Integrodifferential Equations for Systems of Leaky Aquifers and Applications 2. Error Analysis of Approximate Theories

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This paper is the second of a series devoted to developing a method of analysis based on the integrodifferential equations of leaky aquifer systems. In part 1 [*Herrera and Rodarte,* 1973*a*] the integrodifferential equations were derived, and an interpretation for some of the approximate theories was given in terms of the memory functions occurring in the equations. The same equations are quite suitable for a systematic analysis of the errors involved, and therefore in this paper they are used to carry it out.

In recent years the theory of leaky aquifers has experienced dramatic advances that have permitted it to go beyond many of the simplifying assumptions that were used during the initial period of its development. Indeed the theory has continuously been perfected, from the simplest model due to *Jacob* [1946] and later improved by *Hantush and Jacob* [1955] to more sophisticated approximations due to *Hantush* [1960]. Finally, the need to resort to approximate solutions has been removed to a large extent by the construction of some analytical solutions [*Neuman and Witherspoon,* 1969*a, b*] and of numerical methods of analysis [*Javandel and Witherspoon,* 1969].

Because of this progress it is now possible to analyze quantitatively many problems that were not previously amenable to numerical treatment. However, approximate theories frequently have a simpler structure than the exact ones because they may depend on a smaller number of dimensionless parameters and consequently, there are many situations in which it is advisable to use them. To use approximate solutions efficiently, it is essential to know accurately their ranges of applicability. The study of the applicability of approximate theories also may be helpful in acquiring a deeper understanding of the theory of leaky aquifers.

On the other hand, the system of integrodifferential equations governing the dynamics of leaky aquifers that was developed in part 1 [*Herrera and Rodarte,* 1973*a*] has been used to construct a simple approximate numerical method [*Herrera and Figueroa,* 1969; *Herrera,* 1970; *Herrera and Rodarte,* 1973*b*], whose range of applicability is closely connected with that of Hantush's approximation for large values of time [*Neuman and Witherspoon,* 1970; *Herrera and Figueroa,* 1970].

The aquifer system that will be considered is made of two aquifers separated by an aquitard in which the flow is vertical (Figure 1). The following approximations are discussed: (1) neglecting the drawdown at the unpumped aquifer, (2) Hantush's small values of time approximation, and (3) Hantush's large values of time approximation.

The analysis is made for the case of a well pumping at a constant rate, but the results obtained for approximations 1 and 2 are valid whenever the drawdown is a nondecreasing function of time.

The main contributions about this matter are due to *Hantush* [1960, 1964] and to *Neuman and Witherspoon* [1969*c*]. In 1960, Hantush formulated his modified theory of leaky aquifers together with his asymptotic solutions for small and large values of time and proposed a range for its application on the basis of the error analysis of the Laplace transforms. Later, in 1964 he introduced a slight modification of the range corresponding to his solution for large values of time. An important step forward was given by *Neuman and Witherspoon* [1969*c*], who compared their analytical and numerical solutions with the approximate ones.

The main shortcoming of the approach used by Hantush to establish the range of applicability of his theories lies in the fact that the asymptotic behavior of the Laplace transform of a function gives information only about the asymptotic behavior of the function in the time domain but there is no way to establish accurately the error implied by the corresponding approximation.

On the other hand, *Neuman and Witherspoon* [1969*c*] were able to exhibit explicitly the error that approximate solutions yield in the cases that they considered and establish some rules for their application, but they agreed that a more complete analysis of the error would be desirable. A brief discussion of Hantush's approximation for large values of time has already been published [*Neuman and Witherspoon,* 1970; *Herrera and Figueroa,* 1970].

When the integrodifferential equations for leaky aquifers are used, the shape of the memory functions that correspond to each of the approximate theories can be exhibited explicitly, as has been done in part 1. In the present paper by using this approach a direct and accurate analysis is carried out, which has permitted the errors for the drawdown at the different parts of the aquifer system to be estimated.

It is difficult to define the relative error when the drawdown in the unpumped aquifer is neglected. If the standard definition is adopted, the errors at the unpumped aquifer are large at all times because the system starts from an unperturbed situation and neglecting the perturbation implies a 100% relative error even when the perturbation is very small. If one is interested in finding out whether a theory is applicable to a specific problem, this feature of the definition is unsatisfactory because the theory is actually applicable in many situations as long as the perturbation is small. Thus the fact that the relative error is very large is irrelevant in such applications. At the same time it must be observed that such a definition is not informative because in using it, all that can be said is that the relative error at the unpumped aquifer is very large, irrespective of whether the perturbation of the draw-

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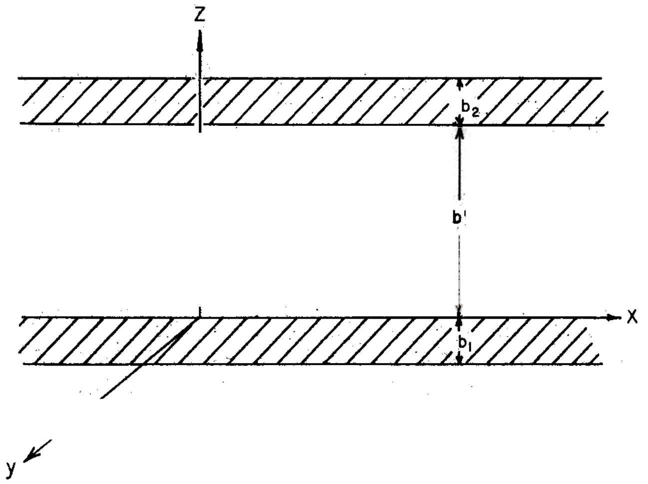

Fig. 1. The aquifer system.

down there is large or small. One can say that the definition erases any information that we may have.

This difficulty, which is very obvious in the case of the unpumped aquifer, is also present in the case of the aquitard. Indeed the drawdown at the aquitard predicted by the approximate solution necessarily tends to zero as one approaches the unpumped aquifer.

Thus in many situations the standard definition of relative error is not satisfactory at the unpumped aquifer and at the aquitard. Consequently, it is necessary to look for alternative definitions. Actually, which definition is the most convenient one depends on the particular application that is going to be made. In this work the relative error considered is obtained by dividing the error by the drawdown at the pumped aquifer. This definition has the advantage over the standard one of not erasing the information that we have on the behavior of the unpumped aquifer. There are applications for which it is relevant, but it must be realized that there are other situations for which it may be unsuited.

Through the use of that definition the bounds obtained for the relative errors made by neglecting the drawdown in the unpumped aquifer and by using Hantush's small time approximation are essentially independent of the distance to the well, but they depend significantly not only on the properties of the aquitard but on the storage capacity of the aquifers as well. Indeed they are monotonically decreasing functions of the storage capacity of the aquifers. In the case of Hantush's approximation for large values of time the relative error has a significant dependence on the properties of the aquitard, on the storage capacity of the pumped aquifer, and on the distance to the well.

As an application the formulas developed here are used to establish the range of applicability of the approximate theories. The range of applicability is defined as that on which each of the approximate theories considered yields errors that are not larger than 5%. Through the use of this criterion for all cases considered the ranges of applicability are larger than what *Hantush* [1960, 1964] had anticipated, a fact that is in agreement with *Neuman and Witherspoon*'s [1969*c*] results. There is a possible exception to this statement: the relative error in the case of *Hantush*'s [1964] approximation for large values of time increases with distance to the well, and probably, the range of applicability given by him is too optimistic for sufficiently large values of R.

Apparently, the method of analysis used in this work can be applied to other problems of interest in groundwater hydrology.

PRELIMINARY RESULTS

In this section some known results that will be used in the sequel are summarized.

When vertical flow is assumed in the aquitard, the drawdowns s_1, s_2, and s' at the aquifers illustrated in Figure 1 are governed by a system of differential equations [*Herrera and Rodarte*, 1973*a*] that can be expressed in terms of the dimensionless variables t', ζ, and R_i ($i = 1, 2$). In these variables this system becomes

$$\frac{\partial^2 s_1}{\partial R_1{}^2} + \frac{1}{R_1}\frac{\partial s_1}{\partial R_1} + \left(\frac{\partial s'}{\partial \zeta}\right)_{\zeta=0} = \frac{1}{\alpha_1}\frac{\partial s_1}{\partial t'} \tag{1a}$$

$$\frac{\partial^2 s'}{\partial \zeta^2} = \frac{\partial s'}{\partial t'} \tag{1b}$$

$$\frac{\partial^2 s_2}{\partial R_2{}^2} + \frac{1}{R_2}\frac{\partial s_2}{\partial R_2} - \left(\frac{\partial s'}{\partial \zeta}\right)_{\zeta=1} = \frac{1}{\alpha_{a2}}\frac{\partial s_2}{\partial t'}$$

For a well pumping at a constant rate these equations must be supplemented by the conditions

$$s'(R, 0, t') = s_1(R, t') \tag{2a}$$

$$s'(R, 1, t') = s_2(R, t') \tag{2b}$$

$$s_1(R, 0) = s_2(R, 0) = s'(R, \zeta, 0) = 0 \tag{3a}$$

together with

$$\lim_{R\to\infty} s_1(R, t') = \lim_{R\to\infty} s_2(R, t') = 0 \tag{3b}$$

$$\lim_{R_1\to 0} R_1 \frac{\partial s_1}{\partial R_1} = \frac{Q}{2\pi T} \tag{3c}$$

$$\frac{\partial s_2}{\partial R} = 0$$

When the drawdowns are subjected to conditions (3*a*), they fulfill (1) and (2) if and only if they satisfy the system of integrodifferential equations [*Herrera and Rodarte*, 1973*a*]:

$$\frac{\partial^2 s_1}{\partial R_1{}^2} + \frac{1}{R_1}\frac{\partial s_1}{\partial R_1} - f * \frac{\partial s_1}{\partial t'} + h * \frac{\partial s_2}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial s_1}{\partial t'} \tag{4a}$$

$$\frac{\partial^2 s_2}{\partial R_2{}^2} + \frac{1}{R_2}\frac{\partial s_2}{\partial R_2} - f * \frac{\partial s_2}{\partial t'} + h * \frac{\partial s_1}{\partial t'} = \frac{1}{\alpha_{a2}}\frac{\partial s_2}{\partial t'} \tag{4b}$$

$$s'(R, \zeta, t') = \int_0^{t'} \frac{\partial s_1}{\partial t'}(R, t' - \tau)\omega(\zeta, \tau)\, d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(R, t' - \tau)\omega(1 - \zeta, \tau)\, d\tau \tag{4c}$$

Here the notation for the convolution has been used; i.e., given any two functions $p(t)$ and $q(t)$,

$$(p * q)(t) = \int_0^t p(t - \tau)q(\tau)\, d\tau \tag{5}$$

The functions f, g, h, and ω were introduced in part 1 and are given in the notation. The first two will be called the memory functions, and the last two the influence functions.

Let $\hat{s}_1$ and $\hat{s}'$ be the approximate values of s_1 and s' that are obtained when the drawdown s_2 in the pumped aquifer is neglected. In this case, (4) become

$$\frac{\partial^2 \hat{s}_1}{\partial R^2} + \frac{1}{R}\frac{\partial \hat{s}_1}{\partial R} - f * \frac{\partial \hat{s}_1}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial \hat{s}_1}{\partial t'} \tag{6a}$$

and

$$\hat{s}'(R, \zeta, t') = \omega(\zeta, t') * \frac{\partial \hat{s}_1}{\partial t'}(R, t') \tag{6b}$$

On the other hand, the value s_s of the drawdown predicted by *Hantush*'s [1960] asymptotic solution for small values of time satisfies [*Herrera and Rodarte*, 1973*a*]

$$\frac{\partial^2 s_s}{\partial R^2} + \frac{1}{R}\frac{\partial s_s}{\partial R} - f_0 * \frac{\partial s_s}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial s_s}{\partial t'} \tag{7}$$

where

$$f_0(t') = (\pi t')^{-1/2} \tag{8}$$

and the drawdown s_L predicted by Hantush's asymptotic solution for large values of time satisfies another modified version of (6*a*), namely,

$$\frac{\partial^2 s_L}{\partial R^2} + \frac{1}{R}\frac{\partial s_L}{\partial R} - f_L * \frac{\partial s_L}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial s_L}{\partial t'} \tag{9}$$

where

$$f_L(t') = \quad + \tfrac{1}{3}\delta(t') = \quad + \delta(t') \int_0^\infty g(\tau)\, d\tau \tag{10}$$

Let $p_i(t)$ and $p_i'(t)$, where $i = 1, 2$, be two functions and their respective first-order derivatives defined for $t \geq 0$. When this notation is used, the main properties of the convolution that will be used in this work can be stated as follows:

$$p_1 * p_2 = p_2 * p_1 \tag{11}$$

$$p_1(0) = 0 \tag{12}$$

then

$$\frac{d}{dt}(p_1 * p_2) = p_1' * p_2 \tag{13}$$

If p_1 satisfies (12) and it is nondecreasing and p_2 is nonnegative, then $p_1 * p_2$ is nondecreasing; if in addition p_2 is nondecreasing, then

$$(p_1' * p_2)(t) \leq p_1(t)p_2(t) \tag{14}$$

The first property (11) is well known, and it is quoted in many textbooks of advanced calculus. The second property (13) can be established easily by applying the usual rules for the derivative of an integral when use is made of (12). The first part of the third property is now obvious in view of the second property (13) and the fact that

$$p_1' * p_2 \geq 0 \tag{15}$$

Finally, when p_2 is nondecreasing

$$\int_0^t p_2(t - \tau)p_1'(\tau)\, d\tau \leq p_2(t) \int_0^t p_1'(\tau)\, d\tau = p_1(t)p_2(t) \tag{16}$$

which can be derived by taking (12) into account.

ERROR ANALYSIS

In the appendix, expressions for the errors implied by the approximate theories considered are developed, and the assumptions under which they were derived are discussed. In this section the main results of the appendix are quoted to supply a basis for the corresponding error analysis.

Neglecting the drawdown at the unpumped aquifer. It is shown in the appendix that the errors

$$\theta_i(R, t') = s_i(R, t') - \hat{s}_i(R, t') \qquad i = 1, 2 \tag{17a}$$

$$\theta'(R, \zeta, t) = s'(R, \zeta, t') - \hat{s}'(R, \zeta, t') \tag{17b}$$

implied by neglecting the drawdown at the unpumped aquifer satisfy the integrodifferential equations

$$\frac{1}{\alpha_{a1}}\frac{\partial \theta_1}{\partial t'} + f * \frac{\partial \theta_1}{\partial t'} = h * \frac{\partial \theta_2}{\partial t'} \tag{18a}$$

$$\frac{1}{\alpha_{a2}}\frac{\partial \theta_2}{\partial t'} + f * \frac{\partial \theta_2}{\partial t'} = h * \frac{\partial s_1}{\partial t'} \tag{18b}$$

$$\theta'(R, \zeta, t') = \omega(\zeta, t') * \frac{\partial \theta_1}{\partial t'}(R, t') + \omega(1 - \zeta, t') \frac{\partial \theta_2}{\partial t'}(R, t') \tag{18c}$$

The solution of this system of equations is

$$\theta_1 = \frac{\partial(1 - q_1)}{\partial t'} * \frac{\partial(1 - q_2)}{\partial t'} * \frac{\partial e_1}{\partial t'} * s_1 \tag{19a}$$

$$\theta_2 = \frac{\partial(1 - q_2)}{\partial t'} * \frac{\partial e_2}{\partial t'} * s_1 \tag{19b}$$

$$\theta' = \left[\frac{\partial(1 - q_1)}{\partial t'} * \frac{\partial e_A^{(1)}}{\partial t'}(\zeta, t') + \frac{\partial e_A^{(1)}}{\partial t'}(\zeta, t')\right] * \frac{\partial(1 - q_2)}{\partial t'} * s \tag{19c}$$

where

$$q_i(t') \approx \hat{q}_i(t' \;= \exp(\alpha_{ai}^2 t') \operatorname{erfc} [\alpha_{ai}(t')^{1/2} \qquad i = \;, 2 \tag{20a}$$

$$e_1(t') = 4 \sum_{n=1}^\infty (-1)^{n+1} n \operatorname{erfc}\left(\frac{n}{t'^{1/2}}\right) \tag{20b}$$

$$e_2(t') = 2 \sum_{n=0}^\infty (-1)^n \operatorname{erfc}\left(\frac{n + \frac{1}{2}}{t'^{1/2}}\right) \tag{20c}$$

$$e_A^{(1)}(\zeta, t') = 2 \sum_{n=1}^\infty \{(-1)^{n+1} n[\omega_0(2n - \zeta, t') + \omega_0(2n + \zeta, t')] - [\omega_0(4n - 2 - \zeta, t') - \omega_0(4n - 2 + \zeta, t')]\} \tag{20d}$$

$$e_A^{(2)}(\zeta, t') = 2 \sum_{n=1}^\infty [\omega_0(4n - 2 - \zeta, t') - \omega_0(4n - 2 + \zeta, t')] \tag{20e}$$

Neglecting the drawdown at the unpumped aquifer yields relative errors that vary at different parts of the aquifer system. At the unpumped aquifer itself they are large at all times because the system starts from an unperturbed situation and neglecting the perturbation implies a 100% relative error even when the perturbation is very small. Therefore it is more informative and relevant to compare the error θ_2 at the unpumped aquifer with the drawdown s_1 at the pumped aquifer.

For similar reasons it is also convenient to compare the error at the aquitard with s_1.

Accordingly, define

$$\epsilon_1(t') = \theta_1/s_1 \tag{21a}$$

$$\epsilon_2(t') = \theta_2/s_1 \tag{21b}$$

$$\epsilon'(\zeta, t') = \theta'/s_1 \tag{21c}$$

Observe that $1 - \hat{q}_i(t')$, $e_i(t')$, and $e_A^{(i)}(\zeta, t')$ are nondecreasing functions of t' that vanish at $t' = 0$. For the case of a steady well pumping at a constant rate the functions s_i and s' of t' are also nondecreasing. Consequently, in view of the second and third properties (12–14) of the section on preliminary results, (19) imply that

$$\epsilon_1(t') \leq \frac{\partial(1 - \hat{q}_1)}{\partial t'} * \frac{\partial(1 - \hat{q}_2)}{\partial t'} * e_1 \tag{22a}$$

$$\epsilon_2(t') \leq \frac{\partial(1 - \hat{q}_2)}{\partial t'} * e_2 \tag{22b}$$

$$\epsilon'(t') \leq \frac{\partial(1 - \hat{q}_2)}{\partial t'} \left[\frac{\partial(1 - \hat{q}_1)}{\partial t'} * e_A^{(1)}(\zeta, t') + e_A^{(2)}(\zeta, t')\right] \tag{22c}$$

For many purposes an estimate of the relative errors less accurate than that given by (22) is satisfactory. It can be derived by repeated application of (14) to (22), which yields

$$\epsilon_1(t') \leq [1 - \hat{q}_1(t')][1 - \hat{q}_2(t')]e_1(t') \tag{23a}$$

$$\epsilon_2(t') \leq [1 - \hat{q}_2(t')]e_2(t') \tag{23b}$$

$$\epsilon'(\zeta, t') \leq [1 - \hat{q}_2(t')]\{[1 - \hat{q}_1(t')]e_A^{(1)}(\zeta, t') + e_A^{(2)}(\zeta, t')\} \tag{23c}$$

In the limit when α_{a1} and α_{a2} tend to infinity, the relations (23) become

$$\epsilon_1(t') \leq e_1(t') \tag{24a}$$

$$\epsilon_2(t') \leq e_2(t') \tag{24b}$$

$$\epsilon'(\zeta, t') \leq e_A^{(1)}(\zeta, t') + e_2^{(2)}(\zeta, t') = e_A(\zeta, t') \tag{24c}$$

by virtue of (A46) when the notation

$$e_A(\zeta, t') = e_A^{(1)}(\zeta, t') + e_A^{(2)}(\zeta, t') \tag{25}$$

is introduced.

Observe that relations (24) hold for arbitrary values of α_{a1} and α_{a2} because

$$0 \leq 1 - \hat{q}_i(t') \leq 1 \tag{26}$$

Therefore relations (23) and (24) together can be interpreted as stating that the relative errors are bounded by those corresponding to the case in which α_{a1} and α_{a2} are infinity and that the relative errors are reduced by some corrective factors that are combinations of products of $1 - q_1$ and $1 - q_2$ when those parameters are finite.

Hantush's approximation for small values of time. *Hantush*'s [1960, 1964] asymptotic solution s_s for small values of time assumes zero drawdown in the unpumped aquifer. So the total error in the drawdown at the pumped aquifer implied by this approximation can be written as

$$\theta_T = s_1(R, t') - s_s(R, t') = (s_1 - \hat{s}_1) + (\hat{s}_1 - s_s) = \theta_1 - \theta_s$$

where θ_1 is given by (17a) and

$$\theta_s = s_s - \hat{s}_1 \tag{28}$$

Observe that (27) can be interpreted as decomposing the total error in one part due to neglecting the drawdown in the unpumped aquifer and another part due to approximating the memory function $f(t')$ in (4a) by

$$f(t') \approx f_0(t') = (\pi t')^{-1/2}$$

In the appendix it is shown that an estimate of θ_s can be obtained by solving the integrodifferential equation

$$\frac{1}{\alpha_{a1}} \frac{\partial \theta_s}{\partial t'} + f_0 * \frac{\partial \theta_s}{\partial t'} = \Delta f * \frac{\partial s_1}{\partial t'}$$

subject to vanishing initial conditions. In this case the solution is

$$\theta_s = \frac{\partial(1 - \hat{q}_1)}{\partial t'} * \frac{\partial e_s}{\partial t'} * s_1 \tag{31}$$

where

$$e_s(t') = 2 \sum_{n=1}^{\infty} \operatorname{erfc}\left(\frac{n}{t'^{1/2}}\right) \tag{32}$$

and $\hat{q}_1(t')$ is given by (20a). Here again the functions $1 - \hat{q}_1$, e_s, and s_1 are monotonically increasing functions of time whose initial values are zero, so that relation (14) can be used. Thus

$$\epsilon_s(t') = \theta_s/s_1 \leq \frac{\partial(1 - \hat{q}_1)}{\partial t'} * e_s \tag{33}$$

A further application of relation (14) to (33) yields

$$\epsilon_s(t') \leq (1 - \hat{q}_1)e_s \tag{34}$$

In the limit when α_{a1} tends to infinity this relation becomes

$$\epsilon_s(t') \leq e_s(t') \tag{35}$$

Again relation (35) is valid for arbitrary values of α_{a1} because of (26). Consequently, relations (34) and (35) together can be interpreted as stating that for finite values of α_{a1} the errors corresponding to infinite α_{a1} are reduced by the factor $1 - \hat{q}_1(t')$, which is smaller than 1.

By virtue of (27) and the fact that θ_1 and θ_s are positive functions it is seen that the total relative error ϵ_T satisfies.

$$\epsilon_T(t') = \frac{|\theta_T|}{s_1} \leq \max(\epsilon_1, \epsilon_2) \tag{36}$$

Hantush's approximation for large values of time. Hantush's approximation for large values of time satisfies (9), and this equation can only be applied when it is possible to neglect the drawdown at the unpumped aquifer. Therefore the analysis of the error for Hantush's approximation for large values of time will be made under this assumption.

Equation (9) can be written in view of (10) as

$$\frac{\partial^2 s_L}{\partial R^2} + \frac{1}{R}\frac{\partial s_L}{\partial R} - s_L - \frac{1}{3}\frac{\partial s_L}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial s_L}{\partial t'}$$

On the other hand, (6a) is

$$\frac{\partial^2 s}{\partial R^2} + \frac{1}{R}\frac{\partial s}{\partial R} - s \qquad g * \frac{\partial s}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial s}{\partial t'} \tag{37b}$$

By subtracting (37a) from (37b) the equation

$$\frac{1}{\alpha_{a1}}\frac{\partial \theta_L}{\partial t'} + f * \frac{\partial \theta_L}{\partial t'} = \frac{\partial^2 \theta_L}{\partial R^2} + \frac{1}{R}\frac{\partial \theta_L}{\partial R}$$
$$+ \frac{1}{3}\frac{\partial s_L}{\partial t'} - g * \frac{\partial s_L}{\partial t'} \tag{38}$$

is obtained. At points where the errors are largest, lateral outflow tends to decrease them, and therefore it is possible to obtain an estimate of the errors by neglecting the radial derivatives in this equation; when this is done, the following equation is obtained:

$$\frac{1}{\alpha_{a1}}\frac{\partial \theta_L}{\partial t'} + \; * \frac{\partial \theta_L}{\partial t'} = \frac{1}{3}\frac{\partial s_L}{\partial t'} - g * \frac{\partial s_L}{\partial t'} \tag{39}$$

To see that lateral outflow tends indeed to decrease the maximum of θ_L, observe that the error Δ made when (38) is replaced by (39) satisfies

$$\frac{1}{\alpha_{a1}}\frac{\partial \Delta}{\partial t'} + f * \frac{\partial \Delta}{\partial t'} = \frac{\partial^2 \theta_L}{\partial R^2} + \frac{1}{R}\frac{\partial \theta_L}{\partial R}$$

At a maximum, $\partial\theta_L/\partial R = 0$, whereas $\partial^2\theta_L/\partial R^2 \geq 0$. Then it can be seen that $\partial\Delta/\partial t' \geq 0$. This fact is in agreement with well-known results for the heat equation.

To obtain an approximate solution valid for large values of time, (39) can be replaced by [*Herrera and Rodarte*, 1973a]

$$\frac{1}{\alpha_{ac}}\frac{\partial \theta_L}{\partial t'} - \theta_L = \frac{1}{3}\frac{\partial s_L}{\partial t'} - g * \frac{\partial s_L}{\partial t'} \tag{40}$$

Therefore

$$\theta_L = \alpha_{ac} G * \frac{\partial s_L}{\partial t'} - \alpha_{ac}^2 p * G * \frac{\partial s_L}{\partial t'} \tag{41}$$

where α_{ac} is given in the notation

$$G(t') = \int_{t'}^{\infty} g(\tau)\, d\tau = \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\exp(-n^2\pi^2 t')}{n^2} \tag{42}$$

and

$$p(t') = \exp(-\alpha_{ac} t' \tag{43}$$

From (40) it follows that

$$\epsilon_L = \frac{\theta_L}{s} \approx \frac{\theta_L}{s_L} = \left(\alpha_{ac} G * \frac{\partial s_L}{\partial t'} - \alpha_{ac}^2 p * G * \frac{\partial s_L}{\partial t'}\right) s_L$$

Here [*Hantush*, 1960, 1964]

$$s_L(R, t') = \frac{1}{4\pi T}\int_0^{t'} \exp\left[-\alpha_{ac}\tau - \frac{R^2}{4\alpha_{ac}\tau}\right]\tau^{-1}\, d\tau$$
$$= \frac{1}{4\pi T} W\left(\frac{R^2}{4\alpha_{ac}t'}, R\right) \tag{45}$$

Range of applicability. This section is devoted to a discussion of the range of applicability of approximate theories. The results obtained in previous works will be compared with the ranges derived by using (23), (34), and (44). The range of applicability will be defined as that for which the bounds for the relative errors given by (23), (34), and (44) are smaller than 5%.

TABLE 1. Range of Time on Which the Drawdown in the Unpumped Aquifer Can Be Neglected

	$\alpha_{a2} = 10^{-1}$	$\alpha_{a2} = 10^0$	$\alpha_{a2} = 10^1$	$\alpha_{a2} = \infty$
		Pumped Aquifer		
$\alpha_{a1} = 10^0$	1.23	0.54	0.43	0.40
$\alpha_{a1} = 10^1$	0.96	0.43	0.34	0.33
$\alpha_{a1} = \infty$	0.93	0.40	0.33	0.32
		Unpumped Aquifer		
	0.50	0.16	0.11	0.10
		Aquitard at ζ = 0.2		
$\alpha_{a1} = 10^0$	1.10	0.44	0.34	0.33
$\alpha_{a1} = 10^1$	0.88	0.40	0.32	0.31
$\alpha_{a1} = \infty$	0.87	0.39	0.31	0.30
		Aquitard at ζ = 0.5		
$\alpha_{a1} = 10^0$	0.88	0.32	0.23	0.22
$\alpha_{a1} = 10^1$	0.80	0.31	0.23	0.22
$\alpha_{a1} = \infty$	0.79	0.31	0.23	0.22
		Aquitard at ζ = 0.8		
$\alpha_{a1} = 10^0$	0.68	0.22	0.15	0.14
$\alpha_{a1} = \infty$	0.65	0.22	0.15	0.14

By proceeding in this manner, Table 1 and the tabulation below as well as Figure 2 were obtained. Table 1 gives the upper limit of the values of time t', for which the drawdown in the unpumped aquifer can be neglected. Hantush's approximation for small values of time can be applied according to this criterion at any time t' smaller than the following values:

α_{a1}	t'

Finally, Figure 2 gives the lower limit T_L' of the interval of time t', on which Hantush's approximation for large values of time can be applied at the pumped aquifer.

It must be recalled that Hantush's approximations assume zero drawdown at the unpumped aquifer. Therefore their ranges of applicability given in the above tabulation and Figure 2 must be restricted further by this condition (Table 1, the pumped aquifer).

The results contained in Table 1, the in-text tabulation, and Figure 1 were tested by comparing them with actual examples computed by *Neuman and Witherspoon* [1969a, b, c], and satisfactory agreement was obtained.

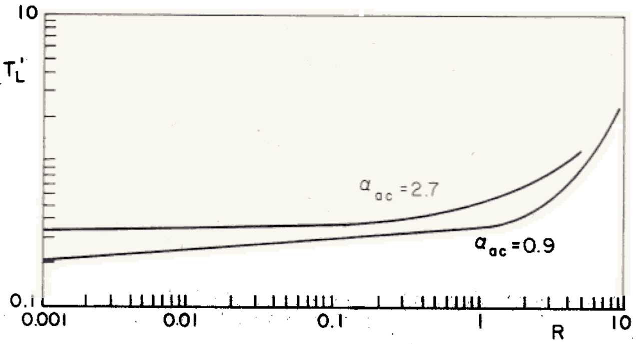

Fig. 2. The range of applicability of Hantush's approximation for large values of time.

Neglecting the drawdown at the unpumped aquifer. The studies by *Neuman and Witherspoon* [1969*a*, *b*, *c*] are the main contributions that have been made up to now to establish the interval of time on which this hypothesis can be applied. Previous studies by the same authors [*Witherspoon and Neuman*, 1967; *Neuman and Witherspoon*, 1968] are also informative.

The main conclusion of *Neuman and Witherspoon* [1969*b*, *c*] was that the drawdown at the unpumped aquifer can be neglected in computing the drawdown at the pumped aquifer and the aquitard whenever

$$t' \leq 10^{-1} \tag{46}$$

Regarding the unpumped aquifer they only asserted that the relative errors are large at all times. In addition, *Neuman and Witherspoon* [1969*b*, *c*] observed that in the case of the pumped aquifer and the aquitard the relative errors implied by this hypothesis depend significantly on the properties of the aquifers. However, this observation was not reflected in relation (46), which is independent of the properties of the aquifers.

In Table 1 the ranges of applicability of the hypothesis of zero drawdown at the unpumped aquifer as implied from the inequalities (23) are given. More precisely, the right-hand members of inequalities (23) are smaller than 0.05 whenever t' is smaller than the figures given in Table 1. It can be seen that this hypothesis can indeed be applied everywhere in the system whenever (46) is fulfilled, independently of the properties of the aquifer. However, this limit has a significant dependence on the properties α_{a1} and α_{a2} of the aquifers, being larger than 10^{-1} for any finite values of these parameters. The upper limit of the range of applicability also varies at the different parts of the aquifer system, being smallest at the unpumped aquifer and largest at the pumped aquifer. The worst situation occurs at the unpumped aquifer when α_{a2} is infinity; it is in this case that the value 10^{-1} for the upper limit of the interval of applicability is achieved.

Hantush's approximation for small values of time. The range of applicability advanced by *Hantush* [1960] for his approximation for small values of time is defined by the condition

$$10^{-} \tag{47}$$

Later, *Neuman and Witherspoon* [1969*b*, *c*] observed that it could be enlarged somewhat, but they did not establish any new limits for it.

The tabulation in the section on range of applicability was derived from the inequality (34) in the manner explained previously. It can be seen that the range of applicability at the pumped aquifer of Hantush's approximation for small values of time depends on the properties of that aquifer and can be 4 or 5 times larger than the range that Hantush had anticipated.

It must be observed that Hantush's solution assumes zero drawdown at the unpumped aquifer, and consequently, its applicability is further restricted by this condition, as discussed in the previous section.

Hantush's approximation for large values of time. *Hantush* [1960] stated that his solution for large values of time could be applied whenever

$$t' \geq 5 \tag{48}$$

Neuman and Witherspoon [1969*b*, *c*] have observed that the actual range of applicability was larger, but also in this case they did not establish new limits for it.

In the present work by using (44) in the manner explained previously it was shown that the interval of time on which this approximate solution can be safely used is a function of the adimensional distance to the well R and of the adimensional parameter

$$\alpha_{ac} = \frac{3\alpha_{a1}}{3 + \alpha_{a1}}$$

Figure 2 exhibits the graphs of the lower limit of the interval of time t' on which this theory can be applied for the values 0.9 and 2.7 of α_{ac}. It must be recalled that the parameter α_{ac} is bounded by 3 and therefore the values chosen for α_{ac} cover an interval sufficiently wide.

Observe that there is a large region relevant in applications in which the lower limit of the interval of applicability is 15 or more times smaller than that given by *Hantush* [1960].

Finally, recall that this theory assumes zero drawdown at the unpumped aquifer, and consequently, the note at the end of the previous section is pertinent here.

Herrera and Figueroa's approximation. The computational method developed by *Herrera and Figueroa* [1969] and *Herrera* [1970] simplifies significantly the computations required in studies of leaky aquifers [*Herrera and Rodarte*, 1973*b*] because it uncouples the corresponding system of differential equations. It can be thought of as a refinement of those methods introduced by *Hantush and Jacob* [1955] and *Hantush* [1967] because it takes into account the storage of the aquitard. On the other hand, it is also a generalization of the approximation for long times [*Herrera and Figueroa*, 1969, 1970; *Herrera*, 1970; *Herrera and Rodarte*, 1973*a*; *Neuman and Witherspoon*, 1970] because it does not require the use of the hypothesis that the drawdown at the unpumped aquifer vanishes.

According to the results obtained in some applications made so far [*Herrera and Rodarte*, 1973*b*] this method predicts correctly the effect of the interaction between the aquifers in many cases of practical interest. The times t' for which it can be applied are limited below by the values given in Figure 2, but they are not limited above because vanishing drawdown in the unpumped aquifer is not assumed. The range of applicability given in Figure 2 was deduced on the basis of the results obtained for an isolated well, but it seems that it is wider in the case of regional studies, as will be discussed elsewhere.

APPENDIX

Neglecting the drawdown in the unpumped aquifer. When the drawdown s_2 in the unpumped aquifer is neglected, (1), (2), and (3) for the corresponding approximate solutions $\hat{s}_1$ and $\hat{s}'$ become

$$\frac{\partial^2 \hat{s}_1}{\partial R_1^2} + \frac{1}{R_1}\frac{\partial \hat{s}_1}{\partial R_1} + \left(\frac{\partial \hat{s}'}{\partial \zeta}\right)_{\zeta=0} = \frac{1}{\alpha_{a1}}\frac{\partial \hat{s}_1}{\partial t'} \tag{A1a}$$

$$\frac{\partial^2 \hat{s}'}{\partial \zeta^2} = \frac{\partial \hat{s}'}{\partial t'} \tag{A1b}$$

$$\hat{s}'(R_1, 0, t') = \hat{s}_1(R_1, t') \tag{A1c}$$

$$\hat{s}'(R_1, 1, t') = 0 \tag{A1d}$$

$$\hat{s}_1(R, 0) = \hat{s}'(R, \zeta, 0) = 0 \tag{A2a}$$

$$\lim_{R\to\infty} \hat{s}_1(R, t') = 0 \tag{A2b}$$

and

$$\lim_{R\to\infty} R \frac{\partial \hat{s}_1}{\partial R} = \frac{Q}{2\pi T_1} \qquad \text{(A2c)}$$

Alternatively, (A1) can be replaced by (6).

On the other hand, the exact solutions s_i and s' satisfy (1), (2), and (3). These equations together with (A1) and (A2) imply the following system of equations for the errors θ_i and θ':

$$\frac{\partial^2 \theta_1}{\partial R_1{}^2} + \frac{1}{R_1}\frac{\partial \theta_1}{\partial R_1} + \left(\frac{\partial \theta'}{\partial \zeta}\right)_{\zeta=0} = \frac{1}{\alpha_{a1}}\frac{\partial \theta_1}{\partial t'} \qquad \text{(A3a)}$$

$$\frac{\partial^2 \theta_2}{\partial R_2} + \frac{1}{R_2}\frac{\partial \theta_2}{\partial R_2} \quad \left(\frac{\partial s'}{\partial \zeta}\right)_{\zeta=1} = \frac{1}{\alpha_{a2}}\frac{\partial \theta_2}{\partial t'} \qquad \text{(A3b)}$$

$$\frac{\partial^2 \theta'}{\partial \zeta^2} = \frac{\partial \theta'}{\partial t'} \qquad \text{(A3c)}$$

$$\theta'(R, 0, t') = \theta_1(R, t') \qquad \text{(A4a)}$$

$$\theta_2(R, t') = s_2(R, t') = s'(R, 1, t') \qquad \text{(A4b)}$$

$$\theta'(R, 1, t') = \theta_2(R, t') \qquad \text{(A4c)}$$

$$\theta_1(R, 0) = \theta_2(R, 0) = \theta'(R, \zeta, 0) = 0 \qquad \text{(A5a)}$$

$$\lim_{R\to\infty} \theta_1(R, t') = 0 \qquad \text{(A5b)}$$

$$\lim_{R\to 0} R \frac{\partial \theta_1}{\partial R} = 0 \qquad \text{(A5c)}$$

Alternatively, by virtue of (4) and (6), (A3) and (A4) can be replaced by

$$\frac{\partial^2 \theta_1}{\partial R_1{}^2} + \frac{1}{R_1}\frac{\partial \theta_1}{\partial R_1} - f * \frac{\partial \theta_1}{\partial t'} + h * \frac{\partial \theta_2}{\partial t'} = \frac{1}{\alpha_{a1}}\frac{\partial \theta_1}{\partial t'} \qquad \text{(A6a)}$$

$$\frac{\partial^2 \theta_2}{\partial R_2} + \frac{1}{R_2}\frac{\partial \theta_2}{\partial R_2} - f * \frac{\partial \theta_2}{\partial t'} + h * \frac{\partial s_1}{\partial t'} = \frac{1}{\alpha_{a2}}\frac{\partial \theta_2}{\partial t'} \qquad \text{(A6b)}$$

$$\theta'(R, \zeta, t') = \omega(\zeta, t') * \frac{\partial \theta_1}{\partial t'}(R, t') + \omega(1 - \zeta, t') * \frac{\partial \theta_2}{\partial t'}(R, t') \qquad \text{(A6c)}$$

Here the errors θ_i and θ' are given by (17).

At points where the errors are largest, lateral outflow tends to decrease them. Consequently, an estimate of the errors can be obtained by neglecting the radial derivatives that occur in (A3) and (A6). When this is done, (A3) yield

$$\left(\frac{\partial \theta'}{\partial \zeta}\right)_{\zeta=0} = \frac{1}{\alpha_{a1}}\frac{\partial \theta_1}{\partial t'} \qquad \text{(A7a)}$$

$$-\left(\frac{\partial s'}{\partial \zeta}\right)_{\zeta=1} = \frac{1}{\alpha_{a2}}\frac{\partial \theta_2}{\partial t'} \qquad \text{(A7b)}$$

$$\frac{\partial^2 \theta'}{\partial \zeta^2} = \frac{\partial \theta'}{\partial t'} \qquad \text{(A7c)}$$

whereas (A6) become (18).

The results to be derived are more easily expressed in terms of the errors for the limiting case in which both α_{a1} and α_{a2} tend to infinity. In this case, (A7) reduce to

$$\left(\frac{\partial l'}{\partial \zeta}\right)_{\zeta=0} = 0 \qquad \text{(A8a)}$$

$$\left(\frac{\partial s'}{\partial \zeta}\right)_{\zeta=1} = 0 \qquad \text{(A8b)}$$

$$\frac{\partial^2 l'}{\partial \zeta^2} = \frac{\partial l'}{\partial t'} \qquad \text{(A8c)}$$

whereas (A4) and (A5a) become

$$l'(R, 0, t') = l_1(R, t') \qquad \text{(A9a)}$$

$$l_2(R, t') = s'(R, 1, t') \qquad \text{(A9b)}$$

$$l'(R, 1, t') = l_2(R, t') \qquad \text{(A9c)}$$

$$l_1(R, 0) = l_2(R, 0) = l'(R, \zeta, 0) = 0 \qquad \text{(A9d)}$$

Here

$$l_i(R, t') = \lim_{\substack{\alpha_{a1}\to 0\\ \alpha_{a2}\to 0}} \theta_i(R, t') \qquad i = \ , 2 \qquad \text{(A10a)}$$

and

$$l'(R, \zeta, t') = \lim_{\substack{\alpha_{a1}\to 0\\ \alpha_{a2}\to 0}} \theta'(R, \zeta, t') \qquad \text{(A10b)}$$

Equations (1b), (2a), and (A8b) constitute a well-posed problem for s', whose solution is given by

$$s'(\zeta, t') = e(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \qquad \text{(A11a)}$$

and consequently,

$$l_2(R, t') = e_2 * \frac{\partial s_1}{\partial t'} \qquad \text{(A11b)}$$

where

$$e(\zeta, t') = \omega_0(\zeta, t') + \sum_{n=1}^{\infty} (-1)^n[\omega_0(2n + \zeta, t') - \omega_0(2n - \zeta, t')] \qquad \text{(A12a)}$$

and e_2 is given by (20c). Observe that

$$e_2(t') = e(1, t') \qquad \text{(A12b)}$$

This well-posed problem for the heat equation and some others that follow were solved by using Duhamel's method together with the method of images in a manner similar to that used in part 1 of this paper. The details will not be given here, but in any case the results can be checked by direct substitution in the equations and boundary conditions of the corresponding well-posed problems.

Equations (A8a) and (A8c) together with (A9d) also constitute a well-posed problem for l'. The method of images can be used to satisfy (A8a) and (A8b). When this is done, Duhamel's method yields

$$l'(R, \zeta, t') = e_A(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \qquad \text{(A13)}$$

where

$$e_A(\zeta, t') = 2 \sum_{n=1}^{\infty} \{(-1)^{n+1} n[\omega_0(2n - \zeta, t') + \omega_0(2n + \zeta, t')]\} = e_A{}^{(1)}(\zeta, t') + e_A{}^{(2)}(\zeta, t') \qquad \text{(A14)}$$

and $e_A{}^{(1)}$ and $e_A{}^{(2)}$ are given by (20d) and (20e). In view of

(A9a), (A13) at $\zeta = 0$ becomes

$$l_1(R, t') = e_1(t') * \frac{\partial s_1}{\partial t'}(R, t') \qquad (A15)$$

where

$$e_1(t') = e_A(0, t') = 4 \sum_{n=1}^{\infty} (-1)^{n+1} n \operatorname{erfc}(n/t'^{1/2}) \qquad (A16)$$

which establishes (20b).

On the other hand, (A6c) implies that

$$l'(\zeta, t') = \omega(\zeta, t') * \frac{\partial l_1}{\partial t'}(R, t') + \omega(1 - \zeta, t') * \frac{\partial l_2}{\partial t'}(R, t') \qquad (A17)$$

It is convenient to recall that the first term of the right-hand member of this equation vanishes at $\zeta = 1$ and takes on the value l_1 at $\zeta = 0$. Similarly, the second term of the same right-hand member vanishes at $\zeta = 0$ and takes on the value l_2 at $\zeta = 1$. Hence

$$\omega(\zeta, t') * \frac{\partial l_1}{\partial t'}(R, t') = e_A^{(1)}(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \qquad (A18a)$$

and

$$\omega(1 - \zeta, t') * \frac{\partial l_2}{\partial t'}(R, t') = e_A^{(2)}(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \qquad (A18b)$$

because both members of these equations satisfy the heat equation (A8c) and assume the same values at $\zeta = 0$ and $\zeta = 1$. This last fact becomes clear when it is observed that

$$e_A^{(1)}(1, t') = e_A^{(2)}(0, t') = 0 \qquad (A19a)$$

$$e_A^{(1)}(0, t') = e_1(t') \qquad (A19b)$$

$$e_A^{(2)}(1, t') = e_2(t') \qquad (A19c)$$

and (A11b) together with (A15) is taken into account.

Equations (A11b), (A13), and (A15) give the errors for the case in which both α_{a1} and α_{a2} are infinity in terms of the drawdown s_1 at the pumped aquifer. We proceed now to generalize the results to the case in which α_{a1} and α_{a2} are finite. To this end, take the limit when α_{a1} and α_{a2} tend to infinity in (18a) and (18b) to obtain

$$h * \frac{\partial l_2}{\partial t'} = f * \frac{\partial l_1}{\partial t'} \qquad (A20a)$$

$$h * \frac{\partial s_1}{\partial t'} = f * \frac{\partial l_2}{\partial t'} \qquad (A20b)$$

Therefore (18b) can be written as

$$\frac{1}{\alpha_{a2}} \frac{\partial \theta_2}{\partial t'} + f * \frac{\partial \theta_2}{\partial t'} = f * \frac{\partial l_2}{\partial t'} \qquad (A21)$$

Let $q_i(t')$, where $i = 1, 2$, be such that

$$\alpha_{ai} q_i * f = 1 - q_i \qquad t' \geq 0 \qquad (A22a)$$

which implies that

$$q_i(0) = \qquad (A22b)$$

Then in view of (18a), (A21), and (A5a) we have

$$\theta_1 = \alpha_{a1} q_1 * h * \frac{\partial \theta_2}{\partial t'} \qquad (A23a)$$

and

$$\theta_2 = \alpha_{a2} q_2 * f * \frac{\partial l_2}{\partial t'} \qquad (A23b)$$

These equations can be transformed by means of (A22a) into

$$\theta_2 = (1 - q_2) * \frac{\partial l_2}{\partial t'} \qquad (A24a)$$

$$\theta_1 = \alpha_{a1} q_1 * h * \frac{\partial(1 - q_2)}{\partial t'} * \frac{\partial l_2}{\partial t'} \qquad (A24b)$$

where it was possible to use (13) by virtue of (A22b). Finally, the error θ_1 in the pumped aquifer can be expressed in terms of l_1 by using (A20a) and (A22a). In this manner,

$$\theta_1 = (1 - q_1) * \frac{\partial(1 - q_2)}{\partial t'} * \frac{\partial l_1}{\partial t'} \qquad (A25)$$

In view of (A6c), (A24a), and (A25) it is possible to write

$$\theta' = \frac{\partial(1 - q_2)}{\partial t'} * \left[\frac{\partial(1}{\partial t'} \right] \qquad (A26)$$

$$= \frac{1}{\alpha_{a1}} \frac{\partial s_e}{\partial t'} \qquad (A27a)$$

$$\frac{\partial^2 s_e'}{\partial \zeta^2} = \frac{\partial s_e'}{\partial t'} \qquad 0 \leq \zeta < \infty \qquad (A27b)$$

$$s_e'(R, 0, t') = s_e(R, t') \qquad (A28a)$$

$$\lim_{\zeta \to \infty} s_e(R, \zeta, t') = 0 \qquad (A28b)$$

together with

$$s_e(R, 0) = s_e'(R, \zeta, 0) = 0 \qquad (A29a)$$

$$\lim_{R \to \infty} s_e(R, t') = 0 \qquad (A29b)$$

$$\lim_{R \to 0} R_1 \frac{\partial s_e'}{\partial R_1} = \frac{Q}{2\pi T_1} \qquad (A29c)$$

Alternatively, (A27) and (A28) can be replaced by (7). By subtracting (A1) and (A2) from (A27), (A28), and (A29a) the following equations are obtained:

$$\left(\frac{\partial \theta_e'}{\partial \zeta} \right)_{\zeta = 0} = \frac{1}{\alpha_{a1}} \frac{\partial \theta_e}{\partial t'} \qquad (A30a)$$

$$\frac{\partial^2 \theta_e'}{\partial \zeta^2} = \frac{\partial \theta_e'}{\partial t'} \qquad 0 \leq \zeta \leq 1 \qquad (A30b)$$

$$\theta_s'(R, 0, t') = \theta_s(R, t') \quad \text{(A31}a\text{)}$$

$$\theta_s'(R, 1, t') = s_s'(R, 1, t') \quad \text{(A31}b\text{)}$$

$$\theta_s(R, 0) = \theta_s'(R, \zeta, 0) = 0 \quad \text{(A32)}$$

which characterize the errors

$$\theta_s = s_s \quad \hat{s}_1 \quad \text{(A33}a\text{)}$$

$$\theta_s' = s_s' \quad \hat{s}' \quad \text{(A33}b\text{)}$$

In (A30a) the terms containing radial derivatives have been eliminated by a reasoning similar to that used to obtain (A7). On the other hand, from (6a) and (7) it follows that

$$\frac{1}{\alpha_{a1}} \frac{\partial \theta_s}{\partial t'} + f_0 * \frac{\partial \theta_s}{\partial t'} = \Delta f * \frac{\partial \hat{s}_1}{\partial t'} \approx \Delta f * \frac{\partial s_1}{\partial t'} \quad \text{(A34}a\text{)}$$

Here

$$\Delta f = f - f_0 = \frac{2}{(\pi t')^{1/2}} \sum_{n=1}^{\infty} e^{-n^2/t} \quad \text{(A34}b\text{)}$$

It is convenient to introduce now the auxiliary functions $\hat{s}''(R, \zeta, t')$, $\theta''(R, \zeta, t')$, and $\theta'''(R, \zeta, t')$; they are defined by the following set of conditions:

$$\frac{\partial^2 \hat{s}''}{\partial \zeta^2} = \frac{\partial \hat{s}''}{\partial t'} \qquad 0 \le \zeta < \infty \quad \text{(A35}a\text{)}$$

$$\hat{s}''(R, 0, t') = \hat{s}_1(R, t') \approx s_1(R, t) \quad \text{(A35}b\text{)}$$

$$\lim_{\zeta \to \infty} \hat{s}''(R, \zeta, t') = 0 \quad \text{(A35}c\text{)}$$

$$\hat{s}''(R, \zeta, 0) = 0 \qquad 0 \le \zeta < \infty \quad \text{(A35}d\text{)}$$

$$\frac{\partial^2 \theta_s''}{\partial \zeta^2} = \frac{\partial \theta_s''}{\partial t''} \qquad 0 \le \zeta < \infty \quad \text{(A36}a\text{)}$$

$$\theta_s''(R, 0, t') = \theta'(R, t') \quad \text{(A36}b\text{)}$$

$$\lim_{R \to \infty} \theta_s''(R, \zeta, t') = 0 \quad \text{(A36}c\text{)}$$

$$\theta_s''(R, \zeta, 0) = 0 \quad \text{(A36}d\text{)}$$

$$\frac{\partial^2 \theta_s'''}{\partial \zeta^2} = \frac{\partial \theta_s'''}{\partial t'} \qquad 0 \le \zeta \le \quad \text{(A37}a\text{)}$$

$$\theta_s'''(R, 0, t') = 0 \quad \text{(A37}b\text{)}$$

$$\theta_s'''(R, 1, t') = \hat{s}''(R, 1, t') \quad \text{(A37}c\text{)}$$

$$\theta_s'''(R, \zeta, 0) = 0 \quad \text{(A37}d\text{)}$$

With these definitions it can be seen that

$$\theta_s' = \theta_s'' + \theta_s''' \qquad 0 \le \zeta \le 1 \quad \text{(A38)}$$

The limits of θ_s, θ_s', θ_s'', and θ_s''' when α_{a1} tends to infinity will be represented by l_s, l', l'', and l''', respectively. With the use of this notation, (A30a) implies in view of (A38) that

$$(\partial l_s''/\partial \zeta)_{\zeta=0} + (\partial l_s'''/\partial \zeta)_{\zeta=0} = 0 \quad \text{(A39)}$$

It is now possible to obtain $\hat{s}''$, l_s''', l_s'', l_s', and l_s successively. From (A35) an expression for $\hat{s}''$ in terms of s_1 can be derived. Then (A37) determine l_s''', which can be used to obtain l_s'' from (A36a), (A36c), (A36d), and (A39). Finally, l_s' and l_s are given by (A38) and (A31a), respectively.

By proceeding in this manner the following equations are obtained:

$$\hat{s}''(R, \zeta, t') = \omega_0(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \quad \text{(A40}a\text{)}$$

$$l_s'''(R, \zeta, t') = e_s'''(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \quad \text{(A40}b\text{)}$$

$$l_s''(R, \zeta, t') = e_s''(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t')$$

$$l_s'(R, \zeta, t') = e_s'(\zeta, t') * \frac{\partial s_1}{\partial t'}(R, t') \quad \text{(A40}d\text{)}$$

$$l_s(R, t') = e_s(t') * \frac{\partial s_1}{\partial t'}(R, t') \quad \text{(A40}e\text{)}$$

where

$$e_s'''(\zeta, t') = \sum_{n=1}^{\infty} [\omega_0(2n - \zeta, t') \quad \omega_0(2n + \zeta, t')] \quad \text{(A41}a\text{)}$$

$$e_s''(\zeta, t') = 2 \sum_{n=1}^{\infty} \omega_0(2n + \zeta, t') \quad \text{(A41}b\text{)}$$

$$e_s'(\zeta, t') = \sum_{n=1}^{\infty} [\omega_0(2n \quad \zeta, t') + \omega_0(2n + \zeta, t')] \quad \text{(A41}c\text{)}$$

whereas e_s is given by (32).

Equation (A34a) implies that

$$f_0 * \frac{\partial l_s}{\partial t'} \approx \Delta f * \frac{\partial s_1}{\partial t'}$$

which allows writing that equation as

$$\frac{1}{\alpha_{a1}} \frac{\partial \theta_s}{\partial t'} + f_0 * \frac{\partial \theta_s}{\partial t'} = f_0 \frac{\partial l_s}{\partial t'}$$

Consequently,

$$\theta_s = \alpha_{a1} \hat{q}_1 * f_0 * \frac{\partial l_s}{\partial t'} \quad \text{(A44)}$$

if $\hat{q}_i(t')$, where $i = 1, 2$, is such that

$$\alpha_{ai} \hat{q}_i * f_0 = \quad - \hat{q}_i \quad \text{(A45}a\text{)}$$

and

$$\hat{q}_i(0) = \quad \text{(A45}b\text{)}$$

The solution of (A45) can be derived from the formulas for the yield given by *Hantush* [1964, p. 336]; it is

$$\hat{q}_i(t) = \exp(\alpha_{ai}^2 t') \operatorname{erfc}[\alpha_{ai}(t')^{1/2}] \quad \text{(A46)}$$

Equation (A44) can be transformed by means of (A40e) and (A45) into (31).

NOTATION

b_i	thickness of ith aquifer, L.
b'	thickness of aquitard, L.
$e_1(t')$	function defined by (20b).
$e_2(t')$	function defined by (20c).
$e_A(\zeta, t')$	function defined by (25).
$e_A^{(1)}(\zeta, t')$	function defined by (20d).

$e_A^{(2)}(\zeta,t')$ function defined by (20*e*).
$e_s(t')$ function defined by (32).
$$f(t') = 1 + 2 \sum_{n=1}^{\infty} \exp(-n^2\pi^2 t').$$
$f_0(t') = (\pi t')^{-1/2}$.
$f_L(t')$ function defined by (10).
$\Delta f(t') = f(t') - f_0(t')$.
$g(t') = f(t') - 1$.
$G(t')$ function defined by (42).
$$h(t') = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp(-n^2\pi^2 t').$$
K_i permeability of ith aquifer, L/T.
K' permeability of aquitard, L/T.
l_i, l', l_s limiting values of θ_i, θ', and θ_s, respectively, when α_{a1} and α_{a2} tend to infinity.
Q pumping rate from aquifer 1, L^3/T.
$q_i(t')$ functions defined by (A22).
$\hat{q}_i(t')$ functions defined by (20*a*).
r radial distance to pumping well, L.
$r/B_i = r(K'/K_i b_i b')^{1/2}$.
$R_i = r/B_i$.
$R = R_1$.
s_i drawdown in ith aquifer, L.
s' drawdown in aquitard, L.
$\hat{s}_i$, $\hat{s}'$ values of s_i and s', respectively, when s_2 is taken to be zero.
s_s, s_s' values of s_1 and s', respectively, predicted by Hantush's approximation for small values of time.
s_L values of s_1, predicted by Hantush's approximation for large values of time.
S_{si} specific storage of ith aquifer, L^{-1}.
S_s' specific storage of aquitard, L^{-1}.
S_i storage coefficient of ith aquifer, equal to $S_{si}b_i$.
S' storage coefficient of aquitards, equal to $S_s'b'$.
t time, T.
t' dimensionless time, equal to $\alpha' t/b'^2$.
$t_{Di} = \alpha_i t/r^2$.
T_i transmissibility of ith aquifer, equal to $K_i b_i$, L^2/T.
T_L' lower limit of the interval of dimensionless time t', on which Hantush's approximation for large values of time can be applied at the pumped aquifer (Figure 2).
$W(u, v)$ well function for leaky aquifers.
z vertical coordinate, L.
$\alpha_i = K_i/S_{si} = T_i/S_i$, L^2T^{-1}.
$\alpha' = K'/S_s'$, L^2T^{-1}.
$\alpha_{ai} = S'/S_i$.
$\alpha_{ac} = 3\alpha_{a1}/(3 + \alpha_{a1})$.
$\beta_i = r/4b_i[(K'S_s'/K_iS_{si})^{1/2}]$.
$\delta(t')$ Dirac's delta function.
ϵ_i relative error, equal to θ_i/s_1.
ϵ' relative error, equal to θ'/s_1.
ϵ_s relative error, equal to θ_s/s_1.
ϵ_L relative error, equal to θ/s_1.
$\zeta = z/b'$.
$\omega_0(\zeta, t') = \text{erfc}\,[\zeta/2(t')^{1/2}]$.
$$\omega(\zeta, t') = 1 - \zeta - 2 \sum_{n=1}^{\infty} (e^{-n^2\pi^2}/n\pi) \sin n\pi\zeta.$$
$\theta_i = s_i - \hat{s}_i$, L
$\theta' = s' - \hat{s}'$, L.
$\theta_T = s_1 - s_s = \theta_1 - \theta_s$, L
$\theta_s = s_s - \hat{s}_1$, L.
$\theta_L = s - s_L$, L.

The dimensionless variables R_i, t', and α_{ai} are related with the more familiar variables r/B_i, t_{Di}, and β_i as follows:

$R_i = r/B_i$.
$t' = [(r/B_i)^4/16\beta_i^2]t_{Di}$.
$\alpha_{ai} = [4\beta_i/(r/\beta_i)]^2$.

Inversely, the more familiar variables r/B_i, t_{Di}, and β_i are related with the dimensionless variables R_i, t', and α_{ai} as follows:

$r/B_i = R_i$.
$t_{Di} = \alpha_{ai}t'/R_i^2$.
$\beta_i = \tfrac{1}{4}\, \alpha_{ai}^{1/2}R_i$.

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