58

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(numerical methods/diffraction/elasticity/seismology/earthquake engineering)

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Reprinted from the Proceedings of The National Academy of Sciences
Vol. 74, No. 7, pp. 2595–2597, July 1977

Reprinted from
Proc. Natl. Acad. Sci. USA
Vol. 74, No. 7, pp. 2595–2597, July 1977
Applied Physical and Mathematical Sciences

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Communicated by Emilio Rosenblueth, April 22, 1977

ABSTRACT Functional valued operators are used to formulate diffraction problems in a general abstract form. Variational principles for this formulation are developed. They are then applied to derive results on the linearized theory of free surface flows previously reported. Applications are also made to potential theory and elasticity.

In recent years the numerical solution of scattering problems has received considerable attention. To treat them it has frequently been necessary to extend the numerical grids to excessively large regions because such problems are formulated in unbounded regions. To avoid this difficulty several procedures have been proposed. One of the most successful is the integral equation method (1); another recent one is the hybrid element method (2). The latter seems very promising because it exhibits great flexibility. In some cases, variational principles, involving a bounded region only, have been reported for problems that are, however, formulated in an unbounded one (2).

It is shown here that this remarkable property depends on simple facts. Two general theorems which are widely applicable to problems of practical interest are given.

To obtain these results we use a theory expressed in terms of functional valued operators and developed in detail elsewhere (3, 4). Application of such operators has been rather restricted in spite of their being best suited for the formulation of variational principles. Indeed, the generality of the results presented here illustrates the advantages of using such theory.

Variational principles for the linearized theory of free surface flows, which were previously reported (2), and for elastic diffraction are derived as examples of the general theorems obtained in this paper.

Diffraction problems

Consider a linear functional valued operator (3, 4) $P{:}D \rightarrow D^*$, where D is a linear space in the field of real or complex numbers (whose elements $u,v,\ldots,$ will be called motions), and D^* is the linear space of all the linear functionals defined on D. Such an operator (3, 4) always possesses an adjoint P^*, and therefore its antisymmetric part $A = (P - P^*)/2$ is well defined. $P - A$ is self adjoint because it is the symmetric part of P. Our discussion refers to the case when P and A are such that

$$\langle Pu,v\rangle = 0 \text{ for every } v \ni Av = 0, \quad \Rightarrow Pu = 0. \qquad [1]$$

The derivative and variation of real or complex valued functionals Ω are used in the sense of additive Gateaux variations (3, 4). In general, $\Omega'{:}D \rightarrow D^*$, while the variation $\delta\Omega{:}\hat{I} \rightarrow I'$, where $\hat{I} \subset D$ is an affine subspace generated by a linear subspace $I \subset D$. In addition, $\delta\Omega$ has the property

$$\langle \Omega'(u),v\rangle = \langle \delta\Omega(u),v\rangle \qquad [2]$$

for every $u \in \hat{I}$ and $v \in I$.

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Definition 1: Let $I \subset D$ be a linear subspace such that for every $u,v \in I$ and $w \in D$:

$$(i) \quad \langle Au,v\rangle = 0 \qquad [3a]$$

$$(ii) \quad A(u - w) = 0 \quad \Rightarrow w \in I. \qquad [3b]$$

Then I is said to be a connectivity condition and motions $u \in I$ are said to be internally generated. When I is such that:

$$(iii) \quad \langle Au,w\rangle = 0 \quad \forall u \in I, \Rightarrow w \in I \qquad [3c]$$

then the connectivity condition is said to be complete. The set $\mathscr{E} \subset D$ is defined by:

$$\mathscr{E} = \{u \in D \mid Pu = 0\}. \qquad [4]$$

Motions $u \in \mathscr{E}$ are said to be externally generated.

Definition 2: Given $U \in D$ and $f \in D^*$, a problem of diffraction consists in finding $u \in D$, such that

$$Pu = f \qquad [5a]$$

and

$$u = U + v \qquad [5b]$$

where $v \in I$. Problems of diffraction in the restricted sense correspond to the case $f = 0$.

Questions of existence will not be discussed. In what follows it will be assumed that the linear functional f belongs to the range of P.

Variational formulations

The following theorems are suitable for applications of the hybrid element method.

THEOREM 1. *Given a connectivity condition* I $\subset$ D, *define the functional* Ω *by:*

$$\Omega(u) = \frac{1}{2}\langle Pu,u\rangle - \langle f - AU,u\rangle \qquad [6]$$

for every u $\in$ D, *and the affine subspace*

$$\hat{I} = \{u \in D \mid U - u \in I\} \qquad [7]$$

Then, for any u $\in$ D, *the following assertions*

(*i*) u is a solution of the diffraction problem.

$$(ii) \quad u \in \hat{I} \text{ and } \delta\Omega(u) = 0 \qquad [8]$$

are equivalent. When I *is complete, they are, in addition, equivalent to*

(*iii*) For every $v \in I$:

$$\langle \Omega'(u),v\rangle = 0. \qquad [9]$$

Proof: A direct computation shows that

$$\langle \Omega'(u),v\rangle = \langle Pu - f,v\rangle + \langle A(U - u),v\rangle. \qquad [10]$$

When $u \in \hat{I}$ and $v \in I$, the term involving A vanishes by Eq. **3a**, and therefore (*i*) is equivalent to (*ii*) by virtue of Eq. **2** and the fact that $\delta\Omega:\hat{I} \rightarrow I^*$. When u is a solution, $u \in \hat{I}$ necessarily; thus, (*i*) $\Rightarrow$ (*iii*) always. Assume I is complete and Eq. **9** holds; then by Eqs. **1** and **10**, $Pu - f = 0$. Also, $U - u \in I$ by Eqs. **10** and **3c**.

In many applications the requirement of a motion being internally generated is expressed as boundary conditions and it is advantageous to formulate variational principles for which both u and its variation are not required to satisfy such conditions.

THEOREM 2. *Define the functional* X, *for every couple* u,u$_E$ $\in$ D, *by*

$$X(u,u_E) = \frac{1}{2}\langle Pu,u\rangle - \langle A\bar{u},[u]\rangle + \langle AU,u_E\rangle - \langle f,u\rangle \qquad [11]$$

where

$$[u] = u_E - u; \bar{u} = \frac{1}{2}(u_E + u). \qquad [12]$$

Let

$$\mathcal{D} = \{(u,u_E)\,|\,u \in D \text{ and } U - u_E \in I\}. \qquad [13]$$

Then, the variation δX *over* $\mathcal{D}$, *satisfies*

$$\delta X(u,u_E) = 0 \qquad [14]$$

if and only if u *is a solution of the diffraction problem and* A[u] = 0.

Proof: Recall that

$$\langle Au_E,v_E\rangle - \langle Au,v\rangle = \langle A[u],\bar{v}\rangle + \langle A\bar{u},[v]\rangle. \qquad [15]$$

Thus,

$$\langle \delta X(u,u_E),(v,v_E)\rangle = \langle Pu - f,v\rangle + \langle A(U - u_E),v_E\rangle + 2\langle A[u],\bar{v}\rangle = \langle Pu - f,v\rangle + 2\langle A[u],\bar{v}\rangle. \qquad [16]$$

This vanishes for every $v \in D$ and $v_E \in I$ if and only if Eq. **5a** and $A(u_E - u) = 0$ hold. In view of Eq. **3b**, this implies that $U - u \in I$ because $U - u_E \in I$.

Applications

Consider the region R illustrated in Fig. 1. Let the members $\phi,\psi,\ldots$ of the linear space D be the complex-valued functions possessing second-order continuous partial derivatives in R. Define $P:D \rightarrow D^*$ by

$$\langle P\phi,\psi\rangle = \int_R \psi\frac{\partial^2\phi}{\partial x_i\partial x_i}dx - \int_{\partial_1R}\psi\left(\alpha\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial\phi}{\partial x_2}\right)dx - \int_{\partial_2R}\psi\frac{\partial\phi}{\partial n}dx \qquad [17]$$

where α is a real constant. Here and in what follows sum over the range of repeated indices is understood. The operator P so defined is not self-adjoint because by integration by parts it can be seen that its antisymmetric part satisfies

$$2\langle A\phi,\psi\rangle = \int_{-h}^{0}\left(\psi\frac{\partial\phi}{\partial x_1} - \phi\frac{\partial\psi}{\partial x_1}\right)\Bigg|_{x_-}^{x_+}dx_2 - \alpha\left(\psi\frac{\partial\phi}{\partial x_1} - \phi\frac{\partial\psi}{\partial x_1}\right)\Bigg|_{x_-}^{x_+}. \qquad [18]$$

The operators P and A have property **1** because the support of Av is contained in ∂R.

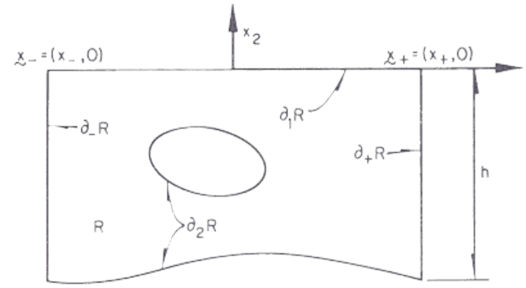

FIG. 1. Region for the linearized free-surface flow.

The functional $f \in D^*$ will be

$$\langle f,\psi\rangle = -\int_{\partial_2R}\psi V dx \qquad [19]$$

where V is a function on ∂_2R. Then the equation for ϕ corresponding to Eq. **5a** is equivalent to the system

$$\nabla^2\phi = 0; \alpha\phi_{x_1x_1} + \phi_{x_2} = 0 \quad \text{on} \quad \partial_1 R; \partial\phi/\partial n = V \text{ on } \partial_2R. \qquad [20]$$

This is the set considered by Mei and Chen (2) when applying the hybrid element method to linearized free-surface flows. To include this kind of problem in our frame-work, let the members of the set $I \subset D$ of internally generated motions be the functions ϕ of D such that ϕ and $\partial\phi/\partial x_1$, for some choice of the constants A_n^- and A_n^+, are identical with the functions

$$\sum_{n=0}^{\infty} A_n^- e^{-iK_nx_1}\cosh k_n(x_2 + h);$$

$$\sum_{n=0}^{\infty} A_n^+ e^{iK_nx_1}\cosh k_n(x_2 + h) \qquad [21]$$

and their x_1-derivatives, on $x_1 = x_-$ and $x_1 = x_+$, respectively. Here, $k_0 = 0$, k_1 = the positive real root of equation:

$$\alpha k = \tanh kh \qquad [22]$$

while $k_2,k_3,\ldots$ = the positive imaginary roots.

With these definitions, subspace $I \subset D$ satisfies conditions (*i*) and (*ii*) of *Definition 1*. Property (*ii*) [**3b**] is clear because Eq. **18** implies that $A\phi = 0$ if and only if ϕ and $\partial\phi/\partial x_1$ vanish identically when $x_1 = x_-$ or $x_1 = x_+$. Property (*i*) can be derived from the orthogonality property given by Mei and Chen (2) for the functions occurring in Eq. **21**. In this problem, as in many other applications of the hybrid element method, the connectivity condition is complete. This can be deduced from the completeness of the system of functions $\cosh k_n x_2$ in the interval $[-h,0]$, which in turn is implied by the fact that they are solutions of a Sturm–Louiville problem.

Theorems 1 and *2* are thus applicable to the corresponding diffraction problem. Let Φ^I be the incident wave potential; then U can be taken as any function of D such that it and its first x_1-derivative are equal to Φ^I and $\partial\Phi^I/\partial x_1$, respectively, on $x_1 = x_-$ and $x_1 = x_+$. *Theorem 2* is more general than two variational principles given by Mei and Chen (2); each of these is obtained by setting $\Phi^I = \partial\Phi^I/\partial x_i = 0$ and $V = 0$, respectively, in *Theorem 2*.

For applications to elasticity and potential theory, consider functions $\mathbf{u} = (u_1, \ldots, u_m)$ defined in a region R of the n-dimensional Euclidean space (Fig. 2), with boundary $\partial_1R \cup \partial_2R \cup \partial_3R$, where ∂_iR ($i = 1,2,3$) are pair-wise disjoint sets, and let $\mathbf{n}$ be its outward unit normal vector. Take D as the set of such

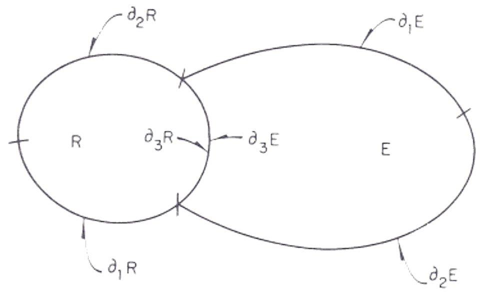

FIG. 2. Regions considered in application to elasticity and potential theory.

functions that are C^2 on R. Define $P:D \to D^*$ by

$$\langle P\mathbf{u},\mathbf{v}\rangle = \int_R v_\alpha \mathcal{L}_\alpha(\mathbf{u})dx + \int_{\partial_1 R} u_\alpha T_\alpha(\mathbf{v})dx - \int_{\partial_2 R} v_\alpha T_\alpha(\mathbf{u})dx \quad [23]$$

where $\mathcal{L}$ and $\mathbf{T}$ are differential operators:

$$\mathcal{L}_\alpha(\mathbf{u}) = \frac{\partial}{\partial x_j}\left(C_{\alpha j\beta q}\frac{\partial u_\beta}{\partial x_q}\right) + \rho u_\alpha; T_\alpha(\mathbf{u}) = C_{\alpha j\beta q}\frac{\partial u_\beta}{\partial x_q}n_j. \quad [24]$$

Latin indices run from 1 to n, while Greek ones from 1 to m. Coefficients ρ and $C_{\alpha j\beta q}$ are smooth functions such that

$$C_{\alpha j\beta q} = C_{\beta q\alpha j}. \quad [25]$$

By integration by parts, it is seen that

$$\langle A\mathbf{u},v\rangle = \int_{\partial_3 R}\{v_\alpha T_\alpha(\mathbf{u}) - \mathbf{u}_\alpha T_\alpha(\mathbf{v})\}dx. \quad [26]$$

Operators P and A satisfy condition 1 because the support of A is contained in the boundary.

Given functions $\mathbf{b},\mathbf{d},\sigma$ on R, $\partial_1 R$, and $\partial_2 R$, respectively, define $\mathbf{f} \in D^*$ by

$$\langle \mathbf{f},\mathbf{v}\rangle = \int_R v_\alpha b_\alpha dx + \int_{\partial_1 R} d_\alpha T_\alpha(\mathbf{v})dx - \int_{\partial_2 R} v_\alpha \sigma_\alpha dx \quad [27]$$

for every $\mathbf{v} \in D$. Thus, frequently (5), Eq. **5a** is equivalent to

$$\mathcal{L}(\mathbf{u}) = \mathbf{b}; \mathbf{u} = \mathbf{d} \text{ and } \mathbf{T}(\mathbf{u}) = \sigma \quad [28]$$

in their respective domains. When $m = 1$, $n = 3$, and $C_{1j1q} = \delta_{jq}$, Eq. **28** is Laplace's equation if $\rho \equiv 0$ and the reduced wave equation if $\rho \equiv 1$. If $m = n = 3$, Eq. **28** is the equilibrium equation of elasticity, if $\rho \equiv 0$ and the reduced equation of elastodynamics if $\rho \equiv 1$.

Diffraction problems are usually formulated in a region R and a region E next to it. Frequently E is assumed to be unbounded, but our theory is equally applicable when E is bounded, as illustrated in Fig 2. The connectivity condition $I \subset D$ can be defined as the subset of motions of D, with the property that they can be extended continuously across $\partial_3 R$ into motions of a set $\mathcal{W}$ whose members are defined on E and satisfy certain regularity conditions (e.g., a radiation condition).

The continuity requirement is frequently expressed by the condition that $\mathbf{u}$ and $\mathbf{T}$ be continuous across the common part $\partial_3 R$ of the boundary.

For a set I defined in this manner to satisfy condition (*ii*) of *Definition 1*, it is enough that A be such that $A\mathbf{u} = 0 \Rightarrow u_\alpha \equiv T_\alpha(\mathbf{u}) \equiv 0$, on $\partial_3 R$. This latter condition is fulfilled in most situations of interest by operator A as defined by Eqs. **26** and **24**. Thus, for example, in applications to potential theory or the reduced wave equation, $T_1(\mathbf{u})$ is $\partial u/\partial n$, and the above condition is fulfilled. In applications to elasticity, it is enough that C_{ijpq} be strongly elliptic (5).

Condition (*i*) of *Definition 1* is

$$\langle A\mathbf{u},\mathbf{v}\rangle = \int_{\partial_3 R}\{u_\alpha T_\alpha(\mathbf{v}) - v_\alpha T_\alpha(\mathbf{u})\}dx = 0 \quad [29]$$

for every $\mathbf{u},\mathbf{v} \in I$. When E is bounded, there is a wide class of problems that satisfy Eq. **29**. Indeed, let $\partial_1 E$, $\partial_2 E$, and $\partial_3 E$ be a mutually disjoint decomposition of the boundary of E such that $\partial_3 E = \partial_3 R$. Take $\mathcal{W}$ as the set of motions $\mathbf{U}$ defined on E such that $\mathcal{L}(\mathbf{U}) = 0$, $\mathbf{U} = 0$, and $\mathbf{T}(\mathbf{U}) = 0$ on E, $\partial_1 E$, and $\partial_2 E$, respectively. Under these conditions, for members $\mathbf{U},\mathbf{V} \in \mathcal{W}$, the reciprocity relation

$$\int_{\partial_3 R}\{U_\alpha T_\alpha(\mathbf{V}) - V_\alpha T_\alpha(\mathbf{U})\}dx = \int_{\partial_1 R + \partial_2 R}\{U_\alpha T_\alpha(\mathbf{V}) - V_\alpha T_\alpha(\mathbf{U})\}dx = 0 \quad [30]$$

implies Eq. **29**.

When E is unbounded, motions of $\mathcal{W}$ are required, in addition, to satisfy a radiation condition. It can be shown that under very general hypotheses radiation conditions of the type introduced previously (6, 7) lead to connectivity conditions satisfying Eq. **30**.

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