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APPLICATIONS OF DUAL PRINCIPLES TO DIFFUSION EQUATIONS

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APPLICATIONS OF DUAL PRINCIPLES TO DIFFUSION EQUATIONS

by

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INTRODUCTION. The results presented in this paper were obtained in the course of an investigation on transient flow of fluids in porous media but they are equally applicable to other problems governed by the heat diffusion equation.

Variational principles have been used extensively in ground water hydrology. Recently, they have served as basis for the development of finite element techniques for fluid flow problems in porous media 1, 2].

A variational principle is an assertion stating that the derivative or variation of some functional vanishes if and only if a given equation is fulfilled. An extremum principle is one which establishes the equivalence between an equation and the fact that some functional attains an extremum value, either a maximum or a minimum. Variational and extremum principles become interrelated when the functionals involved are differentiable, for then a necessary condition for the existence of an extremum is the vanishing of the variation. Naturally this condition is not sufficient and, therefore, the class of all extremum principles is a proper subset of the class of all variational principles. For linear operators, a sufficient condition for the variational principle to be extremal is that the operator involved be positive. This result can be generalized to nonlinear equations introducing the notion of convex functionals; it is then required that the functional involved be convex 3].

Traditionally the theoretical foundations of variational methods have been placed on the theory of differentiation on Hilbert spaces, or more generally, on Banach spaces 4], and, in particular, on the results pertaining to the notion of potential operators [5]. Essentially, one can say that a sufficient condition for an operator equation to admit a variatonal formulation is that the operator be potential, and this will be so if and only if its derivative is a symmetric bilinear functional. When the operator is not potential it is always possible, at least in principle, to transform the problem into an equivalent one for which the operator involved satisfies this condition. Though it has been claimed that this is the key to obtaining a very wide class of variational principles, applicable to any equation 6, 7], one has to be aware of its limited practical value due to difficulties involved in finding such transformations.

Dual principles, also called complementary or reciprocal, form another class of variational principles. Examples are the Lagrange and Hamilton principles in mechanics. In elasticity and the theory of structures the energy principle and the principle of complementary energy have been used extensively. When a dual principle is available, the problem considered is formulated variationally in two different but interrelated ways. In one formulation a solution is characterized by a maximum principle and in the other by a minimum principle. The maximum and minimum values of the respective functionals are the same.

The value of dual variational principles in applications is great, because the difference between the values of both functionals for two different trial functions can be used as a measure of the accuracy of approximate solutions. In many cases the significance of dual variational principles is enhanced because the functionals involved have by themselves some relevant physical meaning, e.g. the energy. Applications of the principles have abounded and accounts of them are available [8-12]

The work by Noble and Sewell [12] is especially relevant. These authors have derived dual variational principles in the form of generalized Lagrange and generalized Hamilton principles. Lagrange principles apply to a system of equations of the form

$$T^* u = Y$$

$$T \frac{\partial Y}{\partial y} = - \frac{\partial Y}{\partial u}$$

where T is a linear operator, T* its adjoint and Y a nonlinear functional

Hamilton principles apply to the system

$$T^* u = \frac{\partial X}{\partial x} \qquad (2a)$$

$$Tx = \frac{\partial X}{\partial u} \qquad (2b)$$

Here X is a nonlinear functional. The extremum principles are established if some functionals defined in terms of Y and X have properties connected with the notions of convex, concave and saddle-shaped functionals.

Variational and extremum principles of the various types described have been developed for steady-state problems in fluid flow through porous media. They are associated with the classical results of potential theory and elliptic differential equations. For initial value problems the development has been less satisfactory. Only variational principles of the simplest type are available, and up to now neither extremum nor dual principles have been obtained.

The available variational principles for transient flow of ground water were formulated by Neuman and Witherspoon 13, 14] using an approach developed by Gurtin [15]. Gurtin's variational principles were originally obtained from considerations regarding the Laplace transforms of the basic differential equations, and he did not establish the connection of his results with the general theory of variational methods. Later, Sandhu and ister [;] and Tonti '] suggested how Gurtin approach he set within this framework and general formulation of inear ir tia val prob given by Herr and Bielak [']

To tional principl for initial-value probl is necess ry to introduce the initial conditions into the governing equatic In Gurtin method the inverse of the time operator is appl ed to obtain system of integro-dif ential equations which tains the initial conditions implic tly, and which iational principles can be derived with of convolutions This transformation is required if the pr bl is ormulated in terms of functional valued opera [']

This pproach which is quite suitable to tr ini ial and boundary condi systematically results in variational pr iples impler than those which Gurtin method vields '] and ha been within functional analytical ramework by Herrer [;] has ecently presented general formulation of iational principl nonlinear prob the main features of the theory being tha only requir the operators to be defined in linear space (no Hilbert Banach spa is assumed This theory incorporates kind of dual iational principl to Sewell 19] which establishes ink between Lagrange and Hamilton principl and iational principles tha not extremal

In this paper apply the advances in the theory of iational principles just described to problems of interest in ground water hydrology the ir part ecal. thematical esults and notati tha wil be used in the seque In the second part two kinds of iational pr ipl tablished of stationary iational pr iples which impl than those already known [ ] and dual extremum princip ilab previously

MATHEMATICAL PRELIMINARIES Each time-dependen probl to be idered in this study ill be iated with system of partial dif tial equations together with uitable boundary and initia conditi There wil be in add tion of systems functions whose elements will be called tates The sets describe inear space We will id subsets whose elements will be called admissib states

It will be assumed systematically tha is af subspa ther exist inear subspace C and elemen such tha

Clearly and incide if

We will deal excl ively with the problem of inding luti tha admissible tates

functional is defined eal valued function whose domain is the inear space D. Gi elemen of say tha the Gateaux derivative U) of exists U if $(d/d\lambda)$ [U λV)] exists for every V of In tha adopt the notation

$$< \Omega'(U), V > = \left[\frac{d}{d\lambda} \Omega (U + \lambda V) \right]_{\lambda=0} \qquad (4)$$

It can be verified readily that for every real α,

$$< \Omega'(U), \alpha V > = \alpha < \Omega'(U), V > \qquad (5)$$

The Gateaux derivative is additive if in addition,

$$< \Omega'(U), V_1 + V_2 > = < \Omega'(U), V_1 > + < \Omega'(U), V_2 > \qquad (6)$$

for every V_1 and V_2 belonging to D.

Given a functional $\Omega : D \to R^1$, for each $U \varepsilon \hat{E}$ and $V \varepsilon E$ we define the variation of Ω in U as a linear functional $\delta\Omega(U)$ on E such that for every $V \varepsilon E$,

$$< \delta\Omega(U), V > = \left[\frac{d}{d\lambda} \Omega (U + \lambda V) \right]_{\lambda=0}$$

Note that

$$< \delta\Omega(U), V > = < \Omega'(U), V >$$

for every $V \varepsilon E$.

For functionals possessing an additive, and consequently linear Gateaux variation, it is convenient to introduce the concept of convexity as an extension of that usually utilized for functions of several variables in elementary calculus [12].

Given two elements U_+ and U_- of $\hat{E}$, the following notation will be adopted.

$$\Delta \Omega = \Omega(U_+) - \Omega(U_-) \tag{9a}$$

$$\Delta U = U_+ - U_- \in E \tag{9b}$$

$$(\delta\Omega)_+ = \delta\Omega(U_+) \tag{9c}$$

$$(\delta\Omega)_- = \delta\Omega(U_-) \tag{9d}$$

Definition. A functional Ω is convex if

$$\Delta \Omega - < (\delta\Omega)_- , \Delta U > \geqslant 0 \tag{10}$$

for every pair of elements U_+ and U_- of $\hat{E}$. It is strictly convex if the inequality is strict whenever $U_+ \neq U_-$. The concept of concave or strictly concave functionals is obtained by reversing the inequality sign in (10).

Alternatively, an equivalent definition of concave and strictly concave functionals can be given by replacing (10) by the inequality

$$\Delta \Omega - < (\delta\Omega)_+ , \Delta U > \geqslant 0. \tag{11}$$

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In defining a saddle functional it is convenient to use, in addition to the previous notation,

$$(\delta_1 \Omega)_- = \delta_1 \Omega(U_-) \tag{16a}$$

$$(\delta_2 \Omega)_+ = \delta_2 \Omega(U_+) \tag{16b}$$

Definition. Let $\{E_1, E_2\}$ be a decomposition of E. Then, the functional Ω is said to be saddle, convex in E_1, and concave in E_2, if

$$\Delta \Omega - \langle (\delta_1 \Omega)_-, (\Delta U)_1 \rangle - \langle (\delta_2 \Omega)_+, (\Delta U)_2 \rangle \geqslant 0 \tag{17}$$

for every U_+ and U_- belonging to $\hat{E}$. The saddle is strict if the inequality holds strictly whenever $U_+ \neq U_-$

This definition is equivalent to requiring Ω to be convex in E_1 when $(\Delta U)_2 = 0$ and concave in E_2 for $(\Delta U)_1 = 0$ [3 and 12].

To derive dual variational principles for transient diffusion problems we will use results obtained by Sewell 18], in the form presented by Herrera 3]. They are contained in the following:

Theorem 1. Let $\{E_1, E_2\}$ be a decomposition of E and Ω a saddle functional, convex in E_1 and concave in E_2. Then, if $U \in \hat{E}$:

i) $\delta\Omega\ (U) = 0$

if and only if

$$\delta_1 \Omega(U) = 0$$

and

$$\delta_2 \Omega(U) = 0.$$

ii) For any U_a, $U_b \in \hat{E}$ that satisfy (19a) and (19b), respectively, we have

$$\Omega(U_a) \leqslant \Omega(U_b)$$

iii) If U is a solution of (18), then:

α) The maximum value of Ω among all admissible states that satisfy (19a) is attained at U;

β) The minimum value of Ω among all admissible states that satisfy (19b) is attained at U;

γ) The respective maximum and minimum values coincide.

Proof. This theorem has been established previously [18,3], but because of its simplicity the proof is presented herein for the sake of completeness. Part i) of the theorem is an immediate consequence of Eq. (14). On the other hand, relation (17) is satisfied for any given pair of admissible functions U_+ and U_-. Thus if we set $U_+ = U_b$ and $U_- = U_a$, this inequality reduces to

$$\Omega(U_b) - \Omega(U_a) = \Delta\Omega \geqslant 0 , \qquad (21)$$

which implies (20). Now, if U is a solution of (18), then U satisfies Eqs. (19); therefore α) and β) follow from (20). Finally γ) is a direct consequence of α) and β).

An attractive feature of many dual principles is that any member U_a satisfying (19a) provides an upper bound and any member U_b satisfying (19b) supplies a lower bound of the common value of the functional Ω at the solution U. The results of Theorem 1, however, do not allow to infer from the difference $\Omega(U_b) - \Omega(U_a)$ an estimate of the closeness between an approximate solution and an exact solution. Theorem 1 states that the minimum value of the functional Ω on solutions of (19a) is equal to its maximum value on solutions of (19b) when a solution U of (18) exists. Though in this case both maximum and minimum are attained at U, there may be elements U' satisfying either (19a) or (19b) on which Ω achieves the same value and which are not solutions of (18). This difficulty does not arise if Ω is strictly saddle, as the following corollary shows.

Corollary 1. If in Theorem 1 Ω is strictly saddle, then:

i) There is at most one solution of Eq. (18);

ii) The equation

$$\Omega(U_a) = \Omega(U_b) \tag{22}$$

holds if an only if $U_a = U_b$ is the solution of (18).

iii) If the solution of (18) belongs to D, the maximum value of Ω among all admissible states that satisfy (19a) is attained exclusively at the solution. The minimum value of the functional among all admissible states that fulfill (19b) also is attained only at the solution.

Proof. In this case relations (21) and (22) imply $U_a = U_b$, and therefore equations (19) are satisfied simultaneously by $U_a = U_b$. This shows ii). The result i) is obviously implied by ii). Finally, to prove iii), let U be the solution of (18) and let $\Omega(U_a)$ be the maximum value of Ω among all admissible states that satisfy (19a); then

$$\Omega(U_a) = \Omega(U) \qquad (23)$$

and consequently $U_a = U$. The second part of iii) can be proved in a similar manner.

This corollary and in particular parts i) and ii) are of special relevance for the construction of approximate solutions, for they show that the functions that yield the maximum and the minimum of the dual principle are unique and equal to each other and correspond to the solution of the problem. This condition is required to assure that the difference $\Omega(U_b) - \Omega(U_a)$ can be used as an estimate of the error of an approximate solution. Thus, the practical usefulness of the results is greater for problems associated with functionals that are strictly saddle.

ALTERNATIVE FORMULATIONS OF THE FLOW PROBLEM. The initial-value problem to be studied is the transient flow of water in a confined flow region, i.e., a completely saturated elastic porous medium that has well defined geometric boundaries. It is assumed that an open region R with boundary A is occupied by a porous and permeable medium completely filled with a slightly compressible liquid such as water or oil. The medium through which the flow occurs has a specific storage $S_s(\underline{x})$ and a symmetric permeability tensor $K_{ij}(\underline{x})$, properties which are dependent upon the position vector $\underline{x}$. $S_s(\underline{x})$ is positive and continuous on $\overline{R}$ and $K_{ij}(\underline{x})$ is assumed to be positive definite and continuously differentiable on $\overline{R}$. The problem consists in finding the dynamic state of the liquid at any instant when an initial state and boundary data are known. The boundary data are given in two complementary parts A_1 and A_2 of A. A_1 is the portion of the boundary on which the head is prescribed and A_2 is the remaining portion of the boundary, on which flux is prescribed.

We do not discuss the existence of solutions, assuming, therefore, that a solution exists, i.e., that the data are in the range of the operator.

Three alternative descriptions of the motion will be considered depending on the variables used to characterize the problem:

i) *Equations in terms of head and velocity*. Let h be the hydraulic head and v_i denote the cartesian componentes of the Darcy velocity vector $\underline{v}$. The physical laws governing the motion of the fluid are the equation of continuity

$$S_s \frac{\partial h}{\partial t} + \frac{\partial v_i}{\partial x_i} = q \quad \text{on} \quad R \qquad (24)$$

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everywhere in R, while v_i are continuous in space and time and have continuous first spatial derivatives. Neuman and Witherspoon 13] have discussed some of the advantages gained by formulating the flow problem as in Eqs. (24) - (28), which permit evaluating head and Darcy velocity simultaneously.

ii) *Equations in terms of head.* In dealing with flow in porous media it has been customary in hydrology to characterize the problem only in terms of head. To arrive at such formulation it is necessary to make use of Eq. (25) to eliminate v_i from the remaining equations in the system (24) - (28). This process leads to

$$S_s \frac{\partial h}{\partial t} - \frac{\partial}{\partial x_i} (K_{ij} \frac{\partial h}{\partial x_j} = q \quad \text{on } R \times [0, t_1 \qquad (29)$$

$$h(\underline{x}, 0) = h_o(\underline{x}) \quad \text{on } R \qquad (30)$$

$$h(\underline{x}, t) = H(\underline{x}, t) \quad \text{on } A_1 \times [0, t_1] \qquad (31$$

$$K_{ij} \frac{\partial h}{\partial x_j} n_i = - W(\underline{x}, t) \quad \text{on } A_2 \times [0, t_1] \qquad (32)$$

The set D of states for this formulation will be made of the continuous functions h $(\underline{x}, t)$ defined on $\bar{R} \times [0, t_1]$, such that their second space derivatives and first time derivatives are continuous everywhere on $R \times [0, t_1]$

iii) *Equations in terms of velocity* The hydraulic head may be eliminated from Eqs. (24) - (28) to obtain a characterization of the flow problem in terms of velocity [14]. The resulting system is:

$$K_{ij}^{-1} \frac{\partial v_j}{\partial t} - \frac{\partial}{\partial x_i} \left(\frac{1}{S_s} \frac{\partial v_j}{} \right) = f_i(\underline{x},t) \text{ on } R \times \; 0,\ t_1 \qquad (33)$$

where K_{ij}^{-1} are the elements of the inverse of K_{ij}, i.e.,

$$K_{ij}^{-1} \; K_{jk} = \delta_{ik}$$

The corresponding boundary and initial conditions are:

$$v_i(\underline{x},\ 0) = N_i(\underline{x}),\ \underline{x} \in R$$

$$\frac{1}{S_s} \frac{\partial v_j}{\partial x_j} = M(\underline{x},\ t) \quad \text{on } A_1 \times \; 0,\ t_1$$

$$v_i\, n_i = W\ (\underline{x},\ t) \text{ on } A_2 \times [0,\ t_1]$$

The functions $f_i(\underline{x},\ t)$, $N_i(\underline{x})$ and $M(\underline{x},\ t)$ appearing in these equations need to be prescribed. If the problem is formulated originally in terms of head and velocity, these function are given by

$$f_i(\underline{x},\ t) = - \frac{\partial}{\partial x_i} [\frac{1}{S_s(\underline{x})} q(\underline{x},\ t) \quad \text{on } R \times [0,\ t_1] \quad (37a)$$

$$N_i(\underline{x}) = - K_{ij}(\underline{x}) \frac{\partial h_o(\underline{x})}{\partial x_j} \qquad \underline{x} \in R \tag{37b}$$

$$M(\underline{x}, t) = - \frac{\partial H(\underline{x}, t)}{\partial t} + q(\underline{x},t) \text{ on } A_1 \times 0, t_1 \tag{37c}$$

The set D of states will be made of the continuous functions $v_i(\underline{x}, t)$ defined on $\bar{R} \times [0, t_1$ such that their second space derivatives and first time derivatives are continuous everywhere on R. The hydraulic head does not enter into this formulation but Eq. (24) can be used to define it.

VARIATIONAL PRINCIPLES IN TERMS OF HEAD AND VELOCITY. In this section we formulate first a simpler version of a stationary variational principle obtained previously by Neuman and Witherspoon 13, 14], the derivation being similar to the procedure proposed in [17]. With this principle and Theorem 1 we establish a dual variational principle in terms of head and Darcy velocity.

Theorem 2. Let the set of admissible states be $\hat{E} = E = D$. For every element $\{h, \underline{v}\}$ of D define the functional

$$\Omega(h, \underline{v}) = \frac{1}{2} \int_R \{S_s h * \frac{\partial h}{\partial t} - 2 v_i * \frac{\partial h}{\partial x_i} - v_i * K_{ij}^{-1} v_j$$

$$+ S_s h(\underline{x}, t_1) [h(\underline{x}, 0) - 2 h_o(\underline{x})] - 2 h*q\} d\underline{x}$$

$$+ \int_{A_1} (h - H) * v_i n_i d\underline{x} + \int_{A_1} h*W d\underline{x} . \tag{38}$$

$$\delta \, \Omega \, (h, \underline{v}) = 0 \tag{39}$$

if and only if $\{h, \underline{v}\}$ is a solution of the problem specified by Eqs. (24)-

A notation motivated by the convolution notation has been adopted in Eq. (28), i.e., for every pair of functions f, g:

$$f * g = \int_0^{t_1} f(\tau) \, g(t_1 - \tau) \, d\tau$$

Proof. Let $U = \{h, \underline{v}\}$ and $V = \{\bar{h}, \bar{\underline{v}}\}$ be two elements of $\hat{E}$. With the notation defined by Eq. (7) we obtain

$$< \delta \, \Omega(U), V > = \int_R \{\bar{h} * \quad S_s \frac{\partial h}{\partial t} + \frac{\partial v_i}{\partial x_i} \qquad q \,]$$

$$- \bar{v}_i * \quad \frac{\partial h}{\partial x_i} + K^{-1}_{ij} \, v_j \quad + S_s \bar{h}(\underline{x}, t_1 \quad [h(\underline{x}, 0) - h_o(\underline{x})]\} \quad d\underline{x}$$

$$+ \int_{A_1} \bar{v}_i n_i * (h - H) \, d\underline{x} + \int_{A_2} \bar{h} * (W - v_i n_i) \, d\underline{x} \, . \tag{40}$$

An analysis which is standard in calculus of variations (see for example [15]) can be used to show that (40) vanishes for every admissible state $\{h, \underline{v}\}$ if an only if Eqs. (24) - (28) are satisfied.

The functional defined by Eq. (38) is simpler than that given by Neuman and Witherspoon 14, Eq. (9)]. The latter is obtained when the former is convoluted with the constant function -1, if q is set equal to

The dual variational principle associated with the present formulation of the problem is given in the following:

Theorem 3. Let the set $\hat{E}$ of admissible states coincide with the whole set D of states. Let Ω be defined by (38) for any admissible state $\{h, \underline{v}\}$. Then:

i) An admissible state $\{h, \underline{v}\}$ is a solution of the system (24) - (28) if and only if

$$\left.\begin{aligned}
& S_s \frac{\partial h}{\partial t} + \frac{\partial v_i}{\partial x_i} - q]^e = 0 \quad \text{on} \quad R \times [0, t \\
& [v_i + K_{ij} \frac{\partial h}{\partial x_j}]^o = 0 \quad \text{on} \quad R \times \ 0, t_1] \\
& h(\underline{x}, 0) = h_o(\underline{x}) \ , \quad \underline{x} \in R \\
& h - H]^o = 0 \quad \text{on} \quad A_1 \times [0, t_1 \\
& v_i n_i - W]^e = 0 \quad \text{on} \quad A_2 \times [0, t_1]
\end{aligned}\right\} \quad (41a)$$

and simultaneously

$$\left.\begin{aligned} &[S_s \frac{\partial h}{\partial t} + \frac{\partial v_i}{\partial x_i} - q]^o = 0 \quad \text{on } R \times [0, t_1] \\ &[v_i + K_{ij} \frac{\partial h}{\partial x_j}]^e = 0 \quad \text{on } R \times [0, t_1] \\ &h(\underline{x}, 0) = h_o(\underline{x}), \quad \underline{x} \in R \\ &[h - H]^e = 0 \quad \text{on } A_1 \times [0, t_1] \\ &v_i n_i - W]^o = 0 \quad \text{on } A_2 \times [0, t_1] \end{aligned}\right\} \qquad (41b)$$

The superscripts e and o denote, respectively, the even and odd components about the midpoint in the interval $[0, t_1]$ of the terms enclosed by brackets.

ii) For any pair of admissible states $\{h_a, \underline{v}_a\}$, $\{h_b, \underline{v}_b\}$ that satisfy (41a) and (41b), respectively, we have

$$\Omega(h_a, \underline{v}_a) \leqslant \Omega(h_b, \underline{v}_b) \qquad (42)$$

iii) Let $\{h, \underline{v}\}$ be an admissible state that satisfies (24-28).

Then:

α) The maximum value of Ω among all admissible states that satisfy (41a) is attained at $\{h, \underline{v}\}$;

β) The minimum value of Ω among all admissible states that satisfy (41b) is attained at $\{h, \underline{v}\}$;

γ) The respective maximum and minimum values coincide.

Proof. Note that E is equal to D. Define E_1 as the subset of E whose elements {h, $\underline{v}$} are such that h is even while $\underline{v}$ is odd, and define E_2 as the subset of E whose elements {h, $\underline{v}$} are such that h is odd while $\underline{v}$ is even. With these definitions {E_1, E_2 is a decomposition of E. The theorem follows now from Theoerms 1 and 2 and the fact that inequality (17) is satisfied. Indeed, if $U_+ = \{h^+, \underline{v}^+\}$ and $U_- = \{h_1^-, \underline{v}^-\}$, then:

$$\Delta\,\Omega - < (\delta_1\Omega)_-,\ (\Delta\,U)_1 > - < (\delta_2\Omega)_+,\ (\Delta\,U)_2 >$$

$$= \frac{1}{2} \int_R [- v_i^o * K_{ij}^{-1} v_j^o + S_s h^o\ (\underline{x},\ 0)\ h^o\ (\underline{x},\ 0)]\ d\underline{x} \geqslant 0. \qquad (43)$$

VARIATIONAL PRINCIPLES IN TERMS OF HEAD. A variational principle in terms of hydraulic head was obtained by Neuman and Witherspoon 13, 14] usin a technique developed by Gurtin 15]. A simpler version of this principle is presented herein and a corresponding extremum principle is established subsequently.

Theorem 4. Let the set $\hat{E}$ of admissible states coincide with the whole set D of states. For every state h, define the functional Ω by

$$\Omega(h) = \frac{1}{2} \int_R \{S_s h * \frac{\partial h}{\partial t} + K_{ij} \frac{\partial h}{\partial x_i} * \frac{\partial h}{\partial x_j} + S_s h(\underline{x}, t_1 \quad h(\underline{x}, 0)$$

$$-2h_o(\underline{x})] \; -2h*q\} \, d\underline{x} - \int_{A_1} (h-H)*K_{ij} \frac{\partial h}{\partial x_j} n_i \, d\underline{x} + \int_{A_2} h*W \, d\underline{x}.$$

Then

$$\delta\Omega(h) = 0$$

if and only if h is a solution of the problem specified by Eqs. (29) - (32).

Proof. This is a particular case of a general theorem given by Herrera and Bielak (Theorem 6.1 of 17]). It can be proved also by calculating the Gateaux variation $\delta\Omega(U)$ directly, as in Theorem 2

In the special case in which the set of admissible states $\hat{E}$ is restricted to satisfy boundary condition (31) the integral over A_1 disappears and the expression (44) for Ω is simplified. The corresponding linear subspace E of D is obtained by requiring that its elements satisfy the boundary condition

$$h(\underline{x}, t) = 0 \text{ on } A_1 \times [0, t_1]$$

Theorem 5. Let the set $\hat{E}$ of admissible states consist of the elements of D that satisfy Eq. (31) and let Ω be given by Eq. (44), which with the constraint imposed on the admissible states, becomes

$$\Omega(h) = \frac{1}{2} \int_R \{S_s h * \frac{\partial h}{\partial t} + K_{ij} \frac{\partial h}{\partial x_i} * \frac{\partial h}{\partial x_j} + S_s h(\underline{x}, t_1)[h(\underline{x}, 0) - 2h_o(\underline{x})$$

$$- 2h * q\} \, d\underline{x} + \int_{A_2} h * W \, d\underline{x} . \tag{47}$$

Then

i) An admissible state h is a solution of the system (29) - (32) if and only if

$$\left. \begin{aligned} & S_s \frac{\partial h}{\partial t} \quad \frac{\partial}{\partial x_i} (K_{ij} \frac{\partial h}{\partial x_j} \quad - q]^e = 0 \quad \text{on } R \times [0, t_1] \\ & h(\underline{x}, 0) = h_o(\underline{x}), \quad \underline{x} \in R \\ & [W + K_{ij} \frac{\partial h}{\partial x_j} \quad n_i^{\ e} = 0 \quad \text{on } A_2 \times [0, t_1] \end{aligned} \right\} \tag{48a}$$

and simultaneously

$$\left. \begin{aligned} & [S_s \frac{\partial h}{\partial t} \quad \frac{\partial}{\partial x_i} \quad (K_{ij} \frac{\partial h}{\partial x_j} \quad - q]^o = 0 \quad \text{on } R \times [0, t_1 \\ & h(\underline{x}, 0) = h_o(\underline{x}), \quad \underline{x} \in R \\ & [W + K_{ij} \frac{\partial h}{\partial x_j} \quad n_i^{\ o} = 0 \quad \text{on } A_2 \times [0, t_1] \end{aligned} \right\} \tag{48b}$$

For any pair of admissible tates and h_b tha satisfy

8a and 8b respect ly

$$(h \leq (h_b$$

the equal ty holding and only if h_b is the solution the sys

is the solution of the system then

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P/L Define the subspace of whose elements

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in tions decompos tion of The theor

Theor and Corollary ince the str ct inequa ty in

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Theorem shows that for the low problem formulated in rms

head the di fe $(h_b$ (be used as an es ima cf the error implied by the pproximate tio and h_b This enhances the value the results in pp tions

VARIATIONAL PRINCIPLES IN TERMS OF VELOCITY sta onary ia-tional princ the Darcy ocity obtained by Neuman and therspoon 13 ing Gurtin approach imp if ed version th: esul tab hed in the owing

heor Let the admi ib states incide with the who set tates For every tate $\underline{v}$ of define the functional Ω by

$$(\underline{v} \quad \int \{ \quad \partial v \quad \partial v \quad \frac{\partial v_j}{\partial x_j} \quad j$$

$$2N \; (\underline{x} \quad 2v \quad d\underline{x} \int_{A_1} M_* \quad d\underline{x} \int_{A_2} [W - v \quad - \frac{\partial v_j}{j} \, d\underline{x} \qquad 50$$

Then

$$\underline{v}$$

and onl is ution of the flow prob speci ied by Eqs

Pr. Let U $\underline{v}$ and V Then with the notation defined by have

$$< \delta\Omega(U), V > = \int_R \{\bar{v}_i * [K_{ij}^{-1} \frac{\partial v_j}{\partial t} - \frac{\partial}{\partial x_i} \frac{1}{S_s} \frac{\partial v_j}{\partial x_j} \qquad f_i]$$

$$+ K_{ij}^{-1} \bar{v}_j(\underline{x}, t_1)[v_i(\underline{x}, 0) - N_i(\underline{x})]\} \, d\underline{x} + \int_{A_1} \bar{v}_i n_i * \frac{1}{S_s} \frac{\partial v_j}{\partial x_j} - M] \, d\underline{x}$$

$$+ \int_{A_2} \frac{1}{S_s} \frac{\partial \bar{v}_j}{\partial x_j} * W - v_i n_i \, d\underline{x} . \tag{52}$$

This equation may be used to show that $\delta\Omega(U)$ vanishes for every admissible $\underline{v}$ if and only if Eqs. (33) - (36) are satisfied.

The dual variational principle associated with this formulation is given by the following.

Theorem 7. Let the set $\hat{E}$ of admissible states consist of the elements of D that satisfy Eq. (36) and let Ω be given by Eq. (50), which, with the constraint imposed on the admissible states, becomes

$$\Omega(\underline{v}) = \frac{1}{2} \int_R \{v_i * K_{ij}^{-1} \frac{\partial v_j}{\partial t} + \frac{1}{S_s} \frac{\partial v_i}{\partial x_i} * \frac{\partial v_j}{\partial x_j} + K_{ij}^{-1} v_j(\underline{x}, t_1)[v_i(\underline{x}, 0)$$

$$- 2 N_i(\underline{x})] - 2 v_i * f_i\} \, d\underline{x} - \int_{A_1} M * v_i n_i \, d\underline{x} . \tag{53}$$

Then

i) An admissible state $\underline{v}$ is a solution of the flow problem described by the system (33) - (36) if and only if

$$\left.\begin{array}{l} \left[K_{ij}^{-1} \dfrac{\partial v_j}{\partial t} - \dfrac{\partial}{\partial x_i} \dfrac{1}{S_s} \dfrac{\partial v_j}{\partial x_j} - f_i \right]^{e} = 0 \text{ on } R \times [0, t_1] \\ v_i(\underline{x}, 0) = N_i(\underline{x}), \quad \underline{x} \in R \\ \left[\dfrac{1}{S_s} \dfrac{\partial v_j}{\partial x_j} - M \right]^{e} = 0 \quad \text{on} \quad A_1 \times [0, t_1] \end{array} \right\} \quad (54a)$$

and simultaneously

$$\left.\begin{array}{l} K_{ij}^{-1} \dfrac{\partial v_j}{\partial t} \quad \dfrac{\partial}{\partial x_i} \quad \dfrac{1}{S_s} \dfrac{\partial v_j}{\partial x_j} - f_i]^{o} = 0 \text{ on } R \times [0, t_1] \\ v_i(\underline{x}, 0) = N_i(\underline{x}), \quad \underline{x} \in R \\ \dfrac{1}{S_s} \dfrac{\partial v_j}{\partial x_j} - M]^{o} = 0 \quad \text{on} \quad A_1 \times [0, t_1] \end{array} \right\} \quad (54b)$$

ii) For any pair of admissible states $\underline{v}_a$ and $\underline{v}_b$ that satisfy (54a) and (54b), respectively,

$$\Omega(\underline{v}_a) \leqslant \Omega(\underline{v}_b) \ ; \qquad (55)$$

iii) If there exists an admissible state $\underline{v}$ which is a solution of the system (33) - (36), then, α) The maximum value of Ω among all admissible states that satisfy (54a) is attained at $\underline{v}$;

β) The minimum value of Ω among all admissible states that satisfy (54b) is attained at $\underline{v}$; and

γ) The respective maximum and minimum values coincide

Proof. Observe that the set E is made of the elements of D that satisfy (36) with vanishing W. Define E_1 as the subset of E whose elements $\underline{v}$ are such that $\underline{v}$ is odd. With these definitions $\{E_1, E_2\}$ is a decomposition of E. The theorem follows from Theorem 1 and from the fact that Ω is saddle, as may be verified readily.

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