_1(x)-u(x,t_1)]\,\delta_1'+\rho[u(x,0)-u_0(x)]\,\delta_0'\rangle\\&=-\int_R\rho\{[u_1(x)-u(x,t_1)]\,\dot v(x,t_1)+[u(x,0)-u_0(x)]\,\dot v(x,0)\}\,dx.\end{aligned}$$

Since this equation is satisfied for every such v, equations (4.23) follow.

A similar reasoning for any $v\in C^2$ with a support whose closure does not have points in common with B_1 proves (4.22b). Thus it has been shown that (4.21), (4.22b) and (4.23) are met by u and therefore for every $v\in C^2$

$$0=\langle v\,|\,\mathscr{L}_{2P}u-f_{2P}\rangle=\langle v\,|\,\beta_1[u-u_{B1}]\rangle;$$

i.e.

$$\beta_1[u-u_{B1}]=0.$$

This establishes (4.22a) since $\mathscr{L}_R$ is non-singular. The proof is now complete.

Hereinafter we shall refer to these time dependent problems as problems i), ii) and iii), respectively.

Finally, we recall that $\mathscr{L}_{2P}$ is self-adjoint, while

$$\mathscr{L}_{2I}^*=\rho\Delta_{2I}^*-\hat{\mathscr{L}}, \tag{4.27a}$$

$$\mathscr{L}_{1I}^*=\rho\Delta_{1I}^*-\hat{\mathscr{L}}. \tag{4.27b}$$

5. Hamilton's Variational Principle

In view of the fact that $\mathscr{L}_{2P}$ is self-adjoint, Theorem 3.1 implies the following

Theorem 5.1 *(Hamilton's variational principle). Let $u\in C^2$ and $\mathscr{L}_B$ be non-singular. Then u is a solution of problem* i) *if and only if*

$$\delta\Omega_{2P}=0 \tag{5.1}$$

at u, where

$$\Omega_{2P}\equiv\langle u\,|\,\mathscr{L}_{2P}u\rangle-2\langle u\,|\,f_{2P}\rangle. \tag{5.2}$$

Proof. We can apply Theorem 4.1 because $\mathscr{L}_B$ is non-singular and consequently problem i) is equivalent to equation (4.18a). Taking into account that $\mathscr{L}_{2P}$ is self-adjoint, straightforward application of Theorem 3.1 yields Theorem 5.1.

As an application of the preceding Theorem, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}-\nabla^2 u=f_R \quad \text{on } R \tag{5.3}$$

with boundary conditions

$$u = u_{B1} \quad \text{on} \quad B_1 \times [0, t_1], \tag{5.4a}$$

$$n_i \frac{\partial u}{\partial x_i} = N \quad \text{on} \quad B_2 \times [0, t_2] \tag{5.4b}$$

and end values

$$u(x, 0) = u_0(x) \quad \text{on } R, \tag{5.5a}$$

$$u(x, t_1) = u_1(x) \quad \text{on } R. \tag{5.5b}$$

It is easily seen that the operator $\mathscr{L}_B$ associated with the wave equation is non-singular. Hence, from Theorem 5.1 we obtain the following

Corollary 5.2. *Let $u \in C^2$. Then u satisfies* (5.3), (5.4) *and* (5.5) *if and only if*

$$\delta\Omega = 0 \tag{5.6}$$

at u, where

$$\begin{aligned}\Omega \equiv & \int_0^{t_1}\int_R \left\{ \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t}\frac{\partial u}{\partial t} - 2 f_R u \right\} dx\, dt \\ & + 2\int_R \{[u(x, t_1) - u_1(x)]\, \dot u(x, t_1) - [u(x, 0) - u_0(x)]\, \dot u(x, 0)\}\, dx \\ & - 2\int_0^{t_1} \left\{ \int_{B1} [u - u_{B1}] \frac{\partial u}{\partial n} dx + \int_{B2} N u\, dx \right\} dt.\end{aligned} \tag{5.7}$$

Proof. The proof is obvious in view of Theorem 5.1.

This result is analogous to Hamilton's Principle; it has the inconvenient feature of being associated with a problem that is not well-posed. The way in which boundary conditions and end-values are incorporated in the functional (5.7) is more general than the manner in which it is usually done, because the set of admissible functions is the whole set C^2; *i.e.* they are not required to satisfy any boundary or initial and final time conditions.

6. Variational Principles for Initial Value Problems

Theorem 4.1 shows that when $\mathscr{L}_B$ is non-singular, the second and first order initial-value problems ii) and iii) are equivalent to equations (4.18b) and (4.18c), respectively. However, $\mathscr{L}_{2I}$ and $\mathscr{L}_{1I}$ are not self-adjoint, and in order to associate with these problems suitable variational principles, we must apply Corollary 3.1. To this end, we introduce a transformation

$$\Pi : C^2 \to C^2$$

such that for every $v \in C^2$ and $(x, t) \in R \times [0, t_1]$ we have

$$(\Pi v)(x, t) = v(x, t_1 - t). \tag{6.1}$$

Then, we define a transformation P of D^* into itself, by means of equations (2.11); *i.e.* for every $v \in C^2 = D$ and $w \in D^*$, we have

$$\langle v \mid P w \rangle = \langle \Pi v \mid w \rangle. \tag{6.2}$$

Taking into account the definitions (4.17b) and (4.17c) of $\mathscr{L}_{2I}$ and $\mathscr{L}_{1I}$, we can verify that for every $u, v \in C^2$

$$\langle v \mid P\mathscr{L}_{2I}u\rangle = \langle u \mid P\mathscr{L}_{2I}v\rangle, \tag{6.3a}$$

$$\langle v \mid P\mathscr{L}_{1I}u\rangle = \langle u \mid P\mathscr{L}_{1I}v\rangle, \tag{6.3b}$$

which show that $P\mathscr{L}_{2I}$ and $P\mathscr{L}_{1I}$ are self-adjoint.

For every $u \in D$ and $w \in D^*$, we shall write

$$\langle u * w\rangle = \langle u \mid Pw\rangle. \tag{6.4}$$

With this notation, we define

$$\begin{aligned}\Omega_2(u) &= \langle u \mid P\mathscr{L}_{2I}u\rangle - 2\langle u \mid Pf_{2I}\rangle \\ &\equiv \langle u * (\mathscr{L}_{2I}u - 2f_{2I})\rangle,\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Omega_1(u) &= \langle u \mid P\mathscr{L}_{1I}u\rangle - 2\langle u \mid Pf_{1I}\rangle \\ &\equiv \langle u * (\mathscr{L}_{1I}u - 2f_{1I})\rangle.\end{aligned} \tag{6.6}$$

Then, from Corollary 3.1 follows

Theorem 6.1. *Let $u \in C^2$ and $\mathscr{L}_B$ be non-singular. Then*

a) *u is a solution of the second order initial value problem* ii) *if and only if*

$$\delta\Omega_2 = 0 \tag{6.7}$$

at u.

b) *u is a solution of the first order initial value problem* iii) *if and only if*

$$\delta\Omega_1 = 0 \tag{6.8}$$

at u.

Proof. Since $\mathscr{L}_B$ is non-singular, the initial value problems ii) and iii) are, by virtue of Theorem 4.1, equivalent to equations (4.18b) and (4.18c), respectively. We can now apply Corollary 3.1 to complete the proof of the theorem because the mapping P is non-singular.

As an example of this Theorem consider the initial value problem for the wave equation and for the heat equation. For the wave equation one must find $u \in C^2$ such that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_i \partial x_i} = f_R \quad \text{on} \quad R \times [0, t_1], \tag{6.9}$$

subject to the boundary conditions

$$u = u_{B_1} \quad \text{on} \quad B_1 \times [0, t_1], \tag{6.10a}$$

$$n_i \frac{\partial u}{\partial x_i} = N \quad \text{on} \quad B_2 \times [0, t_1] \tag{6.10b}$$

and the initial conditions

$$u(x, 0) = u_0(x) \quad \text{on} \ R, \tag{6.11a}$$

$$\dot{u}(x, 0) = \dot{u}_0(x) \quad \text{on} \ R. \tag{6.11b}$$

The initial-value problem for the heat equation requires finding $u\in C^2$ such that

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x_i\partial x_i}=f_R \quad \text{on} \quad R\times[0,t_1], \tag{6.12}$$

subject to the boundary conditions (6.10) and the initial condition (6.11a). Theorem 6.1 implies the variational characterizations stated in

Corollary 6.1. *Let $u\in C^2$. Then*

a) *u is a solution of the initial value problem for the wave equation if and only if*

$$\delta\Omega_2=0, \tag{6.13a}$$

where

$$\begin{aligned}\Omega_2(u)\equiv&\int_R\left\{\frac{\partial u}{\partial x_i}*\frac{\partial u}{\partial x_i}+\frac{\partial u}{\partial t}*\frac{\partial u}{\partial t}-2f_R*u\right\}dx\\&-2\left\{\int_{B_1}[u-u_{B1}]*\frac{\partial u}{\partial n}dx+\int_{B_2}N*u\,dx\right\}\\&+2\int_R\{[u(x,0)-u_0(x)]\,\dot u(x,t_1)-\dot u_0(x)\,u(x,t_1)\}\,dx.\end{aligned} \tag{6.13b}$$

b) *u is a solution of the initial value problem for the heat equation if and only if*

$$\delta\Omega_1=0, \tag{6.14a}$$

where

$$\begin{aligned}\Omega_1(u)\equiv&\int_R\left\{\frac{\partial u}{\partial x_i}*\frac{\partial u}{\partial x_i}+u*\frac{\partial u}{\partial t}-2f_R*u\right\}dx\\&-2\left\{\int_{B_1}[u-u_{B1}]*\frac{\partial u}{\partial n}dx+\int_{B_2}N*u\,dx\right\}\\&+\int_R[u(x,0)-2u_0(x)]\,u(x,t_1)\,dx.\end{aligned} \tag{6.14b}$$

Here for every pair of functions $v,s\in C^0$

$$v*s=\int_0^{t_1}v(\tau)\,s(t_1-\tau)\,d\tau. \tag{6.15}$$

Proof. This Corollary follows immediately from Theorem 6.1 by setting

$$\mathscr{L}u=\frac{\partial^2 u}{\partial x_i\partial x_i}, \tag{6.16a}$$

$$\mathscr{L}_B u=n_j\frac{\partial u}{\partial x_j}. \tag{6.16b}$$

Observe that with this definition, $\mathscr{L}_B$ is non-singular.

The variational characterization for the heat equation presented here is similar to an example given by TONTI [16]. However, our treatment of boundary and initial conditions is slightly more general, because the admissible functions are not required to meet any of these conditions.

7. Connection with Gurtin's Variational Principles

Let f and g be two real valued continuous functions defined on $[0,\infty)$, and let $u, v \in C^0$. We define

$$(f*g)(t)=(g*f)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau; \quad t \geqq 0, \tag{7.1a}$$

$$(f*u)(x,t)=(u*f)(x,t)=\int_0^t f(\tau)\,u(x,t-\tau)\,d\tau; \qquad (x,t)\in R\times[0,t_1], \tag{7.1b}$$

$$(u*v)(x,t)=(v*u)(x,t)=\int_0^t u(x,\tau)\,v(x,t-\tau)\,d\tau; \quad (x,t)\in R\times[0,t_1]. \tag{7.1c}$$

Observe that if $u, v \in C^2$, then $f*u$ and $u*v$ belong to C^2.

We recall the convention adopted in Section 6, equation (6.4), according to which, for every functional $w \in D^*$ and every $u \in C^2 = D$, we write

$$\langle u*w\rangle = \langle u \mid P w\rangle = \langle \Pi u \mid w\rangle, \tag{7.2}$$

where Π is defined by (6.1). In view of the fact that Π is a one to one mapping of C^2 over C^2, a functional w is defined on C^2 when $\langle u*w\rangle$ is known for every $u\in C^2$. Consequently, given the functional w, another functional $f*w$ can be defined on C^2, by

$$\langle u*(f*w)\rangle = \langle (u*f)*w\rangle, \tag{7.3a}$$

which holds for every $u \in C^2$. Taking this definition into account, we can unambiguously drop the parentheses in the above expressions. If L is a functional valued operator defined on C^2, then the functional valued operator $f*L$ for every $u\in C^2$ is given by

$$(f*L)u = f*(Lu). \tag{7.3b}$$

In order to be more precise, in what follows we write $C^M_{t_1}$ for the set C^M. Observe that for every $t_1>0$ there is one such set of functions. The notation C^M_∞ will be used for the set of functions defined on $R\times[0,\infty)$ and possessing continuous derivatives up to order M. Given $u\in C^M_\infty$, the restriction of u to $R\times[0,t_1]$ is a member of $C^M_{t_1}$, and, conversely, every $u\in C^M_{t_1}$ is the restriction to $R\times[0,t_1]$ of some members of C^M_∞. Consequently, whenever $u, v\in C^2_\infty$ we can apply the definitions given previously to their restrictions to $R\times[0,t_1]$.

Two problems will be considered now; they are

G ii) A function $u\in C^2_\infty$ is a solution of this problem if and only if

$$\rho u - t*\mathcal{L}u = t*f_R + \rho(t\dot{u}_0 + u_0) \quad \text{on} \quad R\times[0,\infty), \tag{7.4}$$

$$t*u = t*u_{B1} \quad \text{on} \quad B_1\times[0,\infty), \tag{7.5a}$$

and

$$t*\mathcal{L}_B u = t*N \quad \text{on} \quad B_2\times[0,\infty). \tag{7.5b}$$

G iii) A function $u\in C^2_\infty$ is a solution of this problem if and only if

$$\rho u - 1*\mathcal{L}u = 1*f_R + \rho u_0 \quad \text{on} \quad R\times[0,\infty). \tag{7.6}$$

$$1 * u = 1 * u_{B1} \quad \text{on} \quad B_1 \times [0, \infty), \tag{7.7a}$$

and

$$1 * \mathscr{L}_B u = 1 * N \quad \text{on} \quad B_2 \times [0, \infty), \tag{7.7b}$$

where we have written t and 1 for the functions defined on $[0, \infty)$ whose values for every $t \geq 0$ are t and 1 respectively.

Problems G ii) and G iii) will be called GURTIN's problems, because they can be derived from problems ii) and iii) considered in Theorem 4.1 in a similar manner to that used by GURTIN to derive his integro-differential equations [4]. The equivalence between both sets of problems is established in the following.

Theorem 7.1. *A function u is a solution of problem* G ii) *if and only if u is a solution of problem* ii) *for every* $t_1 > 0$. *Similarly, it is a solution of problem* G iii) *if and only if it is a solution of problem* iii) *for every* $t_1 > 0$.

Proof. The fact that u is a solution of problem G ii) whenever it is a solution of problem ii) for every $t_1 > 0$, is obvious because equations (7.4) and (7.5) are obtained by taking the convolution of equations (4.21) and (4.22) with the function t, using (4.23a) and (4.24).

On the other hand, taking second order derivatives with respect to time of (7.4) and (7.5) gives the set of equations (4.21), (4.22). Finally, (7.4) and the equation derived from it by taking the first order derivative with respect to time yield respectively (4.23a) and (4.24), when they are evaluated at $t = 0$.

This completes the proof of the first part of the Theorem. The second part can be shown in a similar manner.

Given any $t_1 > 0$, consider the equations

$$t * \mathscr{L}_{2I} u = t * f_{2I}, \tag{7.8a}$$

$$1 * \mathscr{L}_{1I} u = 1 * f_{1I} \tag{7.8b}$$

for $u \in C^2_{t_1}$. Then, we prove:

Lemma 7.1. *For any* $u \in C^2_{t_1}$, *we have*

a)
$$t * \mathscr{L}_{2I} u = \overline{\rho u} - \overline{(t * \mathscr{L} u)} - \beta_1 [t * u] - \beta_2 [t * \mathscr{L}_B u], \tag{7.9a}$$

$$1 * \mathscr{L}_{1I} u = \overline{\rho u} - \overline{(1 * \mathscr{L} u)} - \beta_1 [1 * u] - \beta_2 [1 * \mathscr{L}_B u]; \tag{7.9b}$$

b)
$$t * f_{2I} = \overline{t * f_R} - \beta_1 [t * u_{B1}] - \beta_2 [t * N] + \overline{\rho u_0(x)} + \overline{\rho t \dot{u}_0(x)}, \tag{7.10a}$$

$$1 * f_{1I} = \overline{1 * f_R} - \beta_1 [1 * u_{B1}] - \beta_2 [1 * N] + \overline{\rho t u_0(x)}. \tag{7.10b}$$

Proof. This Lemma follows from the fact that the relations

$$\langle v * (t * \mathscr{L}_{2I} u) \rangle = \langle (v * t) * \mathscr{L}_{2I} u \rangle, \tag{7.11a}$$

$$\langle v * (1 * \mathscr{L}_{1I} u) \rangle = \langle (v * 1) * \mathscr{L}_{1I} u \rangle, \tag{7.11b}$$

$$\langle v * (t * f_{2I}) \rangle = \langle (v * t) * f_{2I} \rangle, \tag{7.11c}$$

$$\langle v * (1 * f_{1I}) \rangle = \langle (v * 1) * f_{1I} \rangle \tag{7.11d}$$

are satisfied for every $u, v \in C_{t_1}^2$ by the expressions given in equations (7.9) and (7.10).

By the help of Lemma 7.1, it is easy to prove:

Theorem 7.2. *Let $\mathscr{L}_B$ be non-singular. Then a function u is a solution of problem* G ii) *if and only if it satisfies* (7.8a) *for every $t_1 > 0$. Similarly, it is a solution of problem* G iii) *if and only if it meets* (7.8b) *for every $t_1 > 0$.*

Proof. By use of equations (7.9) and (7.10), the proof is analogous to that of Theorem 4.1 and we dispense with the details of it.

Let $u, v \in C_\infty^2$ and assume that, for every $t > 0$, a functional valued operator L is given on C_t^2. Then by taking the restrictions of u and v to $R \times [0, t]$, a function $\langle v | Lu \rangle$ of t is defined and when this function is continuous, $f * \langle v | Lu \rangle$ is given by equation (7.1a) for every $t > 0$.

Assume now that L stands for either $\mathscr{L}_{1I}$ or $\mathscr{L}_{2I}$. In this case

$$f * \langle v * Lu \rangle = \langle f * v * Lu \rangle = \langle v * f * Lu \rangle = \langle v | P(f * L) u \rangle. \tag{7.12}$$

The first of these equalities can be established by substitution of the definitions of $\mathscr{L}_{1I}$ and $\mathscr{L}_{2I}$ into the respective expressions. The other equalities are implied by equations (7.1–7.3).

Observe that (7.12) shows that the functional valued operator $P(f * L)$ is self-adjoint, because $\langle v * Lu \rangle$ is symmetric in u and v, by virtue of (6.3). Theorem 3.1 can be used now to formulate Gurtin-type variational principles associated with the functionals

$$\Lambda_2 = \langle u * t * \mathscr{L}_{2I} u \rangle - 2 \langle u * t * f_{2I} \rangle, \tag{7.13a}$$

$$\Lambda_1 = \langle u * 1 * \mathscr{L}_{1I} u \rangle - 2 \langle u * 1 * f_{1I} \rangle, \tag{7.13b}$$

defined for every $t > 0$; they are given by

Theorem 7.3. *Assume $\mathscr{L}_B$ is non-singular. Then*

a) *a function u is a solution of problem* G ii) *if and only if*

$$\delta \Lambda_2 = 0; \quad \textit{for every } t > 0. \tag{7.14a}$$

Similarly, u is a solution of G iii) *if and only if*

$$\delta \Lambda_1 = 0; \quad \textit{for every } t > 0. \tag{7.14b}$$

b) *In addition, for every $t > 0$ the following relations hold:*

$$\Lambda_2 = t * \Omega_2, \tag{7.15a}$$

$$\Lambda_1 = 1 * \Omega_1, \tag{7.15b}$$

$$\Omega_2 = \frac{d^2 \Lambda_2}{dt^2}, \tag{7.16a}$$

and

$$\Omega_1 = \frac{d\Lambda_1}{dt}. \tag{7.16b}$$

Proof. Part a) follows from Theorem 3.1. Equations (7.15) are implied by (7.12), (7.13) and the definitions (6.5) and (6.6) of Ω_2 and Ω_1, whereas (7.16) are obtained by taking the second and first order derivatives with respect to time of equations (7.15a) and (7.15b), respectively.

The following schemes summarize our results:

1)
$$\begin{array}{ccccc}
\text{Prob. (G ii)} & \leftrightarrow & t*(\mathscr{L}_{2I}u-f_{2I})=0 & \leftrightarrow & \Lambda_2 \\
t*\uparrow\downarrow\frac{d^2}{dt^2} & & t*\uparrow\downarrow\frac{d^2}{dt^2} & & t*\uparrow\downarrow\frac{d^2}{dt^2} \\
\text{Prob. (ii)} & \leftrightarrow & \mathscr{L}_{2I}u-f_{2I}=0 & \leftrightarrow & \Omega_2
\end{array}$$

2)
$$\begin{array}{ccccc}
\text{Prob. (G iii)} & \leftrightarrow & 1*(\mathscr{L}_{1I}u-f_{1I})=0 & \leftrightarrow & \Lambda_1 \\
1*\uparrow\downarrow\frac{d}{dt} & & 1*\uparrow\downarrow\frac{d}{dt} & & 1*\uparrow\downarrow\frac{d}{dt} \\
\text{Prob. (iii)} & \leftrightarrow & \mathscr{L}_{1I}u-f_{1I}=0 & \leftrightarrow & \Omega_1.
\end{array}$$

These schemes show that simplified versions of variational principles for initial-value problems can be obtained by taking derivatives with respect to time of Gurtin's functionals. This fact has special interest from a practical point of view, because Gurtin-type functionals have been constructed for many problems.

8. A Variational Principle for Linear Elastodynamics

In this section we illustrate the results obtained previously by deriving a variational principle for linear elastodynamics. The initial-value problem to be considered consists in finding a vector valued function $u \in C^2$ which satisfies the equations

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_l}\left(C_{ilkj}\frac{\partial u_k}{\partial x_j}\right) = P_i \quad \text{on} \quad R\times[0,t_1] \tag{8.1}$$

together with the displacement boundary conditions

$$u_i = \hat{u}_i \quad \text{on} \quad B_1\times[0,t_1], \tag{8.2a}$$

the traction boundary conditions

$$T_i \equiv C_{iljk}\frac{\partial u_k}{\partial x_j} n_l = \hat{T}_i \quad \text{on} \quad B_2\times[0,t_1], \tag{8.2b}$$

as well as the initial conditions

$$u_i(x,0) = d_i(x) \quad \text{on } R \tag{8.2c}$$

and

$$\dot{u}_i(x,0) = v_i(x) \quad \text{on } R. \tag{8.2d}$$

In these equations $u_i(x,t)$ and $P_i(x,t)$ denote the Cartesian components of the displacement vector $u(x,t)$ and the body force vector $P(x,t)$, respectively. The function $\rho(x)$ is the mass density of the medium and $C_{ijkl}(x)$ are the components of the elasticity tensor. The latter are assumed to satisfy the symmetry relations

$$C_{ijkl} = C_{jikl} = C_{klij} \tag{8.3a}$$

and

$$C_{ijkl}\,\xi_{ij}\,\xi_{kl} \geqq 0 \tag{8.3b}$$

for every symmetric ξ_{ij}, with equality holding only when ξ_{ij} vanishes identically. The functions d_i and v_i are prescribed initial displacements and initial velocities, while $\hat{u}$ and $\hat{T}$ are given surface displacements and surface tractions.

It may be seen that the mixed problem defined by equations (8.1–8.3) is of the same form as the initial-value problem considered in the second part of Theorem 4.1. The matrix A_{ij} of the operator $\mathscr{L}u$ defined in (4.5) is given, for a fixed i and j, by C_{ikjl}, and F vanishes identically.

One of the theorems that GURTIN obtained for initial-value problems of elastodynamics (Theorems 5.1 of [5]) in our notation reads as follows:

Theorem 8.1. *Assume $u \in C_\infty^2$ satisfies the boundary conditions* (8.2a). *For each $t \in [0, \infty)$ define*

$$\Lambda_2 \equiv \int_R \left\{ \rho(x)[u_i * u_i](x,t) + C_{ijkl}(x)\left[t * \frac{\partial u_k}{\partial x_l} * \frac{\partial u_i}{\partial x_j}\right](x,t) - 2[Q_i * u_i](x,t) \right\} dx - 2\int_{B_2} [t * \hat{T}_i * u_i](x,t)\, dx, \tag{8.4}$$

where

$$Q_i(x,t) = [t * P_i](x,t) + \rho(x)[t v_i(x) + d_i(x)] \quad on \quad R \times [0, \infty). \tag{8.5}$$

Then u is a solution of the initial value problem of elastodynamics if and only if

$$\delta \Lambda_2 = 0 \quad for\ every\ t > 0$$

at u.

According to part c) of Theorem 7.3, a simpler version of this variational principle would be that associated with the functional obtained by taking the second derivative of Λ_2; *i.e.*,

$$\ddot{\Lambda}_2 = \int_R \left\{ \rho(x)[(\dot{u}_i * \dot{u}_i)(x,t) + 2(u_i(x,0) - d_i(x))\,\dot{u}_i(x,t) - 2v_i(x)\,u_i(x,t)] + C_{ijkl}\left(\frac{\partial u_k}{\partial x_l} * \frac{\partial u_i}{\partial x_j}\right)(x,t) - 2(P_i * u_i)(x,t) \right\} dx - 2\int_{B_2} (\hat{T}_i * u_i)(x,t)\, dx. \tag{8.7}$$

An alternative procedure for obtaining this variational principle would be to make use of Theorem 6.1.

Theorem 8.2. *Let $u \in C^2$ and for each $t \in [0, \infty)$ define the functional*

$$\Omega_2 \equiv \int_R \left\{ \rho(x)\{[\dot{u}_i * \dot{u}_i](x,t) + 2[u_i(x,0) - d_i(x)]\,\dot{u}_i(x,t) - 2v_i(x)\,u_i(x,t)\} + C_{ijkl}\left[\frac{\partial u_k}{\partial x_l} * \frac{\partial u_i}{\partial x_j}\right](x,t) - 2[P_i * u_i](x,t) \right\} dx + 2\int_{B_1} [T_i * (\hat{u}_i - u_i)](x,t)\, dx - 2\int_{B_2} [\hat{T}_i * u_i](x,t)\, dx. \tag{8.7}$$

Then u is a solution of the initial-value problem if and only if

$$\delta\Omega_2 = 0 \tag{8.8}$$

at u.

Proof. It follows immediately from part a) of Theorem 6.1 because the operator $\mathscr{L}_B$ implied by equations (8.2b) is non-singular whenever C_{ijpq} satisfies relation (8.3b).*

Observe that the functional Ω_2 given by equation (8.7) reduces to the functional $\bar{A}_2$, when u meets the displacement boundary condition (8.2a).

As mentioned earlier there are many problems for which Gurtin-type variational principles have been formulated. Because of Theorem 7.3 it is sufficient to take the time derivatives of the corresponding functionals to obtain simplified versions of those principles. The functional (8.7) is only one example of many that can be derived in this manner.**

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* This condition is not required when the admissible displacements meet the boundary condition (8.2a). In that case, Theorem 8.1 can be established without its use, as GURTIN did [5].

** An application to dynamic problems of fluid-saturated porous elastic solids was made in [22] to simplify the results obtained in [14].

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