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ADVANCES IN
GROUNDWATER HYDROLOGY

1976 SYMPOSIUM PROCEEDINGS

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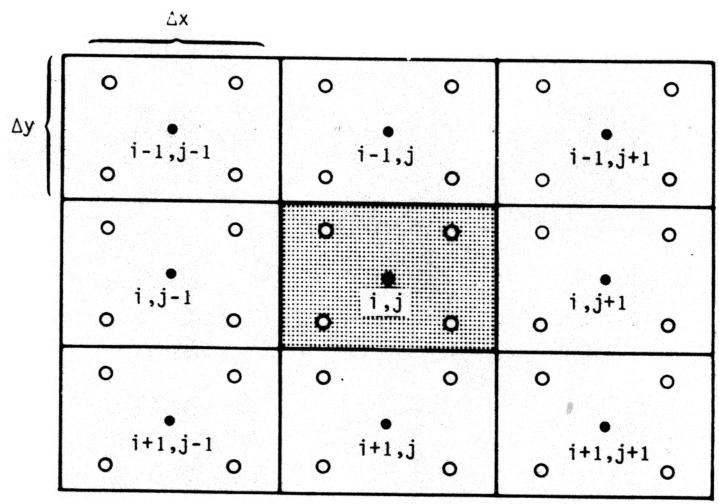

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Minneapolis, Minnesota 55414

ADVANCES IN GROUNDWATER HYDROLOGY
SEPTEMBER AMERICAN WATER RESOURCES ASSOCIATION 1976

A REVIEW OF THE INTEGRODIFFERENTIAL EQUATIONS APPROACH TO LEAKY AQUIFER MECHANICS

Ismael Herrera[1]

ABSTRACT: The integrodifferential equations approach to leaky aquifer mechanics has been developed as part of a program of research which is being carried out at the National University of Mexico. In this paper a review of this theory in its present state of development, is presented. First the integrodifferential equations formulation is given. Then, approximate theories are derived by developing approximations of the memory and influence functions. A critical discussion of these theories is made, from the point of view of its adequacy for numerical treatment. An exact numerical method is presented and its advantages over standard methods is exhibited. An application of the integrodifferential equations to carry out error analysis is also explained.
(KEY TERMS: groundwater; aquifermechanics; modelling; computer applications.)

1. INTRODUCTION

The assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards characterize the mathematical description of the behavior of leaky aquifer systems, because the validity of these assumptions is well established for most cases of practical interest (Neuman and Witherspoon, 1969). Under these assumptions leaky aquifers are governed by a sytem of integrodifferential equations (Herrera and Rodarte, 1973); each of these equations constitute a partial differential equation with memory, because some terms depend on the past history of the drawdown.

The interest of this system stems from several sources. It can be used as a very flexible tool for preliminary analysis before a complex model is advanced. This use is possible because the memory and influence functions have universal shape; i. e., their shape does not depend on the particular problem considered. Therefore much information about a given situation can be derived before carrying out computations; for example, it has been shown (Herrera, 1970, Eq. 30) that the influence function of one aquifer on the next one has the shape of a unit step function (Figure 3) with a lag time t*, given by

$$t^* = b'^2/6a'$$

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Therefore, one can conclude immediately that the system can be treated as uncoupled at times $<t^*$. For a system of two aquifers (Figure 1) this implies that the drawdown in the unpumped aquifer can be neglected at times $<2t^*$. These results agree well with those derived using more elaborate methods (Table 1).

In applications it is frequently advantageous to use approximate expressions for the memory functions instead of the exact ones. Approximate theories developed in the past (Jacob, 1946; Hantush and Jacob, 1954, 1955; Hantush, 1960, 1967) can be shown to correspond to suitable approximations of the memory functions (Herrera and Rodarte, 1973). Some of these approximations were previously limited to the solution of some particular problems (Hantush, 1960); however, by exhibiting the approximate memory functions it is possible to construct corresponding equations (Herrera and Figueroa, 1969; Herrera and Rodarte, 1973) which can be used to solve other problems. The use of modified memory and influence functions as starting point for constructing approximate theories is more direct and convenient than applying the implicit methods previously employed. In this manner new approximate theories have been developed (Herrera, 1970; Herrera, *et al.*, 1976; Herrera and Yates, 1976).

Another advantage of exhibiting the approximate memory and influence functions is that in this manner it is more easy to carry out the error analysis and to establish the applicability of such theories. This can be done comparing the exact and approximate memory functions; this method has been applied to some approximate theories (Herrera, 1974).

In the past, attention was centered mainly on approximations leading to analytical simplifications (Jacob, 1946; Hantush, 1960; Hantush, 1967). Now, the main interest lies on approximations which simplify the numerical treatment of hydrological problems (Herrera and Figueroa, 1969; Herrera, 1970, Herrera, *et al.*, 1976; Herrera and Yates, 1976). The formulation of leaky aquifer mechanics in terms of integrodifferential equations is very well suited to develop simple and accurate numerical methods. This is so, because the drawdowns in the aquitards are eliminated from the basic system and the remaining equations for the aquifers can be easily uncoupled.

In this paper the present state of development of the program of research on this subject, which has been developed at the National University of Mexico since 1968, is revised. In Section 2, the integrodifferential equations and some other preliminary notions are introduced. In Section 3, relevant approximations for the memory and influence functions are discussed. By combining them, approximate theories are constructed in Section 4. A critical discussion of numerical methods derived from these theories is presented in Section 5. Section 6 is devoted to explain an exact numerical method; by this it is meant a method that can be made as accurate as desired. This procedure possesses many advantages over standard methods (Javandel and Witherspoon, 1969), because it requires the introduction of a smaller number of nodes, it uncouples the system and leads to smaller matrices, but at the same time only present values of the drawdowns are required at every stage of the computations and therefore, the computer memory needed is not enlarged. In Section 7 an application (Herrera, 1974) of the integrodifferential equations to carry out error analysis is presented. Although this paper is a review of research most of which has already been published, some new material is also incorporated.

The theory is presented for a system of two aquifers separated by an aquitard (Figure 1) but its modification to a system with more aquifers or with aquitards limited by impermeable layers can be carried out easily (Herrera, 1970).

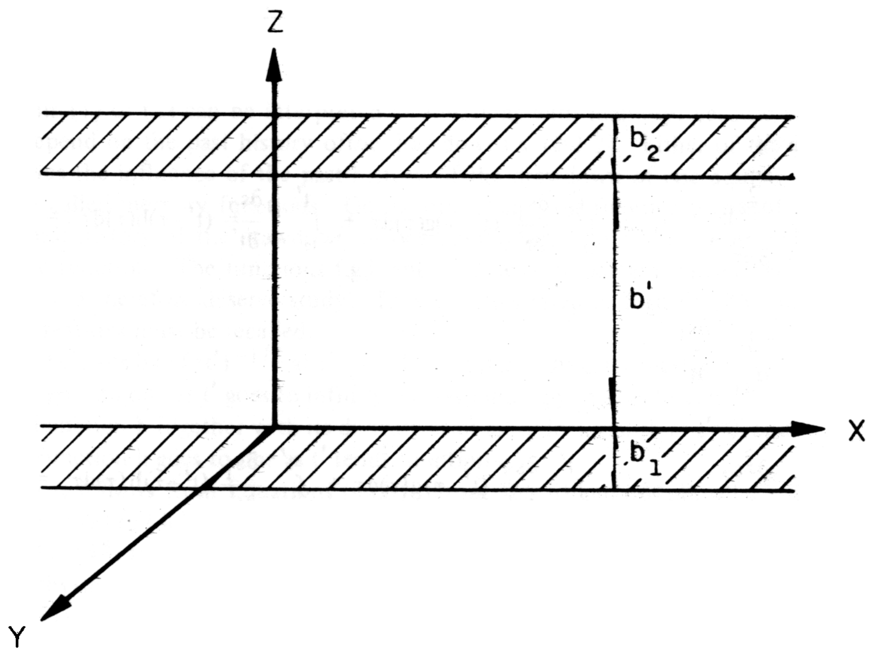

Figure 1. The Aquifer System.

2. THE INTEGRODIFFERENTIAL EQUATIONS

Under the assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards, the mathematical description of leaky aquifers is given by Hantush's (1960) equations, which in turn can be replaced by the integrodifferential equations of leaky aquifer mechanics (Herrera and Rodarte, 1973). For a system of two aquifers separated by an aquitard (Figure 1), when the initial drawdowns vanish, the dimensionless equations are (Herrera and Rodarte, 1973):

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - \int_0^{t'} \frac{\partial s_1}{\partial t'}(t'-\tau)f(\tau)d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(t'-\tau)h(\tau)d\tau$$

$$\frac{1}{a_{a1}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2} + \frac{\partial^2 s_2}{\partial \eta_2^2} \quad \int_0^{t'} \frac{\partial s_2}{\partial t'}(t'-\tau)f(\tau)d\tau + \int_0^{t'} \frac{\partial s_1}{\partial t'}(t'-\tau)h(\tau)d\tau =$$

$$\frac{1}{\alpha_{a2}} \frac{\partial s_2}{\partial t'} \tag{1b}$$

or alternatively

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - s_1 \quad \int^{t'} \frac{\partial s_1}{\partial t'}(t'-\tau)g(\tau)d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(t'-\tau)h(\tau)d\tau$$

$$\frac{1}{\alpha_{a1}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2} + \frac{\partial^2 s_2}{\partial \eta_2^2} - s_2 - \int_0^{t'} \frac{\partial s_2}{\partial t'}(t'-\tau)g(\tau)d\tau + \quad \frac{\partial s_1}{\partial t'}(t'-\tau)h(\tau)d\tau$$

$$\frac{1}{\alpha_{a2}} \frac{\partial s_2}{\partial t'}$$

Equations (1), or alternatively Equations (2), constitute a complete system because a well posed problem for s_1 and s_2 can be formulated by adding to it suitable boundary conditions. Thus, the drawdown s' in the aquitard, does not occur in the basic system of integrodifferential equations governing leaky aquifers. However, once the drawdowns in the main aquifers have been determined, s' is given by

$$s'(\zeta,t') = \int_0^{t'} \frac{\partial s_1}{\partial t'}(t'-\tau)\omega(\zeta,\tau)d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'}(t'-\tau)\omega(1-\zeta,\tau)d\tau \tag{3}$$

In Equations (1) to (3), ξ_i, η_i, ζ and t' are dimensionless variables, while

$$f(t') = 1+2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'} = (\pi t')^{-1/2}\left(1+2\sum_{n=1}^{\infty} e^{-n^2/t'}\right. \tag{4a}$$

$$g(t') = f(t') - \tag{4b}$$

$$h(t') = +2 \sum_{n=}^{\infty} (-1)^n e^{-n^2\pi^2 t'}$$

$$\omega(\zeta, t' \quad -\zeta - 2 \sum_{n=}^{\infty} \frac{e^{-n^2\pi^2 t'}}{n\pi} \sin n\pi\zeta$$

Equations (1) or (2) can be interpreted as equations with memory, because the integral terms depend on the past history ofthe drawdowns s_i (i=1,2). Either of the functions f or g, gives the influence of the past history of the drawdown of one aquifer into itself; they are called memory functions. On the other hand, the function h gives the influence of the past history of the drawdown of one aquifer into the other one; it will be called influence function. The functions f,g,h and ω, play an important role in theory of leaky aquifers and therefore deserve study. They are illustrated in Figures 2, 3 and 4. Some relevant features must be recalled:

i) f behaves like $(\pi t')^{-1/2}$ at $t' = 0$. Thus, g is also singular there;
ii) f goes to one as t' goes to infinity; correspondingly, g goes to zero;
iii) Function h together with its derivatives of all orders, vanish at $t'=0$; and
iv) Function h goes to one as t' tends to infinity.

Property i) follows from Equations (4a,b). Properties ii) and iv) follow by inspection of Equations (4a) to (4c). Property iii) can be proved applying Cesaro summability criterion (Apostol, 1957) to Equation (4c) and its time derivatives of all orders.

The physical meaning of functions f and h is relevant. Assume a constant drawdown of unit magnitude is imposed in one of the aquifers starting at $t'=0$. Then, $f(t')$ gives the rate of flow of water from the aquitard into that aquifer at time t' and $h(t')$ is the rate flow of water from the other aquifer into the aquitard. The steady state value for both of them is 1. Before the steady state condition is achieved the rate of flow from the aquitard into the same aquifer exceeds its asymptotic value by $g(t')$, while the rate of flow from the other aquifer into the aquitard is smaller than its steady state value, the deficiency being $1-h(t')$. If the unit magnitude drawdown is kept fixed indefinitely, the total yield in the corresponding infinite time interval is not finite; however, both the total excess and the total deficiency are finite. They are:

$$\int_0^{\infty} g(t')dt' = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{3} \qquad (5a)$$

$$\int_0^{\infty} \left\{1-h(t')\right\} dt' = \frac{2}{\pi^2} \sum_{n=}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{6}$$

3. APPROXIMATIONS FOR THE MEMORY AND INFLUENCE FUNCTIONS

There are many instances in which it is advantageous to use approximate expressions for the memory and influence functions. Before the formulation of leaky aquifer mechanics in terms of equations with memory, this was done implicitly. Indeed, some

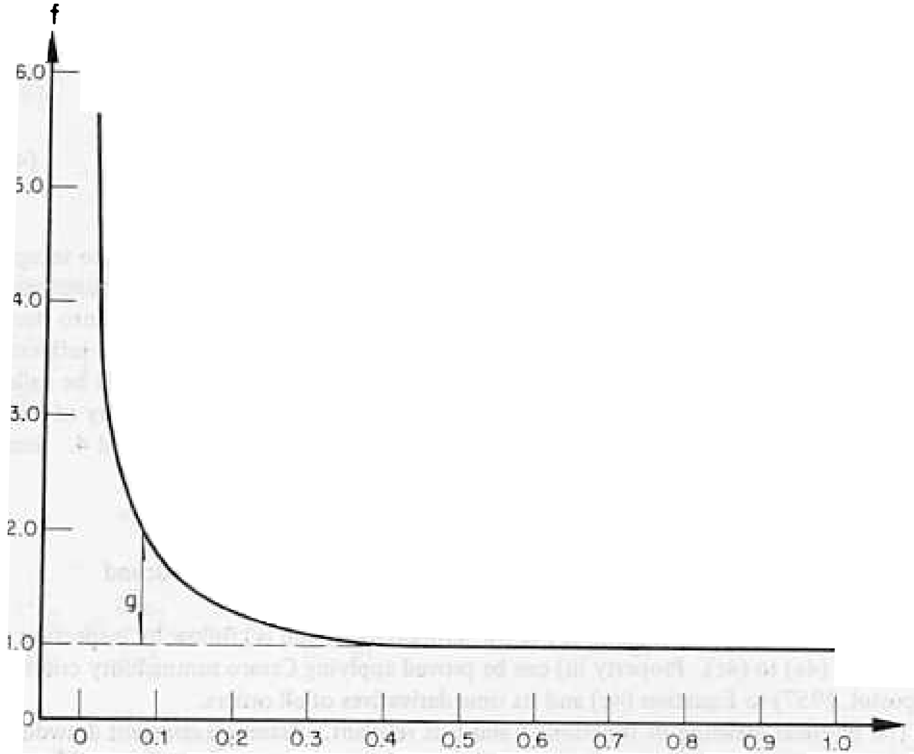

Figure 2. The Memory Functions f and g.

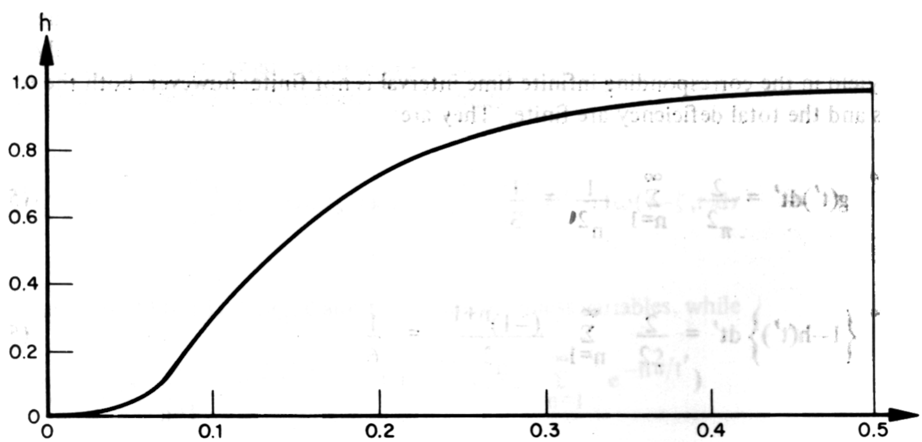

Figure 3. The Influence Function h.

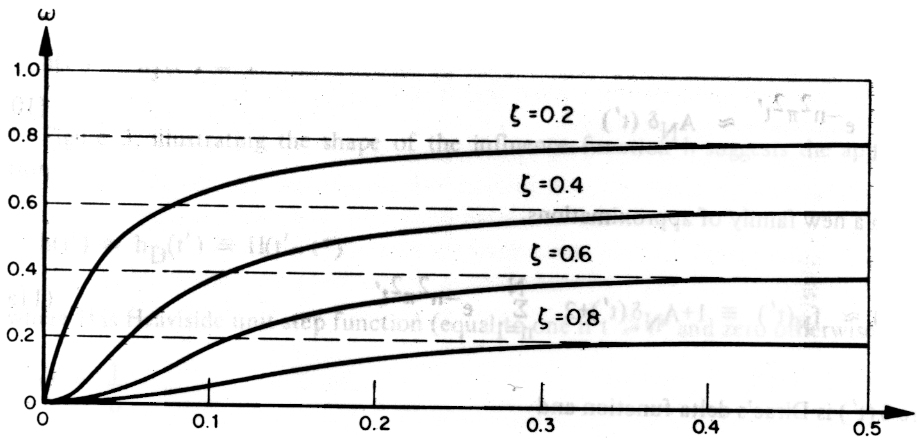

Figure 4. The Function ω when ζ = 0.2, 0.4, 0.6, 0.8.

of the differential equations that have been used (Jacob, 1946; Hantush and Jacob, 1954, 1955; Hantush, 1960, 1967) imply suitable approximations of the memory functions, as has been later recognized (Herrera and Figueroa, 1969; Herrera and Rodarte, 1973). More recently some other approximations have been formulated explicitly (Herrera, 1970; Rodarte, 1976; Herrera, *et al.*, 1976; Herrera and Yates, 1976).

i) The Function f – The simplest approximation is to take it identically zero; this leads to the theory of a confined aquifer.

Another possibility is to take:

$$f(t') \approx f_J(t') \equiv \tag{6}$$

An approximation valid for small values of time suggests itself by inspection of Equation (4a), it is:

$$f(t') \approx f_S(t') \equiv (\pi t')^{-1/2} \tag{7}$$

By truncating the first series expansion in the same Equation (4a) another family of approximations for f(t') is obtained:

$$f(t') \approx \hat{f}_N(t') \equiv 1+2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \tag{8}$$

However, any such expression is unable to reproduce the singular behavior of f(t') at t'=0. The error implied by Equation (8) is

$$f(t') - \hat{f}_N(t') = 2 \sum_{N+1}^{\infty} e^{-n^2\pi^2 t'} \tag{9}$$

This is a singular function which gets sharper as N increases; thus, the larger N, the more suitable to be approximated by a delta function. Writing

$$\sum_{N+1}^{\infty} e^{-n^2\pi^2 t'} \approx A_N \delta(t') \tag{10}$$

leads to a new family of approximations

$$f(t' \approx f_N(t \equiv +A_N\delta(t')+2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'}$$

Here $\delta(t')$ is Dirac's delta function and

$$A_N = \frac{2}{\pi^2} \sum_{N+1}^{\infty} \quad n^2 \qquad 3 \qquad \pi^2 \quad \sum_{n=1}^{N} \quad n^2$$

This choice of A_N is made in order to satisfy (5a); i.e.

$$\int_0^{\infty} g_N(t')dt' = \quad 3$$

where g_N is f_N-1. It is seen that any approximation of this family preserves total yield, but its time evolution is modified by incorporating the contribution associated with the terms eliminated when truncating the series expansion at the initial time. The suitability of the approximation is now evident: *The contribution to the yield of the function approximated by a delta is concentrated in a neighborhood of zero which gets smaller as N increases. Thus, as N is increased the improvement of the approximation stems from two sources: firstly, more terms are included in an exact manner, and secondly, those which are not taken exactly are more suitable to be approximated by a delta function.*

The approximation $f_o(t')$ obtained by setting N=0 has special interest because it has already played an important role in the theory of leaky aquifers. For this case Equations (11) imply that

$$f(t' \approx f_o(t = 1 + \frac{1}{-} \delta(t' \tag{13}$$

This equation corresponds, in view of Equation (5a), to supply the total excess yield instantaneously, at the initial time.

ii) <u>The Function h</u> – For this function the simplest approximation is to take it as identically zero as has been done in all those studies for which the drawdown in the unpumped aquifer has been neglected.

Next in simplicity is to take it as identical to its steady state value

$$h(t') \approx h_J(t') \equiv$$

Figure 3, illustrating the shape of the influence function h suggests the approximation

$$h(t') \approx h_D(t') \equiv H(t'-t^*)$$

where H is Heaviside unit step function (equal to one if $t' \geq t^*$ and zero otherwise) and

$$t^* = \frac{}{6} \quad (15b)$$

The time lag t^* was chosen in this manner to satisfy Equation (5b); i.e.

$$\int^{\infty} \left\{ 1-h_D(t') \right\} dt' = \frac{1}{6}$$

The approxkmation (15a) has the interpretation that the effect of the interaction between aquifers is delayed by the time t^*.

A more systematic manner of approximating the function h is given by

$$h(t') \approx h_N(t') \quad + \sum_{n=1}^{N} B_{Nn} \; e^{-n^2\pi^2 t'} \; ; \; N=1,2.$$

where N conditions can be imposed on h, for each N, in order to determine the coefficients B_{Nn} (n=1, . . . , N). It is natural to require that the total deficiency [Equation (5b)] be preserved. Additional conditions can be that the function $h_N(t')$ and its first N–2 derivatives vanish at $t'=0$.

4. APPROXIMATE THEORIES

Corresponding to every manner of approximating the memory function f and the influence function h, it is possible to construct an approximate theory, but not all of them are equally interesting. In the past, attention was centered mainly on approximations leading to analytical simplifications of the problems (Jacob, 1946; Hantush, 1960; Hantush, 1967). Recently, attention has been shifting toward approximations which simplify the numerical treatment of hydrological problems (Herrera and Figueroa, 1969; Herrera, 1970; Herrera, *et al.,* 1976; Herrera and Yates, 1976). For its discussion theories which are relevant for any of these reasons, will be grouped into two large categories: one for which the influence function is set equal to zero and one for which it is not.

The theories without interaction to be considered are:

a) Confined aquifer;

b) Jacob (1946) approximation;

c) Infinitely thick aquitard;
d) Instantaneous excess yield approximation; and
e) Exponential approximations with instantaneous yield.

The theories with interaction to be considered are:

a) Jacob-Hantush approximation;
b) Instantaneous excess yield with Jacob interaction;
c) Instantaneous excess yield with delayed interaction;
d) Exponential approximations with interaction; and instantaneous yield.

The equations governing each of these theories are obtained by replacing functions f,g and h by their corresponding approximations in Equations (1) or alternatively Equations (2).

4.1 Theories Without Interaction

In all these approximations the influence function h is set equal to zero.

a) Confined Aquifer – As mentioned previously, the behavior of leaky aquifers can be approximated by treating them as if they were confined. The differential equation* is obtained by taking

$$f(t') \equiv 0$$

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} = \frac{1}{a_a} \frac{\partial s}{\partial t'} \qquad (18b)$$

b) Jacob Approximation – This approximation is obtained by taking

$$f(t') \approx f_J(t') \equiv$$

which leads to

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - s = \frac{1}{a_a} \frac{\partial s}{\partial t'} \qquad (19b)$$

Jacob (1946) formulated this approximation, Hantush and Jacob (1954) obtained first, the steady state solution for a well pumping at a constant rate and later (Hantush and Jacob, 1955) the non-steady solution for the same problem. The latter is generally known as Hantush-Jacob formula.

c) Infinitely Thick Aquitard – The approximation

$$f(t' \approx f_S(t') \equiv (\pi t')^{-1/2} \qquad (20a)$$

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*For theories without interaction the subindexes 1 and 2 are not relevant and will be dropped out.

yields the equation

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - \int_0^{t'} \frac{\quad}{(\pi \tau)^{1/2}} \frac{\partial s}{\partial t'} (t'-\tau) d\tau = \frac{1}{a_a} \frac{\partial s}{\partial t'}$$

It has been shown (Herrera and Rodarte, 1973), that the exact fundamental solution of this equation corresponding to the problem of a well pumping at a constant rate, is Hantush (1960) approximation for small times. Equation (20b) is exact when the aquitard is infinitely thick.

d) Instantaneous Excess Yield Approximation – This is obtained by setting

$$f(t') \approx f_o(t') \equiv + \frac{\delta(t')}{3} \tag{21a}$$

which gives the equation

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - s = \frac{1}{\upsilon_o} \frac{\partial s}{\partial t'} \tag{21b}$$

where

$$\upsilon_o = \frac{3a_a}{3+a_a} \tag{21c}$$

The fundamental solution for this equation corresponding to a well pumping at a constant rate is Hantush (1960) approximation for large values of time (Herrera and Rodarte, 1973). Equation (21b) has remarkable simplicity and it has been shown that can be transformed into the equation of a confined (non-leaky) aquifer (Herrera and Figueroa, 1969).

e) Exponential Approximations with Instantaneous Yield – The integrodifferential equations obtained when exponential approximations with instantaneous yield are used for the memory functions, are very well suited for numerical treatment (Herrera, *et al.*, 1976; Herrera and Yates, 1976). In view of Equations (11), they are:

$$\frac{\partial^2 s}{\partial \xi^2} + \frac{\partial^2 s}{\partial \eta^2} - s - 2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^t \frac{\partial s}{\partial t'}(\tau) e^{n^2\pi^2\tau} d\tau = \frac{1}{\upsilon_N} \frac{\partial s}{\partial t'}$$

where for every N=0,1,2, . . .

$$\upsilon_N = \frac{a_a}{1+a_a A_N}$$

Observe that the instantaneous yield approximation is a particular case of this more general family of approximations.

4.2 Theories With Interaction

a) Jacob-Hantush Approximation – In this case

$$f(t') \approx f_J(t') \equiv 1;\ h(t') \approx h_J(t') \equiv 1$$

The corresponding differential equations are:

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - s_1 + s_2 = \frac{1}{a_{a1}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial s_2}{\partial \xi_2^2} + \frac{\partial s_2}{\partial \eta_2^2} - s_2 + s_1 = \frac{1}{a_{a2}} \frac{\partial s_2}{\partial t'}$$

The fundamental solution for this theory was given by Hantush (1967).

b) Instantaneous Excess Yield with Jacob Interaction – Next in complexity is a theory in which the total volume of water due to the excess yield is supplied instantaneously at the initial time. This is obtained by setting

$$f(t') \approx f_o(t') \equiv 1 + \frac{\delta(t')}{3};\ h(t') \approx h_J(t') \equiv$$

The equations governing this theory are:

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - s_1 + s_2 = \frac{1}{\upsilon_{1,0}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2} + \frac{\partial^2 s_2}{\partial \eta_2^2} - s_2 + s_1 = \frac{1}{\upsilon_{2,0}} \frac{\partial s_2}{\partial t'} \qquad (24c)$$

where

$$\upsilon_{1,0} = \frac{3a_{a1}}{3+a_{a1}},\quad \upsilon_{2,0} = \frac{3a_{a2}}{3+a_{a2}}$$

The fundamental solution can be easily deduced from that obtained by Hantush (1967) for Equations (23b,c), but has never been reported.

c) Instantaneous Excess Yield With Delayed Interaction – The previous theory can be improved by using delayed interaction. When this is done

$$f(t') \approx f_0(t') \equiv + \frac{\delta(t')}{3} \qquad h(t') \approx h_D(t') \equiv H(t'-t^*)$$

and the equations become

$$\frac{\partial^2 s_1}{\partial \xi_1^2}(t') + \frac{\partial^2 s_1}{\partial \eta_1^2}(t') - s_1(t') + s_2(t'-t^*) = \frac{1}{\upsilon_{1,0}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2}(t') + \frac{\partial^2 s_2}{\partial \eta_2^2}(t') - s_2(t') + s_1(t'-t^*) = \frac{1}{\upsilon_{2,0}} \frac{\partial s_2}{\partial t'}$$

This approximation was originally proposed by Herrera (1970) and it represents an improved version of Hantush–Jacob approximation, because it takes into account the storage of the aquitard, both through the instantaneous excess yield and the delayed response. A manner of constructing the fundamental solution for this problem was given by Herrera and Rodarte (1972).

d) Exponential Approximations With Interaction and Instantaneous Yield – This approximation is the result of taking

$$f(t') \approx f_N(t') \equiv 1 + A_N\,\delta(t') + 2 \sum_{n+1}^{N} e^{-n^2\pi^2 t'}$$

$$h(t') \approx h_M(t') \equiv 1 + \sum_{n=1}^{M} B_{Mn}\, e^{-n^2\pi^2 t'}$$

which leads to

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} - s_1 - 2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_1}{\partial t'}(\tau)\, e^{n^2\pi^2\tau}\, d\tau$$

$$+ \sum_{n=0}^{M} B_{Mn}\, e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_2}{\partial t'}(\tau)\, e^{n^2\pi^2\tau}\, d\tau = \frac{1}{\upsilon_{1N}} \frac{\partial s_1}{\partial t'}$$

$$\frac{\partial^2 s_2}{\partial \xi_2^2} + \frac{\partial^2 s_2}{\partial \eta_2^2} - s_2 - 2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_2}{\partial t'}(\tau)\, e^{n^2\pi^2\tau}\, d\tau$$

$$\sum_{n=0}^{M} B_{Mn}\, e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_1}{\partial t'}(\tau)\, e^{n^2\pi^2\tau}\, d\tau = \frac{1}{\upsilon_{2N}} \frac{\partial s_2}{\partial t'}$$

Equations (26) are very well suited for numerical solution of problems, as is explained later.

5. APPROXIMATE NUMERICAL METHODS

As mentioned previously the relevance of approximate theories depends at present to a large extent on its adequacy for numerical treatment of problems. If Equations (1) or (2) as they stand, were solved numerically by any of the standard procedures to integrate them step by step in time, it would be necessary to evaluate at each step the convolution terms. For this purpose one could use the exact values of the functions f,g and h. However, if this were done it would be required to carry out the integration form 0 to t' a new on each step, because the integrand depends on t' and this variable is being changed. This is inconvenient by two reasons: firstly, the number of computations required increases beyond reasonable limits and secondly, it is necessary to keep at hand the whole past history of $\partial s_1/\partial t'$ which enlarges very much the memory requirements of the computer.

In view of these facts, the value of approximate theories depends on the extent to which they remove these inconvenient features. In this respect, the exponential approximations are the best suited and they have been used to construct a very efficient exact numerical method (Herrera, *et al.*, 1976; Herrera and Yates, 1977) that will be discussed in Section 6. A brief critical review from this point of view of the other theories, is presented in this section.

5.1 Theories Without Interaction

For numerical purposes the infinitely thick aquitard approximation is the worst. As a matter of fact, Equation (20b) is quite unsuited for its numerical solution, because it does not offer any advantage over the exact Equations (1) or (2); in a step-by-step numerical solution procedure its use implies large memory requirements, as well as recalculation of the convolution terms at every step.

The other approximate theories without interaction (except the exponential approximation that will be discussed in Section 6) are governed by partial differential equations without memory whose degree of difficulty when treated numerically, is the same as the heat equation. Consequently, the choice of a theory for the treatment of a given problem depends only on its applicability; for example, when treating a case for which the instantaneous excess yield approximation can give more accurate results, it would be a gross mistake to use Jacob approximation, only because supposedly this is a simpler theory.

5.2 Theories With Interaction

Again, the discussion of exponential theories will be presented in Section 6. Leaving these aside, from the point of view of complexity for its numerical treatment, the Jacob-Hantush and the instantaneous excess yield with Jacob interaction theories can be put together in one category, because they are governed by the same partial differential equations without memory [Equations (23) and (24)]. As mentioned previously, the fundamental solution of the latter theory can be easily deduced from that obtained by Hantush (1967) for the Jacob-Hantush theory.

The numerical treatment of the instantaneous excess yield approximation with delayed response theory [Equations (25)], is more complicated in one respect; the occurrence in the equations of terms evaluated with delay enlarge the memory requirements. However, in many applications the time steps used are of the order of t* and in such cases, this is not an important limitation. On the other hand, due to the delay the equations for this theory are uncoupled, a fact which represents a definite advantage with respect to Jacob-Hantush equations (23).

6. AN EXACT NUMERICAL METHOD

All numerical methods are approximate; here, however by an exact numerical method it is understood a procedure that can be made as accurate as desired. The exponential approximations developed previously can be used to construct numerical methods of such character (Herrera, *et al.*, 1976, Herrera and Yates, 1976).

Equations (22) and (26) present many advantages over Equations (1) or (2). In this respect the use of exponentials to approximate the functions f, g and h is essential, because it avoids many of the short-comings of systems 1 and 2, mentioned at the beginning of last section.

Systems 22 and 26 possess three features which make the numerical methods derived from them especially valuable. They are:

a) *All the expressions within the integral signs are independent of t';*

b) *From a practical point of view systems 22 and 26 are exact; and*

c) *For a step-by-step numerical solution procedure, the system is uncoupled.*

By inspection it is seen that systems 22 and 26 possess property a). The relevance of this property stems from the fact that when using a step-by-step procedure to solve the equations, the value of the integrals can be brought up to date by simply adding the contribution of the time interval $(t', t'+\Delta t')$; this allows reducing the number of computations and the computer memory requirements, because past values of the variables are not needed. Regarding property b), from a practical point of view Equations (26) constitutes an exact system, equivalent to Equations (1) and (2), because just as in the case of the series expansion of a function, the error can be made arbitrarily small by taking N and M sufficiently large; as a matter of fact, it has been found (Herrera and Yates, 1976) that N and M can be taken less or equal than 2 in most applications and only for studies of a very special character N or M> 5 is required. Property c) is a consequence of the shape of function h (Figure 3), which as mentioned in Section 2, vanishes together with all its derivatives at $t'=0$. Therefore, when applying a step-by-step method of solution (Herrera and Yates, 1976) terms such as

$$\sum_{n=0}^{M} B_{Mn}\, e^{-n^2\pi^2 t'} \int_{t'}^{t'+\Delta t'} \frac{\partial s_2}{\partial t'}(\tau)\, e^{n^2\pi^2\tau}\, d\tau$$

of Equation (26c) can be estimated using the value of $\partial s_2/\partial t'$ at t', which has already been determined.

Numerical methods based on exponential approximations have been developed (Herrera, *et al.*, 1976; Herrera and Yates, 1977) and it has been shown that because of their convenient features they lead to smaller matrices and require much less computational effort than standard methods.

7. ERROR ANALYSIS AND APPLICABILITY

In previous sections, approximate theories were derived making suitable approximations of the memory and influence functions. On the other hand, the exact memory functions are given by Equations (3). This knowledge has been used to carry out the error analysis more simply.

Using this method the ranges of applicability of Hantush (1960) approximations for large and small values of time were discussed (Herrera, 1974); similarly, the range of time on which the drawdown in the unpumped aquifer can be neglected was established and the results are given in Table 1.

TABLE 1. Range of Time on Which the Drawdown in the Unpumped Aquifer can be Neglected.

	$\alpha_{a2}=10^{-1}$	$\alpha_{a2}=10^{0}$	$\alpha_{a2}=10^{1}$	$\alpha_{a2}=\infty$
		Pumped Aquifer		
$\alpha_{a1}=10^{0}$	1.23	0.54	0.43	0.40
$\alpha_{a1}=10^{1}$	0.96	0.43	0.34	0.33
$\alpha_{a1}=\infty$	0.93	0.40	0.33	0.32
		Unpumped Aquifer		
	0.50	0.16	0.11	0.10
		Aquitard at $\zeta=0.2$		
$\alpha_{a1}=10^{0}$	1.10	0.44	0.34	0.33
$\alpha_{a1}=10^{1}$	0.88	0.40	0.32	0.31
$\alpha_{a1}=\infty$	0.87	0.39	0.31	0.30
		Aquitard at $\zeta=0.5$		
$\alpha_{a1}=10^{0}$	0.88	0.32	0.23	0.22
$\alpha_{a1}=10^{1}$	0.80	0.31	0.23	0.22
$\alpha_{a1}=\infty$	0.79	0.31	0.23	0.22
		Aquitard at $\zeta=0.8$		
$\alpha_{a1}=10^{0}$	0.68	0.22	0.15	0.14
$\alpha_{a1}=\infty$	0.65	0.22	0.15	0.14

Hantush's approximation for small values of time can be applied according to this criterion at any time t' smaller than the following values:

α_{a1}	t'
	0.53
	0.41
	0.40

Finally, Figure 5 gives the lower limit t'_L of the interval of time t' on which Hantush approximation for large values of time can be applied at the pumped aquifer.

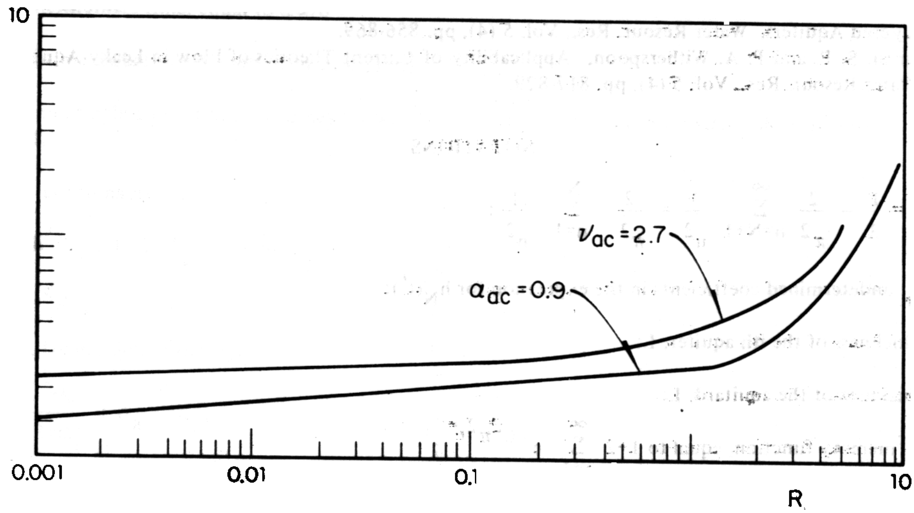

Figure 5. The Range of Applicability of Hantush's Approximation fc Large Values of Time.

A more thorough explanation of the results presented in this section is given in Herrera (1974).

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NOTATIONS

$A_N = \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} = \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2}$;

B_{Nn}, undetermined coefficients in the expression for $h_N(t')$;

b_i, thickness of the ith aquifer, L;

b, thickness of the aquitard, L;

$f(t')$, memory function, equal to $1+2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$;

$f_o(t') \equiv 1+\delta(t')/3$;

$f_J(t') \equiv 1$;

$f_N(t') \equiv 1+A_N\delta(t')+2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'}$;

$f_S(t') \equiv (\pi t')^{-1/2}$

$g(t') \equiv f(t')-1 = 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$

$H(t')$, Heavisede unit step function;

$h(t')$, influence function, equal to $1+2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t'}$

$h_D(t') \equiv H(t'-t^*)$;

$h_J(t') \equiv 1$;

$h_N(t') \equiv 1+ \sum_{n=1}^{N} B_{Nn} e^{-n^2\pi^2 t'}$;

K_i, permeability of ith aquifer, L/T;

K', permeability of the aquitard, L/T;

S_{si}, specific storage of the ith aquifer, L^{-1}

S'_s, specific storage of the aquitard, L^{-1} ;

s_i, drawdown in the ith aquifer, L;

s', drawdown in the aquitard, L;

t, time, T;

t', dimensionless time, equal to $a' t/6'^2$

t^*, lag time, equal to 1/6 when dimensionless;

t'_L, dimensionless lower limit of times for which Hantush approximation for large values of time, is valid;

x,y,z, coordinates, L;

$a' = K'/S'_s$, L^2T^{-1};

$a_{ai} = S'_s/S_{si}$;

$\delta(t')$, Dirac's delta function

$$v_o = \frac{3a_a}{3+a_a} \; ; \; a_a = a_{a1} \; ;$$

$$v_N = \frac{a_a}{1+a_a A_N} \; ;$$

$$v_{iN} = \frac{a_{ai}}{1+a_{ai} A_N}, \quad i=1,2 \; ; N=0,1,2,$$

$$\eta_i = y(K'/K_i b_i b')^{1/2}$$

$$\zeta = z/b'$$

$$\xi_i = x(K'/K_i b_i b')^{1/2}$$

$$\omega(\zeta,t') \doteq 1-\zeta-2\sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2 t'}}{n^2} \sin n\pi\zeta .$$

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