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FINITE ELEMENT METHODS

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FLOW PROBLEMS

Santa Margherita Ligure (Italy) June 14 June 18, 1976

A FAMILY OF APPROXIMATE NUMERICAL PROCEDURES FOR LEAKY AQUIFER SYSTEMS

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(Mexico)

SESSION 4: LAKE CIRCULATION AND AQUIFERS

A FAMILY OF APPROXIMATE NUMERICAL PROCEDURES FOR LEAKY AQUIFER SYSTEMS

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Summary

The integro-differential equations for leaky aquifer systems are here used to formulate a numerical method for studying their dynamical behavior. The advantages of this approach over standard methods are exhibited, applying it some specific examples. The main advantage of the method is that the system of equations is uncoupled and the storage capacity of the computer not large.

1. Introduction

In the study of leaky aquifer systems, the assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards have been extensively used, and Neuman and Witherspoon (1969a) have confirmed their validity for most cases of practical interest. Under these assumptions, it has been shown (Herrera and Rodarte, 1973) that the transient behavior of drawdown is governed by a system of integro-differential equations. It has been shown (Herrera and Rodarte, 1973; Herrera, 1974) that this system constitute a powerful method of analysis.

In a first paper (Herrera and Rodarte, 1973) the nature of approximate theories of leaky aquifer dynamics was analyzed, and in a second one (Herrera, 1974) the error analysis of those theories was carried out. Here, the integro-differential equations are used to construct a numerical method for analyzing the transient behavior of leaky aquifer systems.

Standard numerical methods for this kind of systems (Javandel and Witherspoon, 1969) make use in a direct manner of finite element formulations without profiting of the special features of leaky aquifers. With such approach, the resulting equations for the aquifers and aquitards are coupled. On the contrary, when the integro-differential equations for leaky aquifer dynamics are used, the resulting system of equations is uncoupled and one has to deal with a problem whose complexity is not greater than one corresponding to a single confined aquifer. This implies great advantages from the point of view of computing requirements and precision that can be achieved

2. The integro-differential equations for leaky aquifers

In this section the set of integro-differential equations, equivalent to the partial differential equations governing the transient behavior of multiple-aquifer systems, will be presented for the particular case of a two-aquifer system (Figure 1) separated by a semipervious layer (aquitard or aquiclude).

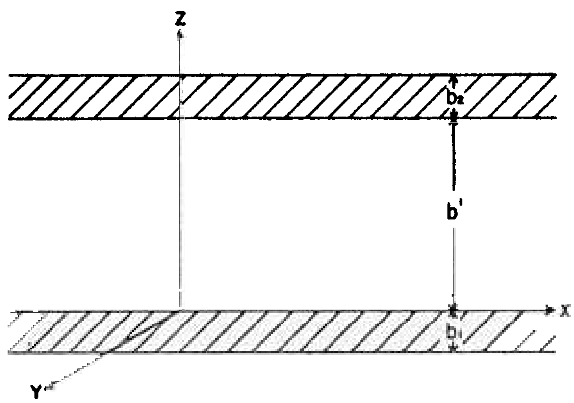

FIGURE 1:- THE AQUIFER SYSTEM

According to Hantush (1960) and Neuman and Witherspoon (1969a,b) the problem can be formulated as follows:

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} + \frac{K}{T_1}\left(\frac{\partial s'}{\partial z}\right)_{z=0} = \frac{1}{\alpha_1}\frac{\partial s_1}{\partial t} \qquad (1a)$$

$$\frac{\partial^2 s'}{\partial z^2} = \frac{1}{\alpha'}\frac{\partial s'}{\partial t} \qquad (1b)$$

$$\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} - \frac{K'}{T_2}\left(\frac{\partial s'}{\partial z}\right)_{z=b'} = \frac{1}{\alpha_2}\frac{\partial s_2}{\partial t}$$

In addition

$$s'(x,y,0,t) \quad s_1(x,y,t) \tag{2a}$$
$$s'(x,y,b',t) \quad s_2(x,y,t) \tag{2b}$$
$$s_1(x,y,0) \quad 0 \tag{2c}$$
$$s_2(x,y,0) \quad 0 \tag{2d}$$
$$s'(x,y,z,0) \quad 0 \tag{2e}$$

Here no contribution to the drawdown by distributed wells has been considered, but the analysis can easily modified to include it. In any particular problem, appropiate boundary conditions have to be added to (1) and (2).

It has been shown (Herrera and Rodarte, 1973) that the system (1,2) is equivalent to

$$\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} - C_1\int_0^t \frac{\partial s_1}{\partial t}(t-\tau)f(\alpha'\tau/b'^2)d\tau$$
$$C_1\int_0^t \frac{\partial s_2}{\partial t}(t-\tau)h(\alpha'\tau/b'^2)d\tau = \frac{1}{\alpha_1}\frac{\partial s_1}{\partial t}$$

$$\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} - C_2\int_0^t \frac{\partial s_2}{\partial t}(t-\tau)f(\alpha'\tau/b'^2)d\tau +$$
$$C_2\int_0^t \frac{\partial s_1}{\partial t}(t-\tau)h(\alpha'\tau/b'^2)d\tau = \frac{1}{\alpha_2}\frac{\partial s_2}{\partial t} \tag{3b}$$

and

$$s'(x,y,z,t) = \int_0^t \frac{\partial s_1}{\partial t}(x,y,t-\tau)\,\omega(z/b',\alpha'\tau/b'^2)d\tau$$
$$+\int_0^t \frac{\partial s_2}{\partial t}(x,y,t-\tau)\,\omega(1-z/b',\alpha'\tau/b'^2)d\tau \tag{3c}$$

where $\omega(\zeta,t) = 1-\zeta-2\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2 t}}{n\pi}\sin n\pi\zeta.$

A dimensionless version of the system (3) is given in (Herrera and Rodarte, 1973), when the drawdown in one of the aquifers can be neglected, i.e. when

$$s_2(x,y,t) = 0 \tag{4}$$

this takes the form

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - \int_0^{t'} \frac{\partial s}{\partial t'} (t'-\tau) f(\tau) d\tau = \frac{1}{\alpha_a} \frac{\partial s}{\partial t'}$$

$$s'(\xi,\eta,\zeta,t') = \int_0^{t'} \frac{\partial s_1}{\partial t'} (\xi,\eta,t'-\tau)\omega(\zeta,\tau) d\tau \qquad (5b)$$

Here, the subindex of s_1 has been droped out because it is irrelevant in this case. The set of dimensionless variables used in these equations are:

$$\xi = x(K'/K_1 b_1 b')^{1/2} \qquad (6a)$$

$$\eta = y(K'/k_1 b_1 b')^{1/2} \qquad (6b)$$

$$\zeta = z/b' \qquad (6c)$$

$$t' = \alpha' t/b'^2 \qquad (6d)$$

together with

$$\alpha_a = \frac{S'}{S_1} \qquad (6e)$$

A dimensionless variable suitable for axisymmetric problems is

$$R = (\xi^2+\eta^2)^{1/2} \qquad (7)$$

Using this variable, equation (5a) becomes

$$\frac{\partial^2 s}{\partial R^2} + \frac{1}{R} \frac{\partial s}{\partial R} - \int_0^{t'} \frac{\partial s}{\partial t'} (t'-\tau) f(\tau) d\tau = \frac{1}{\alpha_a} \frac{\partial s}{\partial t'} \qquad (8)$$

For the developments that follows, it is important to recall that the value f(t') of f at t', gives the rate of flow of water from the aquifer to the aquitard when a constant drawdown of unit magnitude is imposed on the aquifer starting at t'=0. Thus the function

$$F(t') = \int_0^{t'} f(\tau) d\tau \qquad (9)$$

is the total yield of water of the aquitard up to time t' under those conditions.

3. A family of approximations for the memory function

In this section an infinite family of approximations $\{f_N(t');\ N=0,1,2...\}$ for the memory function f(t') will be presented such that every one of them preserves the total yield from the aquitard to the main aquifer; more precisely, such that

$$\lim_{t'\to\infty} \left\{ \int_0^{t'} f(\tau) d\tau - \int_0^{t'} f_N(\tau) d\tau \right\} = 0 \qquad (10)$$

for every $N=0,1,2,\ldots$. This property is important in numerical applications, because it has been observed that the accuracy of approximate solutions are highly sensitive to the fulfilment of such condition.

Observe that definition (9) implies that $F(0)$ vanishes. On the other hand, it can be seen that condition (2c) implies that $\frac{\partial s}{\partial t'}(0)$ also vanishes. Using these conditions, integration by parts yields:

$$\int_0^{t'} \frac{\partial s}{\partial t'}(t'-\tau)f(\tau)d\tau = \int_0^{t'} \frac{\partial^2 s}{\partial t'^2}(t'-\tau)F(\tau)d\tau$$

Therefore, if $F_a(t'$ is any approximation of $F(t')$, equation (11) becomes

$$\int_0^{t'} \frac{\partial s}{\partial t'}(t'-\tau)f(\tau)d\tau \approx \int_0^{t'} \frac{\partial^2 s}{\partial t'^2}(t'-\tau)F_a(\tau)d\tau$$

$$F_a(0)\frac{\partial s}{\partial t'}(t') + \int_0^{t'} \frac{\partial s}{\partial t'}(t'-\tau)F_a'(\tau)d\tau$$

Here, F_a' is the derivative of F_a and the term containing $F_a(0)$ has been retained because for the approximations to be used it will be different to zero in spite of the fact that $F(0)$ vanishes.

The memory function is given by (Herrera and Rodarte, 1973):

$$f(t') = 1+2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'} \qquad (13$$

Integrating this equation from zero to t', it is obtained:

$$F(t') = t'+\frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2 t'}}{n^2} \qquad (14$$

Here the fact that

$$\frac{1}{3} = \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (15)$$

has been used. The family of approximations to be considered is obtained from (14) by truncating the series expansion and defining

$$F_N(t') = t' + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{N} \frac{e^{-n^2\pi^2 t'}}{n^2}; \quad N=1,2,\ldots \qquad (16)$$

The function $F_o(t'$ is defined by

$$F_o(t') = t' + \frac{1}{3}$$

Substitution of F_N in (12) yields

$$\int_0^{t'} \frac{\partial s}{\partial t'}(t'-\tau)f(\tau)d\tau \approx F_N(0)\frac{\partial s}{\partial t'}(t') +$$

$$\int_0^{t'} \frac{\partial s}{\partial t'}(t'-\tau)F_N'(\tau)d\tau \tag{18}$$

In view of (15), here

$$F_N(0) = \frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2} = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2} \tag{19a}$$

and

$$F_N'(t')=1+2\sum_{n=1}^{N} e^{-n^2\pi^2 t'} \quad ; \quad N=0,1,2,... \tag{19b}$$

When N=0, equations (19) become

$$F_N(0) = {}^{1}/3 \tag{20a}$$

and

$$F_N'(t') = 1 \tag{20b}$$

Consequently, equation (5a) reduces to

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - s = \left(\frac{1}{3} + \frac{1}{\alpha_a}\right)\frac{\partial s}{\partial t'} \tag{21}$$

which is the generalization of Hantush's approximation for large values of time that was previously obtained by Herrera and Figueroa (1969), and Herrera and Rodarte (1973).

Everyone of the approximations defined by equations (18) and (19) has a simple interpretation which will now be established.

Let $f_N(t')$ be the approximation of $f(t')$ yielding $F_N(t')$ by means of (10); i.e., such that

$$F_N(t') = \int_0^{t'} f_N(\tau)d\tau \qquad ($$

In view of (16) and (19a), such would be the case, if

$$f_N(t')=1+2\sum_{n=1}^{N} e^{-n^2\pi^2 t'} + \left(\frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2}\right)\delta(t') \qquad ($$

where $\delta(t')$ is Dirac's delta function. Camparing (23) (13), it is clear that the approximation implied is:

$$2 \sum_{n=N+1}^{\infty} e^{-n^2\pi^2 t'} \approx \left(\frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}\right) \delta(t') \qquad (24)$$

Taking into account that

$$\int_0^{\infty} \sum_{n=N+1}^{\infty} e^{-n^2\pi^2\tau}\, d\tau = \frac{1}{\pi^2} \sum_{n=N+1} \frac{1}{n^2}$$

it is clear that the total flow from the aquitard is preserved by approximation (24), but the time evolution of flow is modified by incorporating the contribution coming from the terms neglected in the infinite sum (13) at the initial time. In view of the fact that in each term of the series the negative exponent increases with n, they decrease more rapidly with t as n increases and their contribution to the flow is concentrated in a neighborhood of the initial time which gets smaller as n increases. Thus, when more terms of the series are included (i.e. when N is increased) the improvement of the approximation arises from two sources: first, the terms that have been incorporated in an exact manner, and secondly, the fact that those remaining are more suitable to be approximated by a delta function

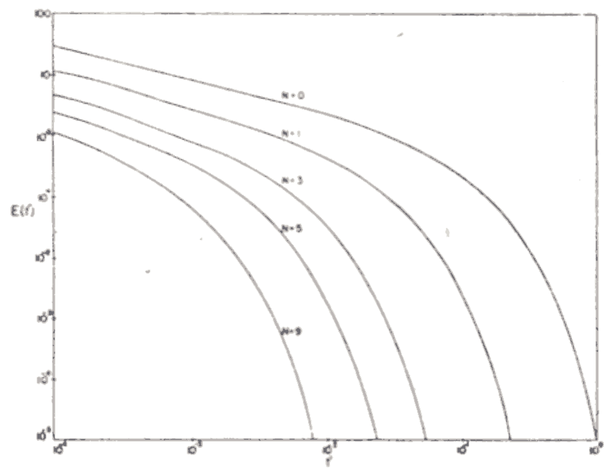

FIGURE 2 - EVOLUTION WITH TIME OF THE RELATIVE ERROR N=0,1,3,5 AND 9

As mentioned before, the exact yield of the aquitard up to time t' is F(t') and therefore $F_N(t')$ is the approximate value of this same yield. Thus, at t' the relative error implied by the approximation is

$$E(t') = \frac{F_N(t') - F(t')}{F(t')} \tag{26}$$

The evolution with time of this quantity is informative and is illustrated in Fig. 2.

4. The numerical treatment

The numerical solution of the aquifer system can be obtained from the differential equations (1) or from the integro-differential equations (3). If we consider the case when $s_2 \equiv 0$, the first formulation requires the simultaneous calculations for s_1 and s' while the latter allows the computation of first s and then s' as an integral of s (5). However, the presence of the convolution integral

$$\int_0^{t'} \frac{\partial s}{\partial t} (t'-\tau) f(\tau) d\tau \tag{27}$$

implies that the determination of the value $s(R,t'+\Delta t')$ is dependent upon all previously calculated values of $s(R,\tau)$, $\tau<t'+\Delta t'$ and would increase both memory and processing time.

The use of the approximation F_N of F (16) resolves this difficulty in that the convolution can be removed. From (18), it is readily obtained that

$$\int_0^{t} \frac{\partial s}{\partial t'} (t'-\tau) f(\tau) \cong F_N(0) \frac{\partial s}{\partial t'}(t') + s(t')$$

$$+ 2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s}{\partial \tau}(\tau) e^{n^2\pi^2\tau} \, d\tau$$

This representation of the integral permits the calculation of $s(R,t'+\Delta t')$ in terms of the value obtained for $s(\mathbf{R},t')$.

5. Comparison of results

To test the validity and efficiency of this method, the numerical solution to (5) was obtained for the case of a steady well, by using a finite element technique together with the approximation (28). A Crank-Nicholson procedure was used for the solution of the resulting system of ordinary integro-differential equations in the time variable t'.

The solution was obtained taking N=5 and compared with the analytical solution for this problem (Neuman and Witherspoon, 1969)

$$s(R,t')=\frac{Q}{2}\int_0^{\infty}\frac{1-e^{-y\ t'}}{y}\begin{Bmatrix}0; & y^2/\alpha<y\cot y\\ J_o(R\sqrt{y^2/\alpha-y\cot y}); & y^2/\alpha\geq y\cot y\end{Bmatrix}dy \qquad (29)$$

The results are illustrated in Figs. 3 and 4, where $t_D=\alpha t'/R$ and $\beta=R$.

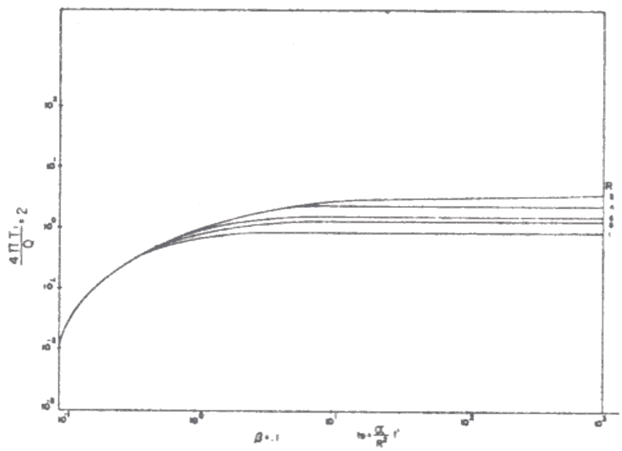

FIGURE 3:- THE NUMERICAL SOLUTION FOR N=5 AND β=1

Comparing these figures with the analytical solution due to Neuman and Witherspoon (1969b), it is seen that there is satisfactory agreement between them.

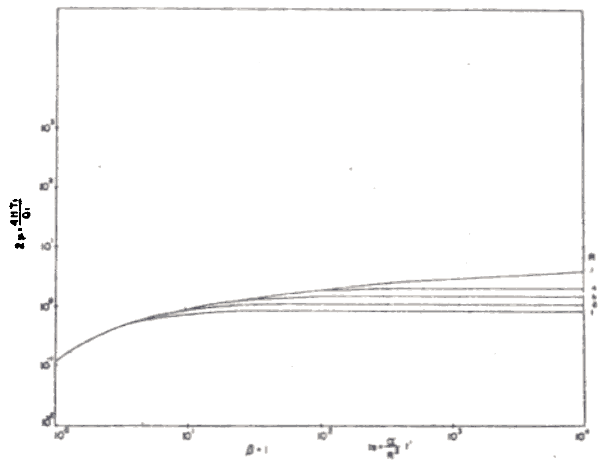

FIGURE 4 - THE NUMERICAL SOLUTION FOR N=5 AND β=1

References

HANTUSH, M.S., (1960): "Modification of the theory of leaky aquifers", J. Geophys. Res., Vol. 65 (11), p. 3713.

HERRERA, I., (1974): "Integro-differential equations for systems of leaky aquifers and applications. 2. Error analysis for approximate theories", Water Resour. Res., Vol. 10 (4), p. 811.

HERRERA, I., and FIGUEROA, G.E., (1970): "A correspondence principle for the theory of leaky aquifers", Water Resour. Res., Vol. 6 (3), p. 1011.

HERRERA, I., and RODARTE, L., (1973): "Integro-differential equations for systems of leaky aquifers and applications. 1. The nature of approximate theories", Water Resour. Res., Vol. 9 (4), p. 995.

NEUMAN, S.P., and WITHERSPOON, P.A., (1969a): "Applicability of current theories of flow in leaky aquifers", Water Resour. Res., Vol. 5 (4), p. 817.

NEUMAN, S.P., and WITHERSPOON, P.A., (1969b): "Theory of flow in a two aquifer system", Water Resour. Res., Vol. 5 (4), p. 803.