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Theory of connectivity for formally symmetric operators

(numerical methods/finite element method/variational principles/elasticity/earthquake engineering)

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ABSTRACT **A previous paper introduced the notion of complete connectivity conditions and developed variational principles for diffraction problems subjected to such restrictions. Here, an abstract definition of formally symmetric operators is given and it is shown that the problem of connecting solutions of equations associated with this kind of operators leads to complete connectivity conditions. The variational principles previously developed as well as a present more general one are thus applicable. The problem of connecting solutions defined in different regions is basic for finite element formulations. Formally symmetric operators occur in many branches of science and engineering. Applications are given here to potential theory, wave propagation, elasticity, and a general class of boundary integral equations.**

Although the finite-element method was originally developed for particular areas of structural mechanics and elasticity, the mathematical basis of the technique is applicable to problems found throughout applied mathematics, continuum mechanics, engineering, and physics. When applying the finite-element method we first seek solutions in the individual elements and then connect those corresponding to different elements (1). The problem of connecting solutions defined in different regions is hence basic in the formulation of the finite element-method.

A previous paper (2) introduced a class of operators, (here called "formally symmetric"), defined a diffraction problem associated with such operators, and derived variational principles for that problem. Applications were given to problems in the linearized theory of free surface flow (3), potential theory, and elasticity. The present paper gives an abstract definition of formally symmetric operators and develops a general theory of connectivity for problems defined in terms of such operators. The main result is that, whenever the problem of connecting is well posed, it leads to complete connectivity conditions of the type heuristically introduced (2). The general variational principles previously reported (2), as well as one given here which is more convenient for numerical treatment, are applicable. The theory is formulated for the problem of connecting solutions defined in two subregions only, but by induction it can be applied to problems formulated in an arbitrary number of subregions.

An important area of application of the results is that of boundary integral equations (4). Boundary integral equations constitute one possible way of specifying a set of solutions defined in one subregion. Results are thus applicable to such equations whenever they are derived from formally symmetric operators of the kind discussed here. Formally symmetric operators occur in potential theory, wave propagation, diffusion problems, elasticity, structural dynamics, and many other branches of science and engineering.

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As in the preceding paper (2), the general formulation of variational principles using functional-valued operators developed by the author (5) is applied. This approach simplifies treatment of continuum mechanics and its partial differential equations because it is applicable in any linear space, not necessarily normed, or with an inner product, or complete. A discussion of the main advantages of its use as well as a summary of the method have been reported (6, 7).

Characterization of complete connectivity conditions

For operators $P{:}D \to D^*$ defined on a linear space D, in the field of real or complex numbers (whose elements $u,v,\ldots,$ will be called motions) and with values in the space D^* of all the linear functionals defined on D, let $\mathcal{N} \subset D$ be

$$\mathcal{N} = \{u \in D \mid Au = 0\}. \qquad [1]$$

Here $A = (P - P^*)/2$ is the antisymmetric part of P. As in ref. 2, in this paper it will be assumed that

$$\langle Pu,v\rangle = 0 \;\forall v \in \mathcal{N} \Rightarrow Pu = 0. \qquad [2]$$

Operators satisfying Eq. 2 will be called "formally symmetric"; for them, the notion of complete connectivity condition has been introduced (2). The following is an equivalent but briefer formulation of this concept.

Definition 1: A linear subspace $\mathcal{J} \subset D$ is said to be a connectivity condition if:

$$(i) \quad \mathcal{N} \subset \mathcal{J} \qquad [3]$$

$$(ii) \quad \langle Au,v\rangle = 0, \forall u,v \in \mathcal{J}. \qquad [4]$$

The connectivity condition is said to be complete if in addition

(*iii*) For any $w \in D$

$$\langle Aw,v\rangle = 0, \quad \forall v \in \mathcal{J} \Rightarrow w \in \mathcal{J}. \qquad [5]$$

The set $\mathcal{E} \subset D$ is defined by

$$\mathcal{E} = \{u \in D \mid Pu = 0\}. \qquad [6]$$

The relation

$$\langle Au,v\rangle = 0, \quad \forall u,v \in \mathcal{E} \qquad [7]$$

is clear, because $2\langle Au,v\rangle = \langle Pu,v\rangle - \langle Pv,u\rangle$. In what follows of this section, it will be assumed that there is in D a connectivity condition $\mathcal{J}$; however, the completeness of $\mathcal{J}$ will not be taken for granted.

The problem of diffraction associated with P is given next.

Definition 2: Given $U \in D$ and $f \in D^*$ in the range of P, a motion $u \in D$ is said to be a solution of the problem of diffraction if

$$Pu = f \text{ and } U - u = v \in \mathcal{J}. \qquad [8]$$

Abbreviation: ebvp, external boundary value problem.

For every symmetric operator $S{:}D \rightarrow D^*$ the operator $B = S + A/2$ satisfies

$$A = B - B^*. \qquad [9]$$

It will be assumed, that operators $B{:}D \rightarrow D^*$ to be considered in what follows, satisfy Eq. **9**. When such B is available, two problems are defined.

Definition 3: Given $U \in D$, the motion $u \in D$ is said to be a solution of the external boundary value problem (ebvp) if $Bu = BU$ and $u \in \mathcal{J}$. The internal boundary value problem (ibvp) is defined by replacing $\mathcal{J}$ with $\mathcal{E}$.

Any of the three problems already introduced will be said to satisfy existence if they possess (a) at least one solution for every admissible datum, (b) uniqueness when the only solution of the problem with vanishing data is the zero solution, and (c) almost uniqueness when any solution of the latter problem belongs to the null set $\mathcal{N}$. For some purposes it is too restrictive to require uniqueness; owing to this fact, a problem will be said to be well posed if it satisfies existence and almost uniqueness.

LEMMA 1. *The problem of diffraction is well posed if and only if every* $u \in D$ *can be written as* $u = u_1 + u_2$ *with* $u_1 \in \mathcal{J}$, $u_2 \in \mathcal{E}$ *and this representation is almost unique.*

Proof: Assume such representation holds. Given any $f \in D^*$ in the range of P, write $f = PW = PW_1$. Then $u = W_1 + U_2$ is a solution of the diffraction problem. Observe also that any solution u of the diffraction problem with vanishing data belongs to $\mathcal{J} \cap \mathcal{E}$; this implies $u \in \mathcal{N}$ by the assumed almost uniqueness of the representation. Conversely, assume the diffraction problem is well posed. Then, given any $U \in D$ let u be a solution of the diffraction problem with $f = 0$ and v be given by Eq. **8**, the relation $U = v + u$ supplies the desired representation and its almost uniqueness is easily seen.

THEOREM 1. *Assume that for some* B *fulfilling Eq* **9** *the* ebvp *satisfies existence. Then* $\mathcal{J}$ *is a complete connectivity condition.*

Proof: Assume $U \in D, \langle AU,v\rangle = 0, \ \forall v \in \mathcal{J}$; let $u \in \mathcal{J}$ be a solution of the ebvp corresponding to U; and write $w = U - u$. Then $Bw = 0$, so that $\langle B^*w,v\rangle = -\langle Aw,v\rangle = 0, \ \forall v \in \mathcal{J}$. Given any $V \in D$, let $v \in \mathcal{J}$ be a solution of the ebvp corresponding to V, so that

$$-\langle Aw,V\rangle = \langle B^*w,V\rangle = \langle Bv,w\rangle = \langle B^*w,v\rangle = 0.$$

Thus, $w \in \mathcal{N} \quad \mathcal{J}$ and therefore $U \in \mathcal{J}$, because $U = u + w$.

THEOREM 2. *If the diffraction problem is well posed, then* $\exists\, B{:}D \rightarrow D^*$ *satisfying Eq.* **9** *and such that for it the* ebvp *satisfies existence.*

Proof: In view of *Lemma 1*, given any $u \in D$, it is possible to write $u = u_1 + u_2$ with $u_1 \in \mathcal{J}$ and $u_2 \in \mathcal{E}$. For every $u \in D$, define

$$Bu = Au_1. \qquad [10]$$

Operator B is well defined because u_1 is almost unique. For every $v = v_1 + v_2 \in D$, the adjoint of B satisfies

$$\langle B^*u,v\rangle = \langle Bv,u\rangle = \langle Av_1,u_2\rangle = -\langle Au_2,v_1\rangle = -\langle Au_2,v\rangle \qquad [11]$$

where use has been made of Eqs. **4** and **7**.

This shows that

$$B^*u = -Au_2. \qquad [12]$$

Eqs. **10** and **12** together imply Eq. **9**. The ebvp is clearly well posed, because for every $U = U_1 + U_2 \in D$ we have $BU_1 = BU$ by Eq. **10** and $U_1 \in \mathcal{J}$.

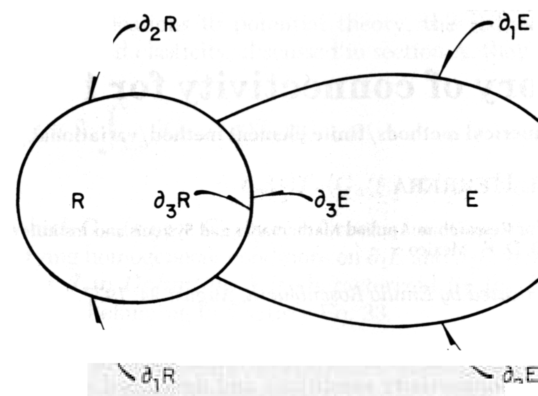

FIG. 1. Regions considered in applications

THEOREM 3. *If the diffraction problem is well posed, then* $\mathcal{J}$ *is complete.*

Proof: This is immediate by *Theorems 1* and *2*.

The following variational principle is more general than those presented previously (2).

THEOREM 4. *Assume* $\mathcal{J}$ *is a complete connectivity condition. Given* $f, f_J \in D^*$, *on the ranges of* P *and* A *respectively, and* $U' \in D$, *define the functional* X, *for every couple* $u,u' \in D$, *by*

$$X(u,u') = \frac{1}{2}\langle Pu,u\rangle + \langle A[u] - 2f_J,\bar{u}\rangle + \langle AU',u'\rangle - \langle f,u\rangle \qquad [13]$$

Then, the couple $u,u' \in D$ *satisfies* $Pu = f$, $A[u] = f_J$ *and* $u' - U' \in \mathcal{J}$ *if and only if*

$$\langle X'(u,u'),(v,v')\rangle = 0 \quad \forall (v,v'),\ v \in D \text{ and } v' \in \mathcal{J}. \qquad [14]$$

Here $[u] = u' - u$ *and* $\bar{u} = (u' + u)/2$.

Proof: The proof of this theorem is similar to the proofs presented previously (2) and will not be given. In numerical applications, *Theorem 4* presents advantages over those given previously (2) because it is not restricted to satisfy the condition $u' - U' \in \mathcal{J}$ from the start and therefore permits satisfying this condition in an approximate manner.

The problem of connecting

In many applications such as the finite-element method, one is interested in connecting solutions defined in two neighboring regions such as R and E in Fig. 1. There are two linearly independent spaces of functions D and D_E, with the property that the functions of D vanish on E, whereas those of D_E vanish on R. Elements $\hat{u} \in \hat{D} = D + D_E$ will be written as $\hat{u} = (u,u_E)$ with $u \in D$ and $u_E \in D_E$. There are two formally symmetric operators $P{:}\hat{D} \rightarrow \hat{D}^*$ and $P_E{:}\hat{D} \rightarrow \hat{D}^*$ satisfying

$$\langle P\hat{u},\hat{v}\rangle = \langle Pu,v\rangle; \quad \langle P_E\hat{u},\hat{v}\rangle = \langle P_E u_E,v_E\rangle \qquad [15]$$

for every $\hat{u},\hat{v} \in \hat{D}$. Due to this property, P and P_E can also be thought of as operators $P{:}D \rightarrow D^*$ and $P_E{:}D_E \rightarrow D_E^*$. The fact that P and P_E are formally symmetric implies that Eq. **2** is satisfied by A and a corresponding relation is satisfied by the antisymmetric part A_E of P_E.

There is a linear subspace $\mathcal{S} \subset \hat{D}$ with the property that $\hat{u} \in \mathcal{S}$ if and only if

$$\langle A\hat{u},\hat{v}\rangle = -\langle A_E\hat{u},\hat{v}\rangle, \quad \forall \hat{v} \in \mathcal{S}. \qquad [16]$$

Elements $\hat{u} \in \mathcal{S}$ are said to be smooth. Motions $u \in D$ and $u_E \in D_E$ will be called smooth extensions of each other whenever $\hat{u} = (u,u_E) \in \mathcal{S}$. It will be assumed that all elements of D and D_E possess smooth extensions. Given any $\hat{u} = (u,u_E) \in D$, the following notation will be adopted:

$$[\hat{u}] = u' - u; \quad \bar{u} = \frac{1}{2}(u' + u) \qquad [17]$$

in which $u' \in D$ is any smooth extension of $u_E \in D_E$. In view of Eq. **16**, $A[\hat{u}] = 0$ if and only if $\hat{u}$ is smooth; using this fact, it can be seen that $A[\hat{u}]$ as well as $\langle A[\hat{u}],\bar{v}\rangle$ depend on $\hat{u}$ and $\hat{v}$ but are independent of the particular smooth extensions chosen. A direct computation shows that $\hat{P}:\hat{D} \to \hat{D}^*$ defined by

$$\langle \hat{P}\hat{u},\hat{v}\rangle = \langle (P + P_E)\hat{u},\hat{v}\rangle + 2\langle A[\hat{u}],\bar{v}\rangle \qquad [18]$$

is symmetric.

Let $f,f_E,f_J \in \hat{D}^*$ be in the ranges of P,P_E, and A_E, respectively, and define $\hat{f} = f + f_E + 2f_J$. Then it can be seen, by using Eq. **2**, that the system

$$P\hat{u} = f; \quad P_E\hat{u} = f_E; \quad A[\hat{u}] = f_J \qquad [19]$$

is equivalent to

$$\hat{P}\hat{u} = \hat{f}. \qquad [20]$$

The symmetry property of $\hat{P}$ implies the following variational principle.

THEOREM 5. *Define the functional*

$$X(\hat{u}) = \frac{1}{2}\langle \hat{P}\hat{u},\hat{u}\rangle - \langle \hat{f},\hat{u}\rangle \qquad [21]$$

for every $\hat{u} \in \hat{D}$. *Then* $\hat{u} \in \hat{D}$ *satisfies* **19** *if and only if*

$$X'(\hat{u}) = 0. \qquad [22]$$

Proof: This is a particular case of a general theorem given by Herrera (5) and Herrera and Bielak (6) for symmetric functional valued operators. *Theorem 5* generalizes results given by Prager (8), Nemat-Nasser (9), and others for discontinuous solutions.

Definition 4: Given $U_E \in D_E$; $f,f_E = PU_E,f_J \in \hat{D}^*$, the problem of connecting consists in finding $u,u' \in D$ with the property that $\exists \hat{u} = (u,u_E) \in \hat{D}$ which satisfies Eq. **19** and such that u' is smooth extension of u_E.

The following definition and lemma permit applying the variational principle of *Theorem 4* to the problem of connecting.

Definition 5: The set $\mathcal{J} \subset D$ of internally generated motions is defined by

$$\mathcal{J} = \{u \in D \mid \exists \hat{u} = (u,u_E) \in \mathcal{S} \ni P_E\hat{u} = 0\}. \qquad [23]$$

LEMMA 2. *The set* $\mathcal{J}$ *of internally generated motions given in* Definition 5 *is necessarily a connectivity condition. When the problem of diffraction is well posed, the connectivity condition is in addition complete.*

Proof: $\mathcal{J}$ is obviously a linear space. $\mathcal{N} \subset \mathcal{J}$, because $A[\hat{u}] = 0$ whenever $\hat{u} = (u,0)$ and $Au = 0$. Given $u,v \in \mathcal{J}$, choose $u_E,v_E \in D_E$ so that, as in *Definition 5*, $\hat{u},\hat{v} \in \mathcal{S}$; then

$$\langle Au,v\rangle = -\langle A_E u_E,v_E\rangle = \langle P_E\hat{u},\hat{v}\rangle - \langle P_E\hat{v},\hat{u}\rangle = 0. \qquad [24]$$

Therefore, $\mathcal{J}$ is a connectivity condition and, by *Theorem 3*, it is complete when the diffraction problem is well posed.

In view of this Lemma, the variational principle of *Theorem 4* is applicable to the problem of connecting.

THEOREM 6. *Consider the problem of connecting given in Definition 4, and let* U′ *be any smooth extension of* U_E. *Assume the problem of diffraction is well posed; then* u,u′ $\in$ D *is a solution of the problem of connecting if and only if* **14** *is satisfied.*

Proof: Proof is immediate becasue $u' - U' \in \mathcal{J} \Rightarrow \exists u_E \in D_E$ smooth extension of $u' \ni Pu_E = PU_E = f_E$. We remark that well-posedness of the problem of connecting implies that of the problem of diffraction, because the latter is a particular case of the former.

Applications

A. Elasticity and Potential Theory. Let us consider again the applications to elasticity and potential theory introduced in ref. 2. The elements of D will be functions $\underset{\sim}{u} = (u_1, \ldots ,u_m)$, while those of D_E will be $\underset{\sim}{u}_E = (u_{E1}, \ldots ,u_{Em})$ defined on regions R and E of Fig. 1, which are now assumed to be in the n-dimensional Euclidean space. The boundaries of these regions are $\partial_1 R \cup \partial_2 R \cup \partial_3 R$ and $\partial_1 E \cup \partial_2 E \cup \partial_3 E$, respectively, in which $\partial_i R$ as well as $\partial_i E (i = 1,2,3)$ are pairwise disjoint sets. The unit normal vector n will be taken pointing out of $R \cup E$; on $\partial_3 R = \partial_3 E$ it points outwards from R. Let $P:\hat{D} \to \hat{D}^*$ be

$$\langle P\hat{\underset{\sim}{u}},\hat{\underset{\sim}{v}}\rangle = \int_R v_\alpha \mathcal{L}_\alpha(\underset{\sim}{u})d\underset{\sim}{x} + \int_{\partial_1 R} u_\alpha T_\alpha(\underset{\sim}{v})dx - \int_{\partial_2 R} v_\alpha T_\alpha(\underset{\sim}{u})d\underset{\sim}{x} \qquad [25]$$

in which $\mathcal{L}$ and T are differential operators

$$\mathcal{L}_\alpha(\underset{\sim}{u}) = \frac{\partial}{\partial x_j}\left(C_{\alpha j\beta q}\frac{\partial u_\beta}{\partial x_q}\right) + \rho u_\alpha; \quad T_\alpha(\underset{\sim}{u}) = C_{\alpha j\beta q}\frac{\partial u_\beta}{\partial x_q}n_j \qquad [26]$$

Latin subscripts run from 1 to n, Greek ones from 1 to m, and summation over the range of repeated subscripts is understood. When R is bounded, integrating by parts leads to

$$\langle A\hat{\underset{\sim}{u}},\hat{\underset{\sim}{v}}\rangle = \int_{\partial_3 R} \{v_\alpha T_\alpha(\underset{\sim}{u}) - u_\alpha T_\alpha(\underset{\sim}{v})\}d\underset{\sim}{x}. \qquad [27]$$

The definition of $P_E:\hat{D} \to \hat{D}^*$ is obtained by replacing u and $\underset{\sim}{v}$ with $\underset{\sim}{u}_E$ and $\underset{\sim}{v}_E$ in Eq. **25**, while R is replaced with E. When E is bounded

$$\langle A_E\hat{\underset{\sim}{u}},\hat{\underset{\sim}{v}}\rangle = -\int_{\partial_3 E} \{v_{E\alpha} T_\alpha(\underset{\sim}{u}_E) - u_{E\alpha} T_\alpha(\underset{\sim}{v}_E)\}d\underset{\sim}{x}. \qquad [28]$$

The set of smooth functions $\mathcal{S} \subset \hat{D}$ is defined by the condition that $\hat{u} = (u,u_E) \in \mathcal{S} \Leftrightarrow$

$$\underset{\sim}{u} = \underset{\sim}{u}_E \text{ and } \underset{\sim}{T}(\underset{\sim}{u}) = \underset{\sim}{T}(\underset{\sim}{u}_E) \text{ on } \partial_3 R = \partial_3 E. \qquad [29]$$

It is assumed that $A\underset{\sim}{u} = 0 \Leftrightarrow \underset{\sim}{u} = \underset{\sim}{T}(\underset{\sim}{u}) = 0$. As mentioned elsewhere (2), in applications to potential theory $\underset{\sim}{T}(\underset{\sim}{u})$ is $\partial u/\partial n$ and the above condition is fulfilled. In applications to elasticity, it is enough that C_{ijpq} be strongly elliptic (10). When this happens, $\hat{u} \in \mathcal{S} \Leftrightarrow$ Eq. **16**.

Given the functions $b,b_E,d,d_E,d_J,\sigma,\sigma_E$, and σ_J on $R,E,\partial_1 R,\partial_1 E,\partial_3 R,\partial_2 R,\partial_2 E$, and $\partial_3 R$, respectively, define $f \in \hat{D}^*$ by

$$\langle f,\hat{\underset{\sim}{v}}\rangle = \int_R v_\alpha b_\alpha d\underset{\sim}{x} + \int_{\partial_1 R} d_\alpha T_\alpha(\underset{\sim}{v})d\underset{\sim}{x} - \int_{\partial_2 R} v_\alpha \sigma_\alpha d\underset{\sim}{x} \qquad [30]$$

for every $\hat{\underset{\sim}{v}} \in \hat{D}$, and $f_E \in \hat{D}^*$ in a corresponding manner. Let, in addition, $f_J \in \hat{D}^*$ be such that for every $\hat{v} \in \hat{D}$:

$$\langle \underset{\sim}{f}_J,\hat{\underset{\sim}{v}}\rangle = \int_{\partial_3 E} \{\sigma_{J\alpha} v_\alpha - d_{J\alpha} T_\alpha(\underset{\sim}{v})\}d\underset{\sim}{x}. \qquad [31]$$

Then, the system of equations

$$\underset{\sim}{\mathcal{L}}(\underset{\sim}{u}) = \underset{\sim}{b}; \quad \underset{\sim}{u} = \underset{\sim}{d}; \quad \underset{\sim}{T}(\underset{\sim}{u}) = \underset{\sim}{\sigma} \qquad [32a]$$

$$\underset{\sim}{\mathcal{L}}(\underset{\sim}{u}_E) = \underset{\sim}{b}_E; \quad \underset{\sim}{u}_E = \underset{\sim}{d}_E; \quad \underset{\sim}{T}(\underset{\sim}{u}_E) = \underset{\sim}{\sigma}_E \qquad [32b]$$

$$\underset{\sim}{u}_E - \underset{\sim}{u} = \underset{\sim}{d}_J \text{ and } \underset{\sim}{T}(\underset{\sim}{u}_E) - \underset{\sim}{T}(\underset{\sim}{u}) = \underset{\sim}{\sigma}_J \qquad [32c]$$

is equivalent to Eq. **19** or **20** when $\hat{P}$ is defined by Eq. **18** and $\hat{f} = f + f_E + 2f_J$.

The general problem of connecting solutions defined on E with those defined in region R is given in *Definition 4*. For the class of differential Eq. **32**, the problem is: Given U_E solution of the differential equation and boundary conditions **32b**, find $\underset{\sim}{u}$ satisfying **32a** and such that there exists a solution u_E of **32b** that fulfills the jump conditions **32c**. According to *Theorem 6*, when the problem of diffraction on R is well posed, that of connecting can be formulated variationally as in *Theorem 4;* furthermore, in that case, by *Lemma 2* the set of solutions of **32b** with vanishing data define (*Definition 5*) a complete connectivity condition, and all the variational principles given in ref. 2 are applicable. As observed in ref. 2, the theory developed in this section includes potential theory and the reduced wave equation, which are obtained by choosing $m = 1$ and $C_{1j1q} = \delta_{jq}$. The equations of linear elasticity correspond to the case $m = n = 3$. Conditions **32c** correspond to prescribed jumps of the function and its normal derivative on $\partial_3 R$ or, alternatively, of the displacements and tractions, in applications to elasticity. When d_J and σ_J vanish identically, Eqs. **32c** imply corresponding conditions of continuity.

Regions R and E illustrated in Fig. 1 are bounded; their boundedness is only required to carry out the integration by parts leading to Eqs. **27** and **28**. These same equations can also be derived when one or both of the regions R and E are unbounded if we assume instead that the functions of D and D_E satisfy a radiation condition such as those introduced in ref. 11.

B. Boundary Integral Equations. There are many ways in which the set $\mathcal{J}$ of internally generated motions of *Definition 5* can be specified. One that has found favor uses boundary integral equations (4). The present results and those in ref. 2 are applicable to these equations whenever they are derived from formally symmetric operators satisfying the conditions of Eq. **2.** For applications to potential theory, the reduced wave equation, and elasticity, discussed in section A, they take the form

$$u_\alpha(\underset{\sim}{x}) = 2 \int_{\partial_3 R} \{G_{\gamma\alpha}(\underset{\sim}{\xi},\underset{\sim}{x}) T_\gamma(\underset{\sim}{u}) - u_\gamma(\underset{\sim}{\xi}) T_\gamma(\underset{\sim}{G}_\alpha(\underset{\sim}{\xi},\underset{\sim}{x}))\} d\underset{\sim}{\xi}; \underset{\sim}{x} \in \partial_3 R \qquad [33]$$

in which $G_{\alpha\beta}(\underset{\sim}{x},y) = G_{\beta\alpha}(y,\underset{\sim}{x})$ is Green's functions of region E satisfying homogeneous conditions on $\partial_1 E$ and $\partial_2 E$. In this case, the set $\mathcal{J}$ in *Definition 5* is characterized by the fact that functions belonging to $\mathcal{J}$ satisfy Eq. **33.**

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