Theory of Flow in Unconfined Aquifers by Integrodifferential Equations

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It is shown that when the diffusion of the deviation of the drawdown from its average value can be neglected, the unsteady flow in unconfined aquifers is governed by an integrodifferential equation. For incompressible flow this equation reduces to Boulton's delayed yield equation with $\epsilon = 3$. When the flow is compressible, the kernel can be approximated by Boulton's delayed factor in a range of times whose lower limit approaches zero with the compressibility.

1. INTRODUCTION

There are three main theories describing the unsteady flow of unconfined aquifers: the nonlinear theory, which is based on the Dupuit assumption and is governed by the *Boussinesq* [1903] equation; the linear theory, whose solutions for incompressible and compressible flows were obtained by *Boulton* [1954] and *Neuman* [1972], respectively; and the delayed yield theory, whose solution for the case of constant delayed index was given by *Boulton* [1963]. Solutions are also available for a partially penetrating well.

In all these theories, no attempt is made to incorporate the inertial effect of the unsaturated zone. This is quite appropriate because it has been definitely shown that such effect in unconfined aquifer problems can be neglected. At present, it is recognized that the delayed yield theory must be understood as an approximation of the linear theory, and attempts have been made to derive it systematically [*Streltsova*, 1972, 1973]. However, Streltsova introduced hypotheses whose physical implications are unknown. Recently, research has been carried out [*Gambolati*, 1976] to settle the matter by comparing available solutions for the linear theory [*Boulton*, 1954; *Neuman*, 1972] with the equations governing delayed yield.

In this paper it is shown that the compressible flow of water in unconfined aquifers is governed by an integrodifferential equation. This equation is derived in a rigorous manner from the linearized equations, and it is shown that when the flow is incompressible, the integrodifferential equation reduces to *Boulton*'s [1955] equation. More precisely, the drawdown is first expressed as the sum of its average plus its deviation from this value, and then the linear equations [*Neuman*, 1972] are formulated in terms of these variables. By analyzing the resulting equations it is shown that when conditions of flow are such that the diffusion of the deviation can be neglected, the average drawdown obeys an integrodifferential equation

The structure of this equation is similar to that of the equations with memory governing the unsteady flow of leaky aquifers [*Herrera and Rodarte*, 1973; *Herrera*, 1976]. In the general case of compressible flow the memory function is given by an infinite series of exponentials which reduces to a single term in the incompressible case; this turns out to be *Boulton*'s [1955] equation with $\epsilon = 3$. In general, for time sufficiently large the series can be approximated by its first term; on the other hand, whenever the flow is compressible, there is a range of sufficiently small times on which this is not a suitable approximation. However, this range is rapidly narrowed as the compressibility diminishes. Thus Boulton's equation is in all cases an approximation of the complete integrodifferential equation whose range of applicability (in time) is bounded below by a quantity that goes to zero with the compressibility.

Different segments of the time-drawdown curve are reported in the literature [*Walton*, 1960; *Boulton*, 1963]. They are (1) the Theis curve, which is obtained at very early times; (2) the 'leaky artesian aquifer curve,' obtained later; and (3) after a longer time the time-drawdown curve merges with the Theis nonequilibrium curve associated with the coefficient of storage for water table conditions. Segments 2 and 3 are joined by a segment whose shape can be accounted for by the delayed response theory. Alternative approximations of the memory function are developed which explain these segments of the time-drawdown curve. It is generally thought that segment 2 is obtained when the yield to the water table becomes effective; however, it is shown here that the properties of the corresponding leaky aquifer are independent of the aquifer yield properties and therefore that such behavior cannot be accounted for by the drainage induced by the free surface during its descent. A precise physical interpretation of Boulton's delay index is given, as well as of its generalization introduced here.

Probably the main interest of the results presented in this paper is theoretical, because it clearly establishes the relation between linearized and Boulton's delay theories. However, it is worth recalling that methods to apply numerically the integrodifferential equations are available [*Herrera et al.*, 1976; *Herrera and Yates*, 1977], which in the case of leaky aquifers reduce significantly time and memory requirements.

2. PRELIMINARY RESULTS

For the sake of definiteness, consider an unconfined aquifer of infinite lateral extent that rests on an impermeable horizontal layer (Figure 1). The aquifer material is homogeneous but anisotropic, and its principal permeabilities are oriented parallel to the coordinate axes. A well completely penetrating the aquifer discharges at a constant rate Q, and water is released from storage by compaction of the porous medium, expansion of the water, and gravity drainage at the water table.

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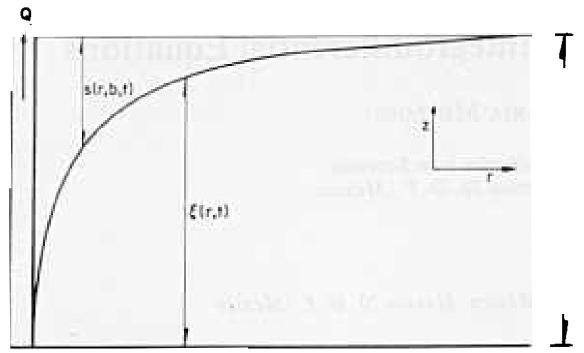

Fig. An unconfined aquifer.

The linearized equations governing such a system are [*Neuman*, 1972]

$$K_r\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) + K_z \frac{\partial^2 s}{\partial z^2} = S_s \frac{\partial s}{\partial t} \qquad 0 < z < b \tag{1}$$

$$s(r, z, 0) = 0 \tag{2}$$

$$\frac{\partial s}{\partial z}(r, 0, t) = 0 \tag{3}$$

$$K_z \frac{\partial s}{\partial z}(r, b, t) = S_y \frac{\partial s}{\partial t}(r, b, t) \tag{4}$$

together with suitable boundary conditions on the well and at infinity. These equations imply in addition the customary assumption that the surface tension of water on the intersection between the free surface and the seepage face is negligible [*Gambolati*, 1976]. The equation for the elevation of the water table is

$$\frac{\partial \xi}{\partial t}(r, t) = -\frac{\partial s}{\partial t}(r, b, t) \tag{5}$$

which can be derived from (7) of *Neuman*'s [1972] paper

Let us define the functions $\bar{s}(r, t)$ and $s_d(r, z, t)$ by

$$\bar{s}(r, t) = \frac{1}{b}\int_0^b s(r, \eta, t)\, d\eta \tag{6}$$

and

$$s_d = s - \bar{s} \tag{7}$$

Integration with respect to z of (1) and use of (6) yield

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) + K_z \left.\frac{\partial s_d}{\partial z}\right|_{z=b} = S\frac{\partial \bar{s}}{\partial t} \tag{8}$$

An equation for s_d is obtained by substitution of (7) into (1) and use of (8):

$$T\left(\frac{\partial^2 s_d}{\partial r^2} + \frac{1}{r}\frac{\partial s_d}{\partial r}\right) - K_z \left.\frac{\partial s_d}{\partial z}\right|_{z=b} + bK_z \frac{\partial^2 s_d}{\partial z^2} = S\frac{\partial s_d}{\partial t} \tag{9}$$

Similarly, (3) and (4) imply

$$S_y \frac{\partial s_d}{\partial t}(r, b, t) + K_z \frac{\partial s_d}{\partial z}(r, b, t) = -S_y \frac{\partial \bar{s}}{\partial t}(r, t) \tag{10}$$

$$\frac{\partial s_d}{\partial z}(r\ 0, t) = 0 \tag{11}$$

In view of (5) and (7) the evaluation of the free surface is given by

$$\frac{\partial \xi}{\partial t}(r, t) = -\frac{\partial \bar{s}}{\partial t}(r, t) - \frac{\partial s_d}{\partial t}(r, b, t) \tag{12}$$

Thus far, no approximation on the linearized equations has been made; (6)–(12) can be used instead of (1)–(5) when they are supplemented by the initial conditions

$$\bar{s}(r, 0) = s_d(r, z, 0) = 0 \qquad 0 < z < b \tag{13}$$

3. APPROXIMATE EQUATIONS OF FLOW

The quantities $\bar{s}$ and s_d are the average and the deviation from this value of the drawdown. The delayed yield theory [*Boulton*, 1954; *Gambolati*, 1976] can be valid only if the term $K_z\,(\partial s_d/\partial z)\;(r, b, t)$ in (8) can be expressed in terms of the average drawdown by a convolution expression of the form

$$K_z \frac{\partial s_d}{\partial z}(r, b, t) = -S_y \int_0^t \mathcal{B}(r, t-\tau)\frac{\partial \bar{s}}{\partial t}(r, \tau)\, d\tau \tag{14}$$

where $\mathcal{B}$ is a function of time and r. By inspection of (9) it is seen that two processes govern the behavior of the deviation s_d of the drawdown from its average value: the diffusion in the vertical direction and the diffusion in the horizontal directions. A relation of the form (14) can hold only if the diffusion in the horizontal directions is much weaker than that in the vertical direction. The conditions for this to be the case require

$$T\left(\frac{\partial^2 s_d}{\partial r^2} + \frac{1}{r}\frac{\partial s_d}{\partial r}\right) \ll K_z\left(b\frac{\partial^2 s_d}{\partial z^2} - \left.\frac{\partial s_d}{\partial z}\right|_{z=b}\right) \tag{15}$$

In general, there are situations when (15) is violated. However, there are many instances of practical interest [*Gambolati*, 1976] where a relation such as (14) (and consequently (15)) is satisfied approximately. Thus one can conclude that the consequences of relation (15) are worth being investigated thoroughly.

When (15) is satisfied, (9) can be replaced by

$$bK_z \frac{\partial^2 s_d}{\partial z^2} - S\frac{\partial s_d}{\partial t} = K_z \left.\frac{\partial s_d}{\partial z}\right|_{z=b} \tag{9'}$$

4. THE CASE OF INCOMPRESSIBLE FLOW: BOULTON'S EQUATION

Because of its simplicity and because it has received most attention, the incompressible case ($S = 0$) will be treated first, leaving for section 5 the more general situation when $S \neq 0$.

If $S = 0$, (8) and (9′) become

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) + K_z \left.\frac{\partial s_d}{\partial z}\right|_{z=b} = 0$$

$$b\frac{\partial^2 s_d}{\partial z^2} - \left.\frac{\partial s_d}{\partial z}\right|_{z=b} = 0 \tag{17}$$

The deviation of the drawdown s_d satisfies in addition (10), (11), and

$$\int_0^b s_d(r, z, t)\, dz = 0 \tag{18}$$

The solution of (17) satisfying (11) and (18) is

$$s_d = \frac{A}{6b}\{3(z-b)^2 + 6b(z-b) + 2b^2\} \tag{19}$$

where A is independent of z and

$$\left.\frac{\partial s_d}{\partial z}\right|_{z_*} = A \tag{20}$$

$$s_d|_{z=b} = Ab/3 \tag{21}$$

The boundary condition (10) yields a differential equation for A:

$$\frac{\partial A}{\partial t} + \frac{3K_z}{S_y b} A = -\frac{3}{b}\frac{\partial \bar{s}}{\partial t} \tag{22}$$

This equation together with the corresponding initial condition (13) implies

$$A = \frac{\partial s_d}{\partial z}(r, b, t) = -\frac{3}{b}\int_0^t \frac{\partial \bar{s}}{\partial t}(\tau) \exp\left[-\frac{3K_z}{S_y b}(t-\tau)\right] d\tau \tag{23}$$

Comparison of this expression with (14) shows that

$$\mathcal{B}(t) = \alpha e^{-\alpha t} \tag{24}$$

where

$$\alpha = 3K_z/S_y b \tag{25}$$

This shows that in the incompressible case the governing equation for the average drawdown is

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) = \alpha S_y \int_0^t \frac{\partial \bar{s}}{\partial t}(\tau) e^{-\alpha(t-\tau)}\, d\tau \tag{26}$$

This is *Boulton*'s [1954] equation. Equations (10), (12), (13), and (14) together with (24) yield the expression for the level of the free surface as

$$\xi(r, t) = b - \alpha \int_0^t \int_0^\tau \frac{\partial \bar{s}}{\partial t}(r, \lambda) e^{-\alpha(\tau-\lambda)}\, d\lambda\, d\tau$$

which is easily transformed into

$$\xi(r, t) = b - \alpha \int_0^t \bar{s}(r, \tau) e^{-\alpha(t-\tau)}\, d\tau \tag{27}$$

5. INTEGRODIFFERENTIAL EQUATIONS FOR AN ELASTIC UNCONFINED AQUIFER

To obtain the integrodifferential equations governing the compressible ($S \neq 0$) flow in unconfined aquifers, it is required to solve (9′):

$$bK_z \frac{\partial^2 s_d}{\partial z^2} - S\frac{\partial s_d}{\partial t} = K_z \left.\frac{\partial s_d}{\partial z}\right|_{z=b} \tag{28}$$

subject to (10), (11), and (13). The solution of this system has been obtained in the appendix by using standard Laplace transform techniques. In this manner it has been shown that (14) holds with $\mathcal{B}$ given by

$$\mathcal{B}(t) = \frac{2K_z}{bS_y}\sum_{n=1}^{\infty} \frac{\rho_n^2}{\rho_n^2 - 1 + \sigma^2} \exp\left(-\frac{K_z}{bS_y}\rho_n^2 t\right) \tag{29}$$

$$\sigma = (S/S_y)^{1/2} \tag{30}$$

and ρ_n are the positive roots of

$$-\rho_n \cot \sigma\rho_n = \sigma - 1/\sigma \tag{31}$$

Table 1, which gives ρ_n^2 ($n = 1, \cdots, 5$) for different values of σ^2, can be derived from the tables given by *Abramowitz and Stegun* [1965].

TABLE 1 Values of ρ_n^2 as a Function of n and σ^2

	$\sigma^2 = 0$	$\sigma^2 = 10^{-2}$	$\sigma^2 = 10^{-1}$	$\sigma^2 = 1$	$\sigma^2 = 10$
	3	2.96	2.93	2.46	0.81
2	∞	4.05×10^2	203.4	22.18	3.31
3	∞	11.95×10^2	599.0	61.62	7.56
4	∞	23.76×10^2	1188.1	120.78	13.69
5	∞	39.53×10^2	1979.6	199.65	21.69

In view of (8) and (14) the average drawdown satisfies

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) = S\frac{\partial \bar{s}}{\partial t} + S_y \int_0^t \mathcal{B}(t-\tau)\frac{\partial \bar{s}}{\partial t}(\tau)\, d\tau \tag{32}$$

with $\mathcal{B}$ given by (29). Equations (10)–(12) together with (14) imply that

$$\xi(r, t) = b - \int_0^t \int_0^\tau \mathcal{B}(\tau - \lambda)\frac{\partial \bar{s}}{\partial t}(r, \lambda)\, d\lambda\, d\tau \tag{33a}$$

The associative and commutative properties of the convolution permit writing the expression for the free surface in the form

$$\xi(r, t) = b - \int_0^t \mathcal{B}(t-\tau)\bar{s}(r, \tau)\, d\tau \tag{33b}$$

The generalized Boulton's kernel $\mathcal{B}$ possesses properties that will be used in the sequel; they are

$$\int_0^\infty \mathcal{B}(\tau)\, d\tau = 1 \tag{34}$$

and

$$\mathcal{B}(t) \approx \frac{1}{S_y}\left(\frac{K_z S_s}{\pi t}\right)^{1/2} \tag{35}$$

when t is small. These relations are shown in the appendix.

6. ALTERNATIVE APPROXIMATIONS OF THE MEMORY FUNCTION ON DIFFERENT RANGES OF TIME

The integrodifferential equation developed in section 5 for unconfined aquifers has a structure which is similar to that for leaky aquifers [*Herrera and Figueroa*, 1969; *Herrera*, 1970, 1974; *Herrera and Rodarte*, 1973]. The main similarities between (32) and the equations corresponding to leaky aquifers [*Herrera and Rodarte*, 1973] are (1) they can be properly described as equations with memory, (2) the memory function is singular at $t = 0$, (3) this singularity behaves as $t^{-1/2}$, (4) the memory possesses an exponential series expansion, and (5) the integral with respect to time of the memory function is finite.

This last property is not always enjoyed by leaky aquifers. Referring to Figure 2, property 5 holds only when the leaky stratum is bounded by an impermeable formation [*Herrera and Figueroa*, 1969]. On the other hand, (34) shows that every unconfined aquifer has this property.

Recently, a systematic discussion of the different approximations suitable for this kind of equation has been presented [*Herrera*, 1976]. In connection with (32) for unconfined aquifers the following approximations are relevant.

Very small times. At very small values of time the integral term in (32) can be neglected. In that case, (32) becomes

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial t}\right) = S\frac{\partial \bar{s}}{\partial t} \tag{36}$$

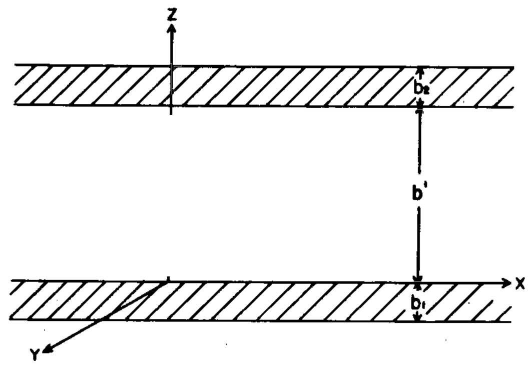

Fig. 2. The leaky aquifer system.

Thus at very small times an unconfined aquifer behaves as a confined aquifer with transmissivity T and storage coefficient S.

Small times. At slightly larger times the integral term in (32) cannot be ignored. However, there is a range of time for which approximation (35) for the kernel can be used. In such a range the governing (32) is

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) = S\frac{\partial \bar{s}}{\partial t} + (K_z S_s)^{1/2}\int_0^t \frac{1}{(\pi\tau)^{1/2}}\frac{\partial \bar{s}}{\partial t}(t-\tau)\,d\tau \qquad (37)$$

Comparison of this equation with (31) and (32) of the paper by *Herrera and Rodarte* [1973] shows that in this range of times the unconfined aquifer behaves as a leaky aquifer with transmissivity and storage coefficient of the main aquifer given by T and S, respectively, while the leaky layer is of infinite thickness with permeability coefficient K_z and specific storage S_s. As has been shown by *Herrera and Rodarte* [1973], the behavior of such an aquifer was used by *Hantush* [1960] to approximate that of a leaky aquifer of finite thickness at small times.

Large values of time. When $\partial\bar{s}/\partial t$ is changing very slowly in comparison with a characteristic time of the generalized Boulton's memory $\mathcal{B}$, one can write

$$\int_0^t \mathcal{B}(\tau)\frac{\partial \bar{s}}{\partial t}(t-\tau)\,d\tau \approx \left\{\int_0^t \mathcal{B}(\tau)\,d\tau\right\}\frac{\partial \bar{s}}{\partial t} = \frac{\partial \bar{s}}{\partial t} \qquad (38)$$

This corresponds to approximating the memory function by a Dirac's delta function [*Herrera and Rodarte,* 1973]. The governing equation is

$$T\left(\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r}\right) = (S + S_y)\frac{\partial s}{\partial t} \qquad (39)$$

The unconfined aquifer behaves in this range of time as a confined aquifer with transmissivity T and storage coefficient $S + S_y$.

7. Discussion

In this section the theories of transient free surface flow will be discussed in the light of the results obtained in this paper. Three aspects will be considered: incompressible flow, compressible flow, and the conditions under which incompressible flow is a suitable approximation to compressible flow.

Incompressible Case

The results obtained in sections 3 and 4 show that the assumption of a delayed yield mechanism leading to *Boulton*'s [1954] empirical equation is concomitant with neglecting the horizontal diffusion in the equation governing the time evolution of the deviation s_d of the drawdown from its mean value (equation (9)). When this is done, Boulton's (26) is obtained. Setting as it is customary, $\alpha = \epsilon K_z/S_y b$, (25) shows that $\epsilon = 3$.

According to *Gambolati* [1976], 'a convincing physical meaning for α' is lacking. The results obtained here supply such a meaning for α. It is as follows. Assume an unconfined aquifer, initially unperturbed, is subjected to an ideal experiment in which a uniform head drop is produced on each of its points and then the average drawdown is kept fixed by extracting water if required, with the only restriction that such extraction must be uniform (i.e., independent of the z-coordinate). Applying (27), it is obtained:

$$\xi(t) = b - \quad + e^{-\alpha t} \qquad (40)$$

It is concluded that when an incompressible unconfined aquifer is subjected to an instantaneous unit head drop, the free surface is not lowered immediately; instead, its descent is delayed by the exponential factor $e^{-\alpha t}$ where α is Boulton's delayed index as given by (25).

It is worth recalling that the diffusion in the horizontal directions vanishes in this ideal experiment, because its solution is necessarily independent of the horizontal coordinates. Thus our theory is exact in this case.

The previous discussion also shows that in order for α to have a definite physical meaning it must be independent of r and t. The same, of course, cannot be said of parameters introduced to match different solutions and lacking a physical significance by themselves. *Gambolati* [1976], by imposing the condition that the delayed yield equation provide an average drawdown equal to that predicted by the linearized theory, obtained values for α which exhibit a significant dependence on r. Such a result is a consequence of the fact that there are regions where the term $\nabla^2 s_d$ that was neglected in the derivation of (9') becomes important. In such regions, however, the value of formulating an equation of the form (14) as well as defining the parameter α is not clear.

Streltsova [1972] obtained the value $\epsilon = 3$ using interesting ad hoc hypotheses. Taking the derivative with respect to time of (27), it is seen that

$$\partial\xi/\partial t = \alpha(\xi \quad \bar{s} - b) \qquad (41)$$

This is her (3), expressing her assumption that the 'specific rate of vertical transfer of flow may be considered to vary linearly with the difference of the average head $b - \bar{s}$ and the free surface head ξ.' In view of (41), our conclusion is that in the incompressible case this condition not only is an approximation but also is satisfied exactly. However, for the compressible case she is correct when stating that this is a first approximation only.

It may be instructive to apply (27) to obtain the free surface head from the average drawdown in the case of a well pumping at a constant rate. *Streltsova*'s [1972] (10) implies

$$\bar{s} = \frac{Q}{2\pi T}\int \frac{J_0\left(\frac{r}{B}x\right)}{x} \left[\quad \frac{}{x^2+1}\exp\quad\frac{\alpha t x^2}{x^2+1}\right)\right] dx \qquad (42)$$

$$)$$

$$\frac{1}{x^2+1} \exp\left(-\frac{\alpha x^2 \tau}{x^2+1}\right)\Bigg]$$

$$\left[\ \ - \exp\left(\ \ \frac{\alpha t x^2}{x^2+1}\right)\right] \tag{43}$$

This is Streltsova's (17).

Compressible Case

In the compressible case the generalized Boulton's kernel does not consist of a single exponential term; instead, it is an infinite series of exponentials given by (29). Equations (36), (37), and (39), respectively, show that for very small times an unconfined aquifer behaves as a confined one with transmissibility T and storage coefficient S; at slightly longer times as a leaky aquifer with permeability and specific storage of the leaky layer equal to K_z and S_s, respectively; and at long times as a confined aquifer with transmissivity T and storage coefficient equal to $S + S_y$.

This corresponds to observations made by Walton and Boulton on pumping tests in unconfined formations and recalled previously by *Gambolati* [1976]. *Walton* [1960] observed that 'three distinct segments of the time-drawdown curve may be recognized under water table conditions,' while *Boulton* [1963] states that 'the very early time drawdown curve follows the Theis curve for an artesian aquifer. However, as the yield to the water table becomes effective, the time-drawdown curve becomes a leaky artesian aquifer curve, the drawdown attaining temporary equilibrium (or near equilibrium). After a longer time . . . the time drawdown curve merges with the Theis non-equilibrium curve associated with the coefficient of storage for water-table conditions.' It must be noticed, however, that the permeability and specific storage of the corresponding leaky aquifer are K_z and S_s, respectively; thus 'the behavior of the aquifer is not yet sensitive to the yield of the water table' at this stage. It is at longer times that the yield of the water table starts to be relevant; indeed, at longer times the generalized Boulton's kernel can be approximated by (see appendix)

$$\left(\frac{K_z S_s}{\pi t}\right)^{1/2} \quad \frac{K_z}{b}(1 - $$

$$\cdot \left\{\mathrm{erfc}\,(\sigma^2 - 1)\left(\frac{K_z t}{S_y b}\right)^{1/2}\right\} \exp\left[(1-\sigma^2)^2 \frac{K_z t}{S_y b}\right] \tag{44}$$

Equations (32) and (44) together imply that the behavior of the aquifer is unaffected by the value of S_y unless time is long enough for the second term in the right-hand member of the latter equation to be significant, a fact that may be relevant when interpreting pumping tests.

Approximation of Compressible Flow by an Incompressible One

Introducing the dimensionless variables,

$$r_a = \frac{r}{b}(K_D)^{1/2} \tag{45}$$

$$\frac{K_z}{bS_y}\, t \tag{46}$$

(32) becomes

$$\frac{\partial^2 \bar{s}}{\partial {r_a}^2} + \frac{1}{r_a}\frac{\partial \bar{s}}{\partial r_a} = \sigma^2 \frac{\partial \bar{s}}{\partial t'} \tag{47}$$

where

$$B(t', \sigma \qquad \frac{{\rho_n}^2}{{\rho_n}^2 - 1 + \sigma^2}\, e^{-{\rho_n}^2 t'} \tag{48}$$

and

$$\int_0^\infty B(t', \sigma)\, dt' = 1 \tag{49}$$

In the appendix it is shown that

$$\lim_{\sigma \to 0} B(t', \sigma) = B(t', 0) = 3e^{-3t'} \tag{50}$$

As a matter of fact, this limit is approached very rapidly, as can be seen in Table 1 and in Figure 3. When $\sigma^2 \neq 0$ (i.e., $S_s \neq 0$), the three distinct segments mentioned previously of the time-drawdown curves are always present, and their extent in terms of the dimensionless time t' depends on the value of σ only. The fact that the limit in (50) is approached rapidly implies that in many cases of practical interest the first two segments are rather short and during most of the transition from confined to unconfined behavior the first term in the series expansion (48) predominates and can be used as an approximation instead of the whole series.

There are two facts that deserve to be mentioned because of their relevance in applications. The smallness of the first two intervals in dimensionless time does not guarantee their smallness in physical time when the factor bS_y/K_z is large. At any time the relative importance of the two terms in the right-hand member of (47) can be estimated by the value of the parameter

$$c(t') = \frac{\sigma^2}{\int_0^{t'} B(\tau, \sigma)\, d\tau} = \frac{S}{S_y \int_0^{t'} B(\tau, \sigma)\, d\tau} \tag{51}$$

In view of (49), $c(t') \to \sigma^2$; however, for small times, c may be much larger than σ^2. By inspection of Table 1 it can be seen that in many cases it is appropriate to approximate $B(t')$ by its first term in the series expansion (48), even at very small values of time t', but it would be misleading to neglect the compressibility (i.e., the term $\sigma^2 \partial \bar{s}/\partial t'$) in (47), because $c(t')$ may be large at those times.

8. Conclusions

The conclusions of this paper can be summarized as follows:

1. The equation governing the linearized theory of unconfined aquifers can be decomposed into equations involving the average drawdown and the deviation of this quantity from its average value.

2. When the horizontal diffusion of the deviation can be neglected, the average drawdown is governed by an integrodifferential equation whose structure is similar to that governing leaky aquifers [*Herrera and Rodarte,* 1973].

3. For the case of incompressible flow this equation reduces to *Boulton*'s [1954] equation with $\epsilon = 3$.

4. The physical interpretation of Boulton's delay factor $e^{-\alpha t}$ is as follows: when an incompressible unconfined aquifer is subjected to an instantaneous unit head drop that is kept fixed, the free surface is not lowered immediately; its descent

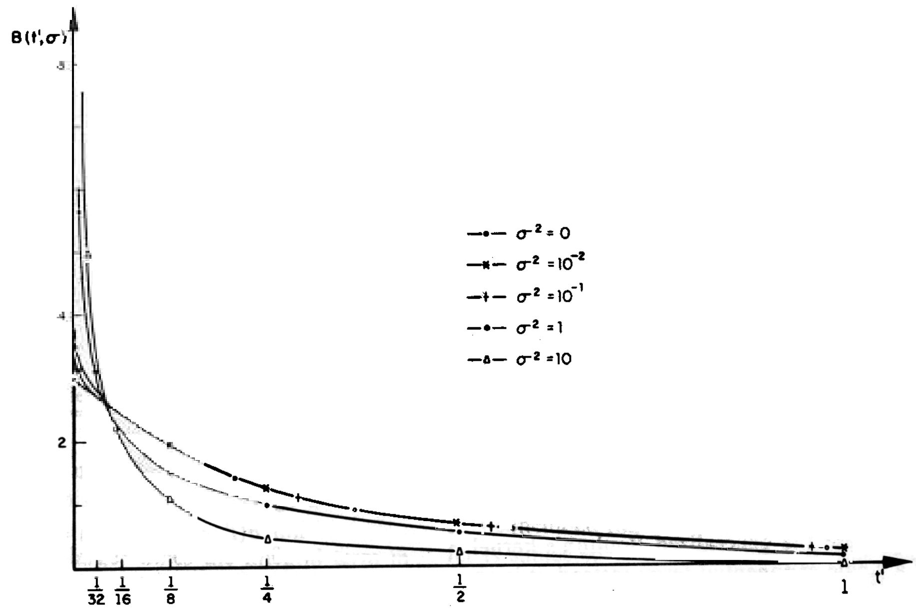

Fig. 3. Generalized (dimensionless) Boulton's kernel $B(t', \sigma)$ for several values of σ

being given by $1 - e^{-\alpha t}$. More generally, if the flow is compressible, it is given by $\int_0^t \mathcal{B}(\tau) d\tau$. This supplies a physical interpretation for the generalized Boulton's kernel $\mathcal{B}$.

5. The three distinct segments of the time-drawdown curve which have been observed under water table conditions, corresponding to a confined aquifer with storage coefficient S, to a leaky aquifer, and to a confined aquifer with storage coefficient $S + S_y$, correspond to different approximations of the memory function $\mathcal{B}(t)$. The leaky aquifer behavior corresponds to a leaky layer with permeability and storage coefficient which are independent of the specific yield, indicating that in this time interval, drainage is not yet significant. In the incompressible case the leaky aquifer behavior does not occur.

6. There is a large range of the parameter $\sigma^2 = S/S_y$, for which the memory functions can be approximated by the Boulton delay factor, which is the first term in the exponential series expansion of the memory function. However, the range on which the aquifer can be treated as incompressible is more restricted because it depends on the values of the coefficient c, as defined by (51), which may be large even if σ^2 is small.

7. In the presence of compressibility, there is always a time interval on which Boulton's generalized kernel cannot be approximated by its first term; however, it decreases rapidly as the compressibility (i.e., σ^2) goes to zero.

APPENDIX

Consider the system (9')–(11); i.e.,

$$\frac{\partial^2 s_d}{\partial z_a^2} - \left.\frac{\partial s_d}{\partial z_a}\right|_{z_a=1} = \sigma^2 \frac{\partial s_d}{\partial t'} \qquad 0 < z_a < \qquad \text{(A1)}$$

$$\frac{\partial s_d}{\partial t'} + \frac{\partial s_d}{\partial z_a} = -\frac{\partial \bar{s}}{\partial t'} \qquad z_a = 1 \qquad \text{(A2)}$$

$$\frac{\partial s_d}{\partial z_a} = 0 \qquad z_a = 0 \qquad \text{(A3)}$$

subject to the initial condition

$$s_d(z_a, 0) = 0 \qquad 0 < z_a < 1 \qquad \text{(A4)}$$

where t' is given by (46), while

$$z_a = z/b$$

Define

$$f(t') = \left.\frac{\partial s_d}{\partial z_a}\right|_{z_1} \qquad g(t') = \frac{\partial \bar{s}}{\partial t'} \qquad \text{(A6)}$$

and take the Laplace transform of system (A1)–(A4) to obtain

$$\frac{\partial^2 \hat{s}_d}{\partial z_a^2} - \sigma^2 p \hat{s}_d = \hat{f} \qquad 0 < z_a < 1 \qquad \text{(A7)}$$

$$p\hat{s}_d = -\hat{f} - \hat{g} \qquad \text{at} \quad z_a = 1 \qquad \text{(A8)}$$

$$\frac{\partial \hat{s}_d}{\partial z} = 0 \qquad \text{at} \quad z_a = 0$$

Thus

$$\hat{s}_d = \frac{\hat{f}(1 - \sigma^2) - \sigma^2 \hat{g}}{\sigma^2 p \cosh \sigma p^{1/2}} \cosh \sigma p^{1/2} z_a - \frac{\hat{f}}{\sigma^2 p} \qquad \text{(A10)}$$

Taking the derivative with respect to z_a of this expression and setting $z_a = 1$ yield

$$\hat{f} = \frac{\partial \hat{s}}{\partial z_a} = \frac{\hat{f}(1 - \sigma^2) - \sigma^2 \hat{g}}{\sigma p^{1/2}} \tanh \sigma p^{1/2} \qquad \text{(A11)}$$

Therefore

$$\hat{f} = -\sigma \hat{g} \frac{\tanh \sigma p^{1/2}}{p^{1/2}} \left[1 + (\sigma^2 - 1) \frac{\tanh \sigma p^{1/2}}{\sigma p^{1/2}}\right]^{-1} \qquad \text{(A12)}$$

Taking the inverse Laplace transform of (A12), it is seen that

$$\left.\frac{\partial s_d}{\partial z_a}\right|_{z_a=1} = -\int_0^{t'} B(\tau, \sigma) \frac{\partial \bar{s}}{\partial t'}(t' - \tau) \, d\tau \qquad \text{(A13)}$$

where B is given by (48) and ρ_n are the positive roots of (31). When (A13) is expressed in terms of the variables z and t, (14) is obtained with $\mathcal{B}$ given by (29).

From (A12) we have

$$\hat{\mathcal{B}}(p) = \sigma^2[\sigma p^{1/2} + (\sigma^2 - 1) \tanh \sigma p^{1/2}]^{-1} \tanh \sigma p^{1/2} \qquad \text{(A14)}$$

Thus

$$\int_0^\infty B(t)\,dt = \lim_{p\to 0} \bar{B}(p) = 1 \tag{A15}$$

An approximation for small t' can be obtained by considering p to be large in (A14). When p is large, tanh $\sigma p^{1/2}$ can be approximated by 1. Thus

$$\bar{B}(p) \approx \sigma^2[\sigma p^{1/2} - 1 + \sigma^2]^{-1} \tag{A16}$$

for large p. The inverse Laplace transform of this function is

$$B(t') = \sigma\left\{\frac{1}{(\pi t')^{1/2}} + \left(\frac{1}{\sigma} - \sigma\right)\left[\operatorname{erfc}\left(\sigma - \frac{1}{\sigma}\right)(t')^{1/2}\right]\right.$$

$$\left.\exp\left(\sigma - \frac{1}{\sigma}\right)^2 t'\right\} \tag{A17}$$

For very small times this expression reduces to

$$B(t') \approx \frac{}{(\pi t')^{1/2}} \tag{A18}$$

The generalized Boulton's kernel is

$$\mathcal{B}(t) = \frac{K_z}{bS_y} B(K_z t/bS_y) \tag{A19}$$

Taking into account (A19), relations (34), (35), and (44) easily follow from (A16), (A18), and (A17), respectively.

Finally, write (31) with $n = 1$ in the form

$$\sigma\rho_1 \cos \sigma\rho_1 - \sin \sigma\rho_1 + \sigma^2 \sin \sigma\rho_1 = 0 \tag{A20}$$

Expand in powers of $\sigma\rho_1$, divide by $\sigma^3\rho_1$, and take the limit when $\sigma \to 0$ to obtain $\frac{1}{3}\rho_1^2 - 1 = 0$, which implies

$$\lim_{\sigma\to 0} \rho_1^2 = 3$$

NOTATION

- $\mathcal{B}(t, \sigma)$ Boulton's generalized kernel, T^{-1}.
- $B(t', \sigma)$ dimensionless version of Boulton's generalized kernel.
- b initial saturated thickness of the aquifer, L.
- b_1, b_2 thickness of the aquifers in Figure 2, L.
- b' thickness of the aquiclude in Figure 2, L.
- erfc complementary error function.
- $\bar{f}$ Laplace transform of f.
- J_0 Bessel function of the first kind.
- K_r horizontal permeability, LT^{-1}.
- K_z vertical permeability, LT^{-1}.
- $K_D = K_z/K_r$ dimensionless permeability.
- p parameter of the Laplace transform.
- Q pumping rate, L^3T^{-1}.
- r radial distance from pumping well, L.
- $r_z = (r/b)(K_D)^{1/2}$ dimensionless radial distance from pumping well.
- s drawdown, L.
- $S = S_s b$ storage coefficient.
- S_s specific (elastic) storage, L^{-1}.
- S_y specific yield.
- t pumping time, T.
- $t' = (K_z/bS_y)t$ dimensionless pumping time.
- $T = bK_r$ transmissibility, L^2T^{-1}.
- x Cartesian horizontal coordinate, L.
- y Cartesian horizontal coordinate, L.
- z vertical coordinate measured from the bottom of the aquifer.
- α Boulton's delay index, T^{-1}.
- $\alpha_s = K_r/S_s$, L^2T^{-1}.
- $\alpha_y = K_z/S_y$, LT^{-1}.
- $\epsilon = \alpha S_y b/K_z$.
- ξ elevation of the free surface, L.
- $\sigma = (S/S_y)^{1/2}$.

REFERENCES

Abramowitz, M., and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

Boulton, N. S., The drawdown of the water table under non-steady conditions near a pumped well in an unconfined formation, *Proc. Inst. Civil Eng.*, *3*(3), 564–579, 1954.

Boulton, N. S., Unsteady radial flow to a pumped well allowing for delayed yield from storage, *Publ. 37*, pp. 472–477, Int. Ass. Sci. Hydrol., Rome, 1955.

Boulton, N. S., Analysis of data from nonequilibrium pumping tests allowing for delayed yield from storage, *Proc. Inst. Civil Eng.*, *26*, 469–482, 1963.

Boussinesq, J., Recherches théoriques sur l'écoulement des nappes d'eau infiltrées dans le sol et sur débit de sources, *J. Math. Pures Appl.*, *10*, 5–78, 1903.

Gambolati, G., Transient free surface flow to a well: An analysis of theoretical solutions, *Water Resour. Res.*, *12*(1), 27–39, 1976.

Hantush, M. S., Modification of the theory of leaky aquifers, *J. Geophys. Res.*, *65*(11), 3713–3725, 1960.

Herrera, I., Theory of multiple leaky aquifers, *Water Resour. Res.*, *6*(1), 185–193, 1970.

Herrera, I., Integrodifferential equations for systems of leaky aquifers and applications, 2, Error analysis of approximate theories, *Water Resour. Res.*, *10*(4), 811–820, 1974.

Herrera, I., A review of the integrodifferential equations approach to leaky aquifer mechanics, in *Proceedings of the Symposium on Advances in Groundwater Hydrology*, American Water Resources Association, Chicago, Ill., 1976.

Herrera, I., and G. E. Figueroa, A correspondence principle for the theory of leaky aquifers, *Water Resour. Res.*, *5*(4), 900–904, 1969.

Herrera, I., and L. Rodarte, Integrodifferential equations for systems of leaky aquifers and applications, 1, The nature of approximate theories, *Water Resour. Res.*, *9*(4), 995–1005, 1973.

Herrera, I., and R. Yates, Integrodifferential equations for systems of leaky aquifers and applications, 3, A numerical method of unlimited applicability, *Water Resour. Res.*, *13*(4), 725–732, 1977.

Herrera, I., R. Yates, and C. Pérez de Celis, A family of approximate numerical procedures for leaky aquifer systems, in *Proceedings of the Second International Symposium on Finite Element Methods in Flow Problems*, pp. 355–364, International Center for Computer-Aided Design, S. Margherita Ligure, Italy, 1976.

Neuman, S. P., Theory of flow in unconfined aquifers considering delayed response of the water table, *Water Resour. Res.*, *8* (4), 1031–1045, 1972.

Streltsova, T. D., Unsteady radial flow in an unconfined aquifer, *Water Resour. Res.*, *8*(4), 1059–1066, 1972.

Streltsova, T. D., Flow near a pumped well in an unconfined aquifer under nonsteady conditions, *Water Resour. Res.*, *9*(1), 227–235, 1973.

Walton, W. C., Application and limitations of methods used to analyze pumping test data, 2, *Water Well J.*, 45–52, March 1960.

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