Integrodifferential Equations for Systems of Leaky Aquifers and Applications 3. A Numerical Method of Unlimited Applicability

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The integrodifferential equation formulation of leaky aquifer mechanics is used to develop a numerical method of unlimited applicability based on the finite element technique. This approach reduces the dimensionality of the problem and effectively uncouples the equations corresponding to each of the aquifers. Thus the number of nodes required and the bandwidth of the matrices involved are significantly reduced. Consequently, storage and computer time are decreased by a factor greater than 30 in axially symmetric problems and by a considerably greater factor in the absence of such symmetry.

1. INTRODUCTION

The assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards characterize the mathematical description of the behavior of leaky aquifer systems because the validity of these assumptions is well established for most cases of practical interest [*Neuman and Witherspoon, 1969a, b*]. Under these assumptions, leaky aquifers are governed by a system of integrodifferential equations [*Herrera and Rodarte, 1973*]; each of these equations constitutes a partial differential equation with memory because some terms depend on the past history of the drawdown.

The shapes of the memory functions do not depend on the properties of the particular aquifers or aquitards considered. Because of this property and other convenient features the conviction that the integrodifferential equations could be used as a powerful method of analysis was expressed, and a series of papers was devoted to it. In the first of them [*Herrera and Rodarte, 1973*], after the integrodifferential equations were derived, an analysis of approximate theories was carried out, proving that they can be obtained by making suitable approximations of the memory functions. *Jacob*'s [1946] approach and *Hantush*'s [1960] approximations for small and large values of time were included in the discussion. The nature of these theories was clarified in this manner.

In the second paper of the series [*Herrera, 1974*] a systematic error analysis making use of the exact and approximate memory functions was carried out. Previously [*Herrera and Figueroa, 1969*], by incorporating the corresponding memory functions in the integrodifferential equations the applicability of theories developed for specific problems was extended to apply to more general geometries and boundary conditions. New and more convenient theories were developed by proposing alternative approximations for the memory functions [*Herrera, 1970*]. In addition, it was shown that many problems of practical interest can be handled in this manner [*Neuman and Witherspoon, 1970; Herrera and Figueroa, 1970; Herrera and Rodarte, 1972*].

Especially attractive has been the possibility of constructing approximations of unlimited applicability suitable for numerical treatment. From this point of view the integrodifferential equations have the following convenient features: First, the dimensionality of the problem is reduced, and second, the equations corresponding to the aquifers are uncoupled. The first property is due to the fact that the basic system of equations involves the drawdowns in the main aquifers only and These are treated as being two dimensional. The second property is due to the fact that interaction between aquifers is delayed; therefore when the time numerical integration is carried out step by step, the equations corresponding to the aquifers are effectively uncoupled.

The adequacy of systems (1) or (2) below for numerical computations has already been demonstrated by treating the problem of a single aquifer [*Herrera et al., 1976*]. In this paper a numerical method of unlimited applicability is presented; it is based on the integrodifferential equations and possesses the two features mentioned above. In its formulation, finite elements were used.

The main difficulty encountered in developing such a method was the handling of the convolution terms occurring in the integrodifferential equations (1) or (2). If these equations were solved numerically by any of the standard procedures to integrate them step by step in time, it would be necessary to evaluate these terms at each step. For this purpose one could use the exact values of the functions f, g, and h. However, if this were done, it would be necessary to carry out the integration from zero to t' anew on each step because the integrand depends on t' and this variable is being changed. This is inconvenient for two reasons: first, the number of computations required increases beyond reasonable limits and second, it is necessary to keep at hand the whole past history of the drawdown, the memory requirements of the computer thus being very much enlarged.

These shortcomings were overcome by using exponential series expansions for the memory and influence functions. The efficiency of the expansion for the memory function g is improved by using a generalization of an approximation proposed by one of the authors [*Herrera and Figueroa, 1969; Herrera, 1970*] and motivated by *Hantush*'s [1960] approximation for large values of time. Its main characteristic consists of approximating the residual of the series expansion by a delta function which has the physical interpretation of being an instantaneous yield of water from the aquitard into the aquifer

In section 2 of this paper the integrodifferential equations are presented. In section 3 the exponential approximations with instantaneous yield are developed. They are then substituted into the equations to obtain the approximate version to be used in the method. In section 4 the numerical treatment

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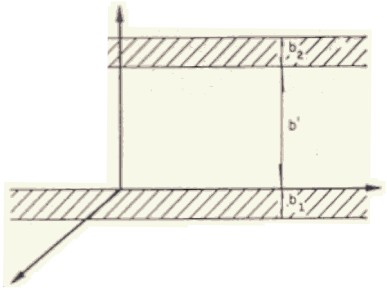


Fig. 1. The aquifer system.

is described. Section 5 is devoted to discussing the method and comparing it with standard procedures. It is shown that important reductions in both processing and memory requirements are achieved by its use. An example with axial symmetry, for which the factor of reduction is 30, is given, and it is shown that this factor is significantly larger in the absence of axial symmetry. It is concluded that the method effectively reduces the dimensionality and permits the treatment of problems which otherwise would not be manageable.

2. The Integrodifferential Equations

Under the assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards, the mathematical description of leaky aquifers is given by the *Hantush* [1960] equations, and this in turn can be reformulated in terms of the integrodifferential equations of leaky aquifer mechanics [*Herrera and Rodarte,* 1973]. A dimensionless version of these equations suitable for numerical treatment and applicable in the case of two aquifers separated by an aquitard (Figure 1) is, when the initial drawdowns vanish,

$$\frac{\partial^2 s_1}{\partial \xi^2} + \frac{\partial^2 s_1}{\partial \eta^2} - f * \frac{\partial s_1}{\partial t'} + h * \frac{\partial s_2}{\partial t'} = \frac{1}{\alpha_{a1}} \frac{\partial s_1}{\partial t'} \tag{1a}$$

$$\lambda\left(\frac{\partial^2 s_2}{\partial \xi^2} + \frac{\partial^2 s_2}{\partial \eta^2}\right) - f * \frac{\partial s_2}{\partial t'} + h * \frac{\partial s_1}{\partial t'} = \frac{1}{\alpha_{a2}} \frac{\partial s_2}{\partial t'} \tag{1b}$$

or, alternatively,

$$\frac{\partial^2 s_1}{\partial \xi^2} + \frac{\partial^2 s_1}{\partial \eta^2} - s_1 - g * \frac{\partial s_1}{\partial t'} + h * \frac{\partial s_2}{\partial t'} = \frac{1}{\alpha_{a1}} \frac{\partial s_1}{\partial t'} \tag{2a}$$

$$\lambda\left(\frac{\partial^2 s_2}{\partial \xi^2} + \frac{\partial^2 s_2}{\partial \eta^2}\right) - s_2 - g * \frac{\partial s_2}{\partial t'} + h * \frac{\partial s_1}{\partial t'} = \frac{1}{\alpha_{a2}} \frac{\partial s_2}{\partial t'} \tag{2b}$$

where ξ, η, and t' are dimensionless variables (see the notation list). The derivation of (1) and (2) from the equations given by *Herrera and Rodarte* [1973] is accomplished by expressing the space coordinates in terms of the dimensionless variables associated with only one of the aquifers. This leads to the introduction of the additional dimensionless parameter $\lambda = T_2/T_1$. Although (1) and (2) do not consider possible contributions from distributed wells, they can be easily modified to include them.

Either (1) or (2) can be considered to constitute a complete system of equations because a well-posed problem for s_1 and s_2 can be formulated with the addition of suitable boundary conditions. Consequently, the drawdown in the aquitard s' does not occur in the basic system of integrodifferential equations of leaky aquifer mechanics. However, s' is given in terms of the drawdowns of the main aquifers by

$$s'(\zeta, t') = \int_0^{t'} \frac{\partial s_1}{\partial t'} (t' - \tau)\omega(\zeta, \tau)\, d\tau + \int_0^{t'} \frac{\partial s_2}{\partial t'} (t' - \tau)\omega(1 - \zeta, \tau)\, d\tau \tag{3}$$

Here ζ is the dimensionless height of the points of the aquitard. The kernels of the integrodifferential equations are

$$f(t') = +2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t'} = (\pi t')^{1/2}\left(1 + 2 \sum_{n=1}^{\infty} e^{-n^2/t'}\right) \tag{4a}$$

$$g(t') = f(t') - = 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t'} \tag{4b}$$

$$h(t') = +2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t'} \tag{4c}$$

$$\omega(\zeta, t') = -\zeta - 2 \sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2 t'}}{n\pi} \sin n\pi\zeta \tag{4d}$$

Equations (1) or (2) can be interpreted as having memory because the integral terms depend on the past history of s_1 and s_2. The past values of the drawdown of one aquifer influence the behavior of both of them; the influence on the aquifer itself is given by f or g, and the influence on the other by h. The first two are called memory functions, while the latter is called the influence function.

The functions f, g, h, and ω play an important role in leaky aquifer mechanics; they are illustrated in Figures 2–4. Some of their most relevant features are the following: (1) f behaves as $(\pi t')^{-1/2}$ at $t' = 0$, and thus g is also singular there, (2) f goes to 1 as t' goes to infinity, and correspondingly, g goes to zero, (3) h and ω together with their derivatives of all orders vanish at $t' = 0$, and (4) h goes to 1 as t' tends to infinity, while ω goes to $1 - \zeta$. These properties follow immediately from their definitions (equations (4)). The third property, however, requires the application of the Cesaro summability criterion [*Apostol,* 1957] to (4*c*) and (4*d*) and their time derivatives of all orders.

The physical meaning of the functions f, g, and h is relevant. Assume that a constant drawdown of unit magnitude is imposed in one of the aquifers starting at $t' = 0$. Then $f(t')$ is the rate of flow of water from the aquitard into that aquifer at time t', while $h(t')$ is the rate of flow of water from the other aquifer into the aquitard. The steady state value for both of them is 1,

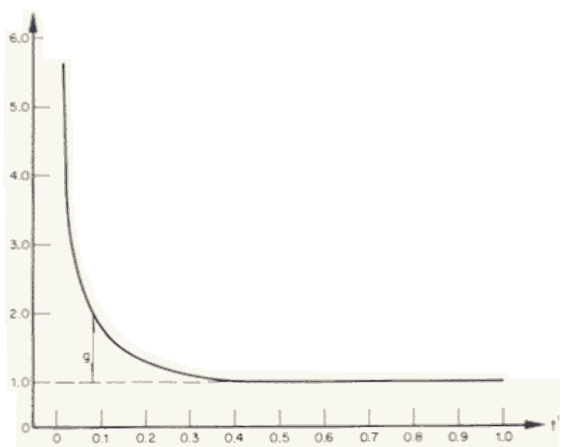


Fig. 2. The memory functions f and g

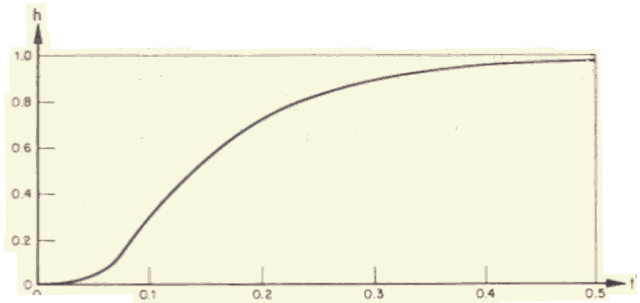


Fig. 3. The influence function h.

but before this condition is achieved, the rate of flow from the aquitard into the same aquifer exceeds its asymptotic value by $g(t')$, while the rate of flow from the other aquifer into the aquitard is smaller by the amount $1 - h(t')$. If the unit magnitude drawdown is kept fixed indefinitely, then the total yield in the corresponding infinite time interval is not finite; however, both the total excess and the total deficiency are finite. These are

$$\int_0^\infty g(t')\,dt' = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{3} \tag{5a}$$

$$\int_0^\infty [1 - h(t')]\,dt' = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{1}{6} \tag{5b}$$

On the other hand, the functions

$$F(t') = \int_0^{t'} f(\tau)\,d\tau = t' + \int_0^{t'} g(\tau)\,d\tau = t' + G(t') \tag{6a}$$

and

$$H(t') = \int_0^{t'} h(\tau)\,d\tau \tag{6b}$$

give, respectively, the total yield of the aquitard and the total volume of water that passes from one of the aquifers into the aquitard up to time t'.

3. Approximations of the Memory and Influence Functions

There are some difficulties which hinder the direct use of the integrodifferential equations in numerical problems. If (1) or alternatively (2) as they stand were to be solved numerically by any of the standard procedures to integrate them step by step in time, it would be necessary to evaluate the convolution terms at each step. To this end, one could use the exact values of the functions f, g, and h. However, if this were done, it would be necessary to carry out the integration from zero to t' anew on each step, since the integrand depends on t', which is being changed. This is inconvenient for two reasons: first, the number of computations required would increase beyond reasonable limits, and second, it would be necessary to keep at hand the entire past history of $\partial s_1/\partial t'$, the memory requirements of the computer thus being greatly increased.

In order to develop useful numerical methods based on systems (1) or (2) it is therefore necessary to find ways of overcoming these inconvenient features. A critical review from this point of view of approximate theories has been presented separately [*Herrera*, 1976]. Here attention will be restricted to a family of exponential approximations that removes the difficulties just mentioned.

It is important that approximations of memory and influence functions preserve total yield, since the accuracy of approximate solutions has been observed to be highly sensitive to the fulfilment of such a condition. Two sequences of functions $\{g_N;\ N = 0, 1, 2, \cdots\}$ and $\{h_N;\ N = 0, 1, 2, \cdots\}$ approximating g and h, respectively, will be constructed satisfying (5), i.e., such that

$$\int_0^\infty g_N(t')\,dt' = \tfrac{1}{3} \tag{7a}$$

$$\int_0^\infty [1 - h_N(t')]\,dt' = \tfrac{1}{6} \tag{7b}$$

The Memory Function

Well-known properties of the heat equation and the assumption of vanishing initial drawdowns imply that the initial values of their time derivatives also vanish. On the other hand, $G(0)$ vanishes by virtue of its definition (equation (6a)). Therefore integration by parts yields

$$\int_0^{t'} \frac{\partial s_i}{\partial t'}(t' - \tau)g(\tau)\,d\tau = \int_0^{t'} \frac{\partial^2 s_i}{\partial t'^2}(t' - \tau)G(\tau)\,d\tau$$

$$\approx \int_0^{t'} \frac{\partial^2 s_i}{\partial t'^2}(t' - \tau)G_N(\tau)\,d\tau = G_N(0)\frac{\partial s_i}{\partial t'}(t'$$

$$+ \int_0^{t'} \frac{\partial s_i}{\partial t'}(t' - \tau)G_N'(\tau)\,d\tau \qquad i = 1, 2 \tag{8}$$

Here G_N' is the derivative with respect to time of G_N, and the term $G_N(0)$ has been retained because preservation of total yield requires that this quantity be nonvanishing.

From (4a), (4b), (5a), and (6a) the exact expression for G is

$$G(t') = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2 t'}}{n^2} \tag{9}$$

The sequence of approximations for G is obtained by truncating this series expansion:

$$G_N(t') = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{e^{-n^2\pi^2 t'}}{n^2} \qquad N = 0, 1, 2, \cdots \tag{10}$$

As

$$G_N(\infty) = \tfrac{1}{3} \qquad N = 0, 1, 2, \cdots \tag{11a}$$

every member of this family does, in fact, preserve total yield. The notation

$$A_N = G_N(0) = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2} = \frac{2}{\pi^2}\sum_{N+1}^{\infty}\frac{1}{n^2} \tag{11b}$$

will be adopted.

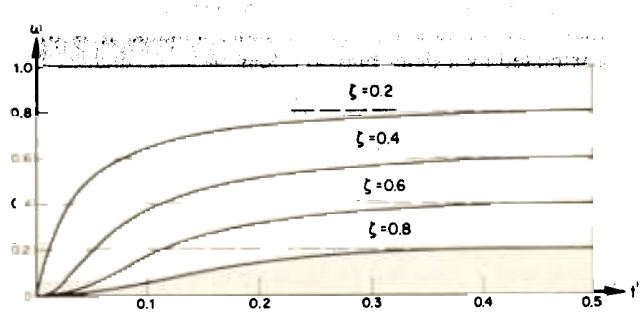


Fig. 4. The function ω when $\zeta = 0.2, 0.4, 0.6$, and 0.8.

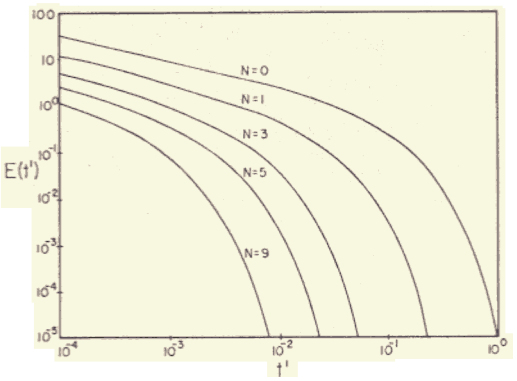


Fig. 5. Error evolution $E(t')$ for $N = 0$, 1, 3, 5, and 9.

Equation (10) implies an implicit definition of g_N. This function must be such that

$$\int_0^{t'} \frac{\partial s_l}{\partial t'}(t' - \tau) g_N(\tau)\, d\tau = G_N(0) \frac{\partial s_l}{\partial t'} + \int_0^{t'} \frac{\partial s_l}{\partial t'}(t' - \tau) G_N'(\tau)\, d\tau \qquad (12)$$

$$G_N(0)\delta(t') + G_N'(t') = A_N\delta(t') + 2\sum_{n=1}^{N} e^{-n^2\pi^2 t'} \qquad (13)$$

where $\delta(t')$ is Dirac's delta function. The function

$$g_0(t') = \tfrac{1}{3}\delta(t') \qquad (14)$$

is the approximation that yields the *Hantush* [1960] solution for large values of time and was used in approximate theories previously developed by one of the authors [*Herrera and Figueroa*, 1969; *Herrera*, 1970]. It has now been embedded in a sequence of approximations which can be made as accurate as is desired by taking N to be sufficiently large.

Alternatively, this family of approximations for the memory function g can be derived by a less systematic but intuitively more enlightening procedure. By truncating the series expansion in (4*b*), we obtain

$$g(t') \approx g_N(t') \equiv 2\sum_{n=1}^{N} e^{-n^2\pi^2 t'} \qquad (15)$$

However, any such expression is unable to reproduce the singular behavior of $g(t')$ at $t' = 0$. The error implied by (15) is

$$g(t') - g_N(t') = 2\sum_{N+1}^{\infty} e^{-n^2\pi^2 t'} \qquad (16)$$

This is a singular function which becomes sharper as N increases; thus the larger N is, the more suitable it is to be approximated by a delta function. Setting

$$\sum_{N+1}^{\infty} e^{-n^2\pi^2 t'} \approx A_N\delta(t') \qquad (17)$$

leads to (13), where A_N is chosen as it is in (11*b*) in order to satisfy the condition of total yield preservation (7*a*). Therefore any approximation of this family preserves total yield, but its time evolution is modified by incorporating the contribution associated with the terms eliminated in truncating the series expansion at the initial time.

The suitability of this family of approximations is now evident: The contribution to the yield of the function approximated by a delta is concentrated in a neighborhood of zero which becomes smaller as N increases. Thus when N is enlarged, the improvement of the approximation stems from two sources: first, more terms are included in an exact manner, and second, those which are not taken exactly are more able to be approximated by a delta function.

The exact and approximate values of the yield from the aquitard up to time t' are given by $F(t')$ and $F_N(t')$, respectively. Consequently, the relative error for this quantity is

$$E(t') = [F_N(t') - F(t')]/F(t') \qquad (18)$$

The evolution with time of this error is informative and is illustrated in Figure 5.

In previous work [*Herrera*, 1974] the applicability of the approximation corresponding to $N = 0$ (the Hantush approximation for large values of time) was discussed. There it was shown that the relative error of the drawdown is less than 5% in most parts of the pumped aquifer when $t' > 3.3 \times 10^{-1}$. This fact can be used to establish a 'rule of thumb' for selecting the value of N to be used in applications. When the 5% error is used as a criterion for defining the range of applicability and it is assumed that this is in turn determined by $E(t')$, Figure 5 shows that $F_N(t')$ can be used whenever $E(t') < 10^{-1}$. It follows from Figure 5 that $N = 1, 3, 5$, and 9 can be used whenever t' is greater than 5×10^{-2}, 8×10^{-3}, 3×10^{-3}, and 1×10^{-3}, respectively. Although the results on which this discussion is based refer to the problem of a single well [*Herrera*, 1974], the situation can only improve in regional studies. In fact, for many of these studies, $N = 0$ is already satisfactory [*Herrera and Rodarte*, 1972].

The Influence Function

An approximation motivated by (4*c*) will be used for h:

$$h_N(t') = 1 + \sum_{n=1}^{N} B_{Nn} e^{-n^2\pi^2 t'} \qquad (19)$$

Here for every N, $\{B_{N1}, B_{N2}, \cdots, B_{NN}\}$ are N coefficients to be determined by imposing suitable conditions. The fulfilment of the third property of section 2, which requires the vanishing of h_N at $t' = 0$, together with its derivatives of all orders, would be very desirable because its violation leads to spurious responses when one of the aquifers remains unpumped. However, it is impossible to satisfy this condition exactly, since there are infinitely many time derivatives and h_N contains N undetermined coefficients only. This is the main difficulty that has to be overcome in order to construct a satisfactory approximation for the influence function h.

Any of the functions h_N given by (19) take the asymptotic value 1 at infinity. Two other conditions which are also important are that the total yield be preserved and that h be zero at $t' = 0$. The first of these leads to

$$\int_0^{\infty} [1 - h_N(t')]\, dt' = \tfrac{1}{6} \qquad (20)$$

The requirement of the vanishing at $t' = 0$ of the first $N - 2$ time derivatives of h_N is enough to complete a set of N conditions necessary to determine the coefficients B_{Nn}.

The functions used to carry out the actual computations were obtained by means of a simpler procedure. The functions p_N were defined by

$$p_N(t') = \sum_{n=0}^{N-1} (-1)^n \{e^{-n^2\pi^2 t'} - \exp[-(n+1)^2\pi^2 t']\}$$

$$N = 1, 2, \qquad (21)$$

No definition was given for p_0. The function h_N ($N = 2, 3,$) was defined by

$$h_N(t') = \alpha p_{N-1}(t') + \beta p_N(t') \qquad (22)$$

with $\alpha + \beta = 1$. Then all these functions vanish at $t' = 0$ and take the asymptotic value 1 at infinity. The coefficients α and β were determined by imposing condition (20).

4. Approximations of the Integrodifferential Equations

Only approximations of the same order for f, g, and h will be considered. Replacing g and h by g_N and h_N as given in (13) and (19) in (2) leads to

$$\frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \eta_1^2} + \sum_{n=0}^{N} B_{Nn} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_2}{\partial t'}(\tau) e^{n^2\pi^2\tau}\, d\tau = \frac{1}{\beta_{1N}} \frac{\partial s_1}{\partial t'} \qquad (23a)$$

$$\lambda \left(\frac{\partial s_2}{\partial \eta_1^2} \right) - s_2 - 2 \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_2}{\partial t'}(\tau) e^{n^2\pi^2\tau} d\tau + \sum_{n=0}^{N} B_{Nn} e^{-n^2\pi^2 t'} \int_0^{t'} \frac{\partial s_1}{\partial t'}(\tau) e^{n^2\pi^2\tau} d\tau = \frac{1}{\beta_{2N}} \frac{\partial s_2}{\partial t'} \qquad (23b)$$

where $B_{N0} = 1$ and

$$\beta_{iN} = \alpha_{ai}/(1 + A_N \alpha_{ai}) \qquad i = 1, 2 \qquad (24)$$

For numerical purposes, (23) presents many advantages over (1) or (2). In this respect the use of exponentials to approximate the functions g and h is essential. Indeed, the use of these functions removes the difficulties encountered in trying to solve (1) and (2) and was discussed at the beginning of section 3. Those difficulties were, first, the need to carry out anew at each step the integration from zero to t' of the convolution terms and, second, the necessity of keeping at hand the past history of the drawdowns s_1 and s_2. The first of these increases the number of computations, while the second one enlarges the memory required.

The system (23) possesses a feature which permits these difficulties to be removed. It is that all the integrands are independent of t'. The usefulness of the system is enhanced by two additional characteristics: When N is increased, the system approaches the exact equations as nearly as is desired, and when a step-by-step numerical solution procedure is applied, the system can be treated as being uncoupled. The first feature, concerning the integrands, follows by inspection of (23). When a step-by-step numerical procedure is used to solve the equations, because of this property, the value of the integrals can be brought up to date by simply adding the contribution of the time interval $(t', t' + \Delta t')$. Thus the past histories of the drawdowns are not needed.

Equations (23) can be made as accurate as is desired by taking N to be sufficiently large because this corresponds to using a more precise approximation for the functions f and g. Our experience indicates that N can be taken to be less than or equal to 2 in many applications and only for studies of a very special character is $N \geq 5$ required.

Finally, the fact that the system can be treated as uncoupled is a consequence of the shape of function h (Figure 3), which vanishes together with all its derivatives at $t' = 0$. Therefore when a step by step method of solution is applied, terms such as

$$\sum_{n=1}^{N} B_{Nn} \exp[-n^2\pi^2(t' + \Delta t')] \int_{t'}^{t'+\Delta t'} \frac{\partial s_2}{\partial t'}(\tau) e^{n^2\pi^2\tau} d\tau$$

in (23a) can be estimated by using the value of $\partial s_2/\partial t'$ at t', which has already been determined (see section 5).

5. The Numerical Method

In this section a method for applying the finite element technique to a two-aquifer system is developed. The results are easily generalized to an arbitrary n layered system [*Herrera*, 1970]. Starting from (23), the Galerkin method is applied in the spatial coordinates, and the Crank-Nicholson procedure is then used to solve step by step the resultant system of ordinary integrodifferential equations in the time variable.

Let D be a finite domain in the $\xi - \eta$ plane (Figure 6) with boundary ∂D. The boundary conditions may be $s_\gamma = 0$ in part of ∂D and $\partial s_i/\partial n = 0$ in another part; also some wells may be located in the interior of D. A finite element mesh is assumed to cover D, and the basis functions $\psi_i(\xi, \eta)$, $i = 1, \cdots, N'$, are defined there. The approximate solutions to (23) are given by

$$s_\gamma(\xi, \eta, t') = \sum_{i=1}^{N'} p_{\gamma i}(t') \psi_i(\xi, \eta) \qquad (25a)$$

$$s'(\xi, \eta, \zeta, t') = \sum_{i=1}^{N'} \hat{p}_i(\zeta, t') \psi_i(\xi, \eta) \qquad (25b)$$

Here and in what follows, Greek indices identify the aquifers and take the values 1 and 2. If the mass and stiffness matrices M and K are defined as

$$M_{ij} = \int_D \psi_i(\xi, \eta) \psi_j(\xi, \eta)\, d\xi\, d\eta \qquad (26a)$$

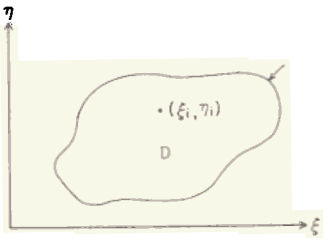
$$K_{ij} = \int_D \nabla\psi_i(\xi, \eta) \cdot \nabla\psi_j(\xi, \eta)\, d\xi\, d\eta \qquad (26b)$$

then the result of multiplying (23) by ψ_j for each $j =$ N and integrating over D gives

$$-KP_1 + C_1 - MP_1' - 2M \sum_{n=1}^{N} e^{-n^2\pi^2 t'} \int_0^{t'} P_1'(\tau) e^{n^2\pi^2\tau} d\tau + M \sum_{n=0}^{N} B_{Nn} e^{-n^2\pi^2 t'} \int_0^{t'} P_2'(\tau) e^{n^2\pi^2\tau} d\tau = \frac{1}{\beta_{1N}} MP_1' \qquad (27a)$$

$$P_2'(\tau) e^{n^2\pi^2\tau} d\tau$$

$$+ M \sum_{n=0}^{N} B_{Nn} e^{-n^2\pi^2 t'} \int_0^{t'} P_1'(\tau) e^{n^2\pi^2\tau} d\tau = \frac{1}{\beta_{2N}} MP_2' \qquad (27b)$$



where $P_\gamma(t') = \langle p_{\gamma 1}(t'), \cdots, p_{\gamma N'}(t')\rangle$, P_γ' stands for the time derivative of these matrices, and

$$C_\gamma = \sum_{i=1}^{N} \frac{Q_{i\gamma}}{T_\gamma} \langle \psi_1(\xi_i, \eta_i), \cdots, \psi_N(\xi_i, \eta_i)\rangle \tag{28}$$

Here it is assumed that all the wells are located at nodes and that $Q_{i\gamma}$ is the rate of extraction from aquifer γ of the well located at (ξ_i, η_i).

This last coupled system of ordinary integrodifferential equations for P_1 and P_2 can be solved using a step by step Crank-Nicholson technique in which the approximations

$$f[t' + (\Delta t'/2)] = \tfrac{1}{2}[f(t') + f(t' + \Delta t')] + O(\Delta t'^2)$$

$$f'\left(t' + \frac{\Delta t'}{2}\right) = \frac{f(t' + \Delta t') - f(t')}{\Delta t'} + O(\Delta t'^2)$$

are used. Evaluating (27a) and (27b) at $t' + \Delta t'/2$ and using the above approximations give a second-order determination of $P_\gamma(t' + \Delta t')$ in terms of $P_\gamma(t')$. The only points which require a special treatment are the evaluation of the integral terms and the coupled nature of the two equations.

Defining

$$D_{\gamma n}(t') = e^{-n^2\pi^2 t'} \int P_\gamma'(\tau) e^{n^2\pi^2\tau}\, d\tau \tag{29}$$

gives

$$D_{\gamma n}(t' + \Delta t'/2) = \frac{\exp\,[-n^2\pi^2(t' + \Delta t'/2)]}{2}$$

$$\left[\int_0^{t'} P_\gamma'(\tau) e^{n^2\pi^2\tau}\, d\tau + \int_0^{t'+\Delta t'} P_\gamma'(\tau) e^{n^2\pi^2\tau} d\tau\right] + O(\Delta t'^2)$$

$$= \exp\,(-n^2\pi^2\Delta t'/2)\, D_{\gamma n}(t') + \frac{\exp\,[-n^2\pi^2(t' + \Delta t'/2)]}{2}$$

$$\int_{t'}^{t'+\Delta t'} P_\gamma'(\tau) e^{n^2\pi^2\tau}\, d\tau + O(\Delta t'^2)$$

$$= \exp\,(-n^2\pi^2\Delta t'/2)\, D_{\gamma n}(t') + \frac{\Delta t'}{2} P_\gamma'(t + \Delta t'/2) + O(\Delta t'^2)$$

$$\exp\,(-n^2\pi^2\Delta t'/2)\, D_{\gamma n}(t') + \tfrac{1}{2}\Gamma_\gamma(t') + O(\Delta t'^2) \tag{30}$$

where

$$\Gamma_\gamma(t') = P_\gamma(t' + \Delta t') - P_\gamma(t') \tag{31}$$

Similarly,

$$D_{\gamma n}(t' + \Delta t') = \exp\,(-n^2\pi^2\Delta t')\, D_{\gamma n}(t') + \exp\,(-n^2\pi^2\Delta t'/2)\, \Gamma_\gamma(t') + O(\Delta t'^2) \tag{32}$$

The use of this approximation in (27) would still result in a pair of coupled equations for $P_\gamma(t' + \Delta t')$. However, an uncoupled system can be obtained by considering the terms

$$\sum_{n=0}^{N} B_{Nn} e^{-n^2\pi^2 t'} \int_0^{t'} P_\gamma'(\tau) e^{n^2\pi^2\tau}\, d\tau$$

occurring in (27), which stem from the approximation for h introduced in section 3. If the exact expression for h had been used, these sums would be

$$I_\gamma(t') = \int_0^{t'} P_\gamma'(\tau) h(t' - \tau)\, d\tau \tag{33}$$

From the fact that $h(t')$ and all its time derivatives vanish at $t' = 0$, it follows that

$$I_\gamma(t' + \Delta t') = \int_0^{t'+\Delta t'} P_\gamma'(\tau) h(t' + \Delta t' - \tau)\, d\tau$$

$$= \int_0^{t'} P_\gamma'(\tau) h(t' + \Delta t' - \tau)\, d\tau + \int_0^{\Delta t'} P_\gamma'(t' + \Delta t' - \tau) h(\tau)\, d\tau$$

$$= \int_0^{t'} P_\gamma'(\tau) h(t' + \Delta t' - \tau)\, d\tau + O(\Delta t'^2)$$

This equation shows that when the Crank-Nicholson procedure is applied to (27), one can write

$$\sum_{n=0}^{N} B_{Nn} \exp\,[-n^2\pi^2(t' + \Delta t')] \int_0^{t'+\Delta t'} P_\gamma'(\tau) e^{n^2\pi^2\tau} d\tau$$

$$= \sum_{n=0}^{N} B_{Nn} \exp\,[-n^2\pi^2(t' + \Delta t')] \cdot \int_0^{t'} P_\gamma'(\tau) e^{n^2\pi^2\tau} d\tau + O(\Delta t'^2)$$

Evaluation of (27) at $t' + \Delta t'/2$ using these results leads to

$$\{M[(\beta_{1N})^{-1} + (N + \tfrac{1}{2})\Delta t'] + (\Delta t'/2)K\}\Gamma_1(t')$$

$$= \left\{C_1 + M\left[\sum_{n=0}^{N} B_{Nn} \exp(-n^2\pi^2\Delta t'/2)\, D_{2n}(t') - 2\sum_{n=1}^{N} \exp(-n^2\pi^2\Delta t'/2)\, D_{1n}(t')\right] - (K + M)P_1(t')\right\}\Delta t' + O(\Delta t'^3)$$

$$\{M[(\beta_{2N})^{-1} + (N + \tfrac{1}{2})\Delta t'] + (\lambda\Delta t'/2)K\}\Gamma_2(t')$$

$$= \left\{C_2 + M\left[\sum_{n=0}^{N} B_{Nn} \exp\,(-n^2\pi^2\Delta t'/2)\, D_{1n}(t') - 2\sum_{n=1}^{N} \exp\,(-n^2\pi^2\Delta t'/2)\, D_{2n}(t')\right] - (\lambda K + M)P_2(t')\right\}\Delta t' + O(\Delta t'^3)$$

In a more compact form these last equations can be rewritten as

$$\tilde{A}_1\Gamma_1(t') = C_1\Delta t' + \tilde{B}_1 P_1 + M[R_2(t) - S_1(t)] \tag{36a}$$

$$\tilde{A}_2\Gamma_2(t') = C_2\Delta t' + \tilde{B}_2 P_2 + M[R_1(t) - S_2(t)] \tag{36b}$$

where

$$\tilde{A}_1 = M[(\beta_{1N})^{-1} + (N + \tfrac{1}{2})\Delta t'] + (\Delta t'/2)K$$

$$\tilde{A}_2 = M[(\beta_{2N})^{-1} + (N + \tfrac{1}{2})\Delta t'] + (\Delta t'/2)\lambda K$$

$$\tilde{B}_1 = -(K + M)\Delta t' \qquad \tilde{B}_2 = -(\lambda K + M)\Delta t'$$

$$R_\gamma(t') = \sum_{n=0}^{N} B_{Nn} \exp\,(-n^2\pi^2\Delta t'/2)\, D_{\gamma n}(t')\Delta t'$$

$$S_\gamma(t') = 2\sum_{n=1}^{N} \exp\,(-n^2\pi^2\Delta t'/2)\, D_{\gamma n}(t')\Delta t'$$

Equations (36) together with (31) and (32) provide a step by step numerical procedure of order $\Delta t'^2$ for P_1 and P_2 which gives the drawdowns s_1 and s_2 by means of (25*a*).

The drawdown in the aquitard can be easily computed by means of (3). When the approximation

$$\omega(\zeta, t) \approx \omega_N(\zeta, t') \equiv -\zeta - \sum_{n=1}^{N} \frac{e^{-n^2\pi^2 t'}}{n\pi} \sin \pi\zeta \qquad (37)$$

is used, we obtain

$$s'(\xi, \eta, \zeta, t') = \sum_{j=1}^{N'} \psi_j(\xi, \eta) \left\{ p_{1j}(t')(1 - \zeta) + p_{2j}(t')\zeta + \sum_{n=1}^{N} [(-1)^n D_{2n}{}^j(t') - D_{1n}{}^j(t')] \sin n\pi\zeta \right\} \qquad (38)$$

When (37) is compared with (25*b*), it is seen that

$$\hat{p}_i(t, \zeta) = p_{1i}(t')(1 - \zeta) + p_{2i}(t')\zeta + \sum_{n=1}^{N} [(-1)^n D_{1n}{}^i(t') - D_{2n}{}^i(t')] \sin n\pi\zeta \qquad (39)$$

6. Discussion

As can be seen from the above discussion, the drawdowns s_1, s_2, and s' for the aquifers and the aquitard can be calculated by using a finite element mesh which extends over the horizontal ξ-η coordinates only. The elimination of the aquitard region from the numerical treatment results in a significant reduction in both processing time and memory requirements. As the aquitard is three dimensional, in the conventional numerical treatment for leaky aquifers [*Javandel and Witherspoon*, 1969] the corresponding mesh is also three dimensional; thus the number of elements required is several times (of the order of 10 or more times) the number used in the aquifers. In the finite element treatment of this kind of problem the matrices are banded so that the computation time and memory needed depend crucially on the bandwidth. However, the bandwidth corresponding to a three-dimensional mesh is significantly larger than that corresponding to a two-dimensional one.

When the numerical method based on the integrodifferential equation approach is used, attention is restricted to the aquifer regions only, and on each time step one can treat the equations as being uncoupled. Therefore the matrices required have the same sizes and bandwidths as those in the treatment of a single two-dimensional confined aquifer. Moreover, these sizes and bandwidths are independent of the degree of approximation used for the memory function g and influence function h. These facts can be corroborated by analyzing (35).

To illustrate the order of magnitude of the reduction in both processing and memory requirements achieved by the use of our method, an example will be discussed. In the problem of an isolated pumping well for which axial symmetry is assumed, if 100 elements are used for the $R = (\xi^2 + \eta^2)^{1/2}$ axis and 10 vertical nodes are used corresponding to each horizontal element, then 1000 nodes would be required for the aquitard region [*Neuman and Witherspoon*, 1969*b*]. The use of linear elements gives a bandwidth of 3 for the confined aquifer and hence for our method. On the other hand, for the conventional aquitard the bandwidth is necessarily greater than 10 (i.e., the number of nodes in the vertical direction). Consequently, storage and computer time, which are roughly proportional to the product of bandwidth and number of nodes, are reduced by a factor of 30.

Problems without axial symmetry treated by the conventional method, where a three-dimensional aquitard is assumed, become intractable if fine meshing is used. On the other hand, when the method presented in this paper is used, the problem is two dimensional and can still be handled. Thus the ability to calculate the drawdowns without recourse to the aquitard region and simultaneously uncoupling the system of equations effectively reduces the dimensionality and permits the treatment of problems which otherwise would not be manageable.

For small values of time the drawdowns increase rapidly; when finite elements are used in the aquitard, a greater refinement is necessary to obtain accurate results. However, such refinements lead to larger matrices; at the same time it is difficult to determine the degree of refinement needed. In the method presented here, such improved accuracy is obtained more easily by simply increasing the number N of terms used in the series expansions for g and h. This number in turn can be estimated by applying the rule of thumb given in section 3.

Finally, it is also possible to estimate the memory and processing time required in terms of the number N' of elements used, the order N of the approximations for g and h, and the bandwidth k of the mass and stiffness matrices. As kN' words of memory are needed for each of the matrices $\tilde{A}_i$, $\tilde{B}_i$, and M, and N' words are needed for the vectors Γ_i, P_i, and D_{ni}, the total space required can be given as

$$S = N' \cdot (5k + 2N + 10) \qquad (40)$$

The time T needed to complete an iteration from t' to $t' + \Delta t'$ can be estimated in terms of the number of elementary operations (multiplications and additions) to be done. As a matrix multiplication takes $2kN'$ operations and a Gauss elimination for the determination of $\Gamma_i(t')$ requires $4kN'$ operations, in view of (36), T can be given approximately as

$$T = 2N' \cdot (10k + 8N + 8) \qquad (41)$$

These results are in close agreement with values obtained from computer runs in which the time for an elementary operation was set to include indexing, fetching, and storing.

For most problems of practical interest the order N of the approximation used for the memory and influence function can be taken to be less than 5, and there are many cases for which $N = 1$ or 2 is completely adequate. Thus N is usually small in relation to the bandwidth parameter k.

Notation

$A_N = \frac{1}{3} - (2/\pi^2)\sum_{n=N+1}^{\infty}(n^2)^{-1} = (2/\pi^2)\sum_{n=1}^{N}(n^2)^{-1}$.
B_{Nn} undetermined coefficients in the expression for $h_N(t')$.
b_i thickness of the ith aquifer, L.
b thickness of the aquitard, L.
$D_{\gamma n}(t')$ auxiliary vector given by (29).
$E(t')$ relative error in the estimation of F, equal to $[F_N(t') - F(t')]/F(t')$.
$F(t') = \int_0^{t'} F(\tau)\, d\tau$.
$f(t')$ memory function, equal to $1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$.
$f_N(t') \equiv 1 + A_N\delta(t') + 2\sum_{n=1}^{N} e^{-n^2\pi^2 t'}$.
$G(t') = \int_0^{t'} g(\tau)\, d\tau$.
$g(t') \equiv f(t') - 1 = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t'}$.
$g_N(t') = f_N(t') - 1 = A_N\delta(t') + 2\sum_{n=1}^{N} e^{-n^2\pi^2 t'}$.
$H(t') = \int_0^{t'} h(\tau)\, d\tau$.
$h(t')$ influence function, equal to $1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t'}$.
$I_\gamma(t')$ auxiliary vector given by (33).
K stiffness matrix.
M mass matrix.

P_γ vector of the coefficients, equal to

$p_{\gamma i}(t')$ coefficient in the finite element representation of s_γ.
$\hat{p}_i(t', \zeta)$ coefficients in the finite element representation of s_i'.
$Q_{i\gamma}$ rate of extraction from the aquifer $\gamma = 1, 2$ of a well located at node i.
S_{si} specific storage of the ith aquifer, L^{-1}.
S_s' specific storage of the aquitard, L^{-1}.
S_i storage of the ith aquifer.
S' storage of the aquitard.
s' drawdown in the aquitard, L.
s_i drawdown in the ith aquifer, L.
s_γ drawdown in the γth aquifer, L.
T_i transmissibility of the ith aquifer, L^2/T.
T' transmissibility of the aquitard, L^2/T.
t time, T.
t' dimensionless time, equal to $\alpha' t/b'^2$.
x, y, z coordinates, L.
$\alpha' = T'/S', L^2/T$.
$\alpha_{ai} = S'/S_i$.
$\beta_{iN} = \alpha_{ai}/(1 + A_N\alpha_{ai})$, where $i = 1, 2$ and $N = 0, 1, 2, \cdots$.
$\delta(t')$ Dirac's delta function.
$\psi_i(\xi, \eta)$ basis functions, where $i = 1, \cdots, N'$.
$\Gamma_\gamma(t')$ auxiliary vector given by (31).
$\lambda = T_2/T_1$.
$\eta_i = y(K'/K_i b_i b')^{1/2}$.
$\zeta = z/b'$.
$\xi_i = x(K'/K_i b_i b')^{1/2}$.
$\omega(\zeta, t') = 1 - \zeta - 2\sum_{n=1}^{\infty}(e^{-n^2\pi^2 t'}/n^2) \sin n\pi\zeta$.

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