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On the variational principles of mechanics

Abstract

Recently, the author has developed a general formulation of variational principles using functional-valued operators which simplifies the treatment of problems of continuum mechanics and its partial differential equations because it is applicable in any linear space, not necessarily normed, or with an inner product, or complete. This method has been applied to obtain dual extremum principles for initial-value problems and, more generally, for non-negative asymmetric operators. More recently, it has been used to obtain general variational principles applicable to diffraction problems and to formulate a general theory of connectivity for formally symmetric operators, which is basic in the formulation of the finite element method. Here, a systematic presentation of this method and these results is given.

1 Introduction

The numerical methods of mechanics and, more generally, of mathematical physics make extensive use of variational formulations. The modern approach to such methods is based on functional analysis. However, in many applications to mechanics, functional analysis is not used systematically, in spite of the fact that it permits the achievement of greater generality, rigour and clarity; to a large extent this is due to the fact that the applicability of functional analysis is frequently hindered by the complicated structures which are assumed in many of its theories.

The complexity of structures limits, in at least two ways, the usefulness of functional analysis:

(a) it makes it difficult (or impossible) to treat complicated situations;
(b) it diminishes the number of people able to apply it efficiently.

This paper is devoted to a presentation of a general framework which simplifies the formulation of variational principles of mechanics and, more generally, of mathematical physics [1]. It is based on the systematic use of functional-valued operators. Advantages of this approach [1–3] are as follows:

1. Problems are formulated in the most general kind of linear spaces, which are not necessarily normed, or with an inner product, or complete. Most work in this field has been done either in inner product spaces [4] or in Hilbert spaces [5, 6] and it is generally thought that this is desirable, if not essential, for the results to hold. In many applications, the introduction of the Hilbert-space structure leads to unwarranted complications which do not occur when functional-valued operators are used.
2. The removal of superfluous hypotheses in the development of a theory is always convenient, because it enlarges its applicability.
3. The symmetry condition for the potentialness of an operator can be extended to linear spaces for which no inner product or norm have to be defined [1]. This fact makes it possible to formulate a theory which is rigorous and at the same time not complicated.
4. Error bounds for approximate solutions are among the most important results that the theory yields [7]. They depend, however, on simple properties which are independent of any Hilbert space structure, and, therefore, can be obtained in the simple setting that the author has developed [3].
5. The generality achieved is greater. This fact has been especially useful in the development of variational principles for diffraction problems and the theory of connectivity to be presented in this paper.

This method has been used to obtain variational [8] and truly extremum principles [1–3] for initial-value problems,† and, more generally, for non-negative asymmetric operators [3]. A procedure which permits extending variational principles, derived by the mirror method into truly extremal principles, has also been constructed [3]. The notion of formally symmetric operators, usually applicable to differential operators only, is extended to functional-valued operators; for such operators, general variational principles have been developed for diffraction problems [9] and for a theory of connectivity [10] basic to the formulation of the finite-element method.

The description of the framework and of the main developments is brief; therefore, many details have been left out. This has required that, in some cases, the results are not presented in their greatest generality. Thus, for example, diffraction and related problems with prescribed

† These results were obtained in 1974 [1, 2]. Later, in a less general setting, Brezis and Ekeland [6], obtained related results.

jumps are not considered, although they have been treated in this more general form in the corresponding references [10]. Similarly, problems defined in affine subspaces are not considered here.

2 Preliminary notions and notations

All linear spaces to be considered will be defined on the field of real or complex numbers $\mathcal{F}$. The outer sum of two such spaces D_1 and D_2 will be represented by $D_1 \oplus D_2$. On the other hand, if a linear space D is spanned by two linearly independent subspaces D_1 and D_2, the space D is isomorphic to $D_1 \oplus D_2$ and we will write $D = D_1 \oplus D_2$; the subspaces D_1, D_2 are called a decomposition of D. In this case, given any $u \in D$, there is a unique pair of elements (u_1, u_2) such that $u_i \in D_i$ $(i = 1, 2)$ and $u = u_1 + u_2$; the elements of this pair are called projections of u on D_1 and D_2, respectively. The notation D^n will be used for the outer sum $D_1 \oplus \cdots \oplus D_n$ when $D_i = D$ $(i = 1, \ldots, n)$.

The value of an n-linear† functional $\alpha : D^n \to \mathcal{F}$ at an element $u = (u_1, \ldots, u_n) \in D^n$ will be represented by $\langle \alpha, u_1, \ldots, u_n \rangle$.‡ The notation D^{n*} will be used for the linear space of all n-linear functionals. Alternatively, D^{1*} will be written as D^*; D^{0*} is defined as $\mathcal{F}$. Notice that $D^{n*} \neq (D^n)^*$.

Functional-valued operators $P : D \to D^{n*}$ are considered in this work. Special attention will be given to the case $n = 1$, i.e., to operators of the form $P : D \to D^*$. When P is linear, its adjoint $P^* : D \to D^*$ is defined by $\langle P^* u, v \rangle = \langle Pv, u \rangle$, which holds for every $u, v \in D$; thus, the adjoint of such linear operators always exists.

For operators of the type $P : D \to D^{n*}$, the notion of continuity can be introduced without a topology in D. The operator $P : D \to D^{n*}$ is said to be bidimensionally continuous at $u \in D$ if, for every v, w, $\xi^{(1)}, \ldots, \xi^{(n)} \in D$, the function $f(\eta, \lambda) = \langle P(u + \eta v + \lambda w), \xi^{(1)}, \ldots, \xi^{(n)} \rangle$ is continuous at $\eta = \lambda = 0$.

The concept of a derivative of an operator will be used in the sense of additive Gateaux variation [11]. More precisely, an element $P'(u) \in D^{(n+1)*}$ will be called the derivative of P at $u \in D$, if, for every $v, \xi^{(1)}, \ldots, \xi^{(n)} \in D$,

$$g'(0) = \langle P'(u), v, \xi^{(1)}, \ldots, \xi^{(n)} \rangle \tag{1}$$

whenever the function $g(t) = \langle P(u + tv), \xi^{(1)}, \ldots, \xi^{(n)} \rangle$. Partial derivatives $P,_1(u)$, $P,_2(u) \in D^{(n+1)*}$ will be considered when $D = D_1 \oplus D_2$. Using the

† Russian authors usually include continuity in the definition of linearity [12]. On the other hand, for most American authors, this concept includes only additivity and homogeneity. In this paper, we follow the latter terminology.

‡ This notation does not imply the existence of an inner product.

unique representation (v_1, v_2), $v_i \in D_i$ $(i=1,2)$, of every $v \in D$, they are defined by

$$\langle P_{,i}(u), v, \xi^{(1)}, \ldots, \xi^{(n)} \rangle = \langle P'(u), v_i, \xi^{(1)}, \ldots, \xi^{(n)} \rangle \qquad (i=1,2), \tag{2}$$

which holds for every $\xi^{(1)}, \ldots, \xi^{(n)} \in D$.

An operator $P: D \to D^*$ is said to be potential if there exists a functional $\psi: D \to \mathscr{F} = D^{0*}$ such that $\psi'(u) = P(u)$ for every $u \in D$. It is well known [12] that a sufficient condition for potentiality is that $P'(u)$ be symmetric for every $u \in D$. This result remains valid for the class of operators $P: D \to D^*$ considered here, if $P'(u)$ is assumed to be bidimensionally continuous at each $u \in D$, as has been shown [1] in a manner related to that suggested by Vainberg [12]. For a linear operator P, this requirement reduces to the condition that P be symmetric. Such P will be said to be non-negative if $\langle Pu, u \rangle \geq 0$ for every $u \in D$ and positive if, in addition, $\langle Pu, u \rangle = 0$ only when $u=0$. Non-positive and negative operators are defined similarly.

The general problem to be considered consists in finding solutions to an equation

$$P(u) = f, \tag{3}$$

where P is a functional-valued operator $P: D \to D^*$ and $f \in D^*$. The solutions u will be restricted to be in a subset $\hat{E} \subset D$. In applications [2, 3], the elements belonging to $\hat{E}$ satisfy some additional boundary or initial conditions and frequently constitute an affine subspace; i.e., there is a subspace $E \subset D$ and an element $w \in D$, $\hat{E} = w + E$. The results to be presented can be easily extended to the case $\hat{E} \neq D$ [2, 3]; however, for simplicity, it will be assumed $\hat{E} = D$, which corresponds to taking $w \in E = D$.

Following Noble and Sewell [4], given any two elements $u_-, u_+ \in D$, define

$$\Delta X = X(u_+) - X(u_-), \tag{4a}$$

$$\Delta u = u_+ - u_-. \tag{4b}$$

If a decomposition D_1, D_2 of D is available, let $\Delta_i u \in D_i$ $(i=1,2)$ be the unique representation of Δu in terms of an element of D_1 plus an element of D_2.

When the functional X is differentiable, it is said to be convex if

$$\Delta X - \langle X'(u_-), \Delta u \rangle \geq 0 \tag{5}$$

or, equivalently,

$$\Delta X - \langle X'(u_+), \Delta u \rangle \leq 0 \tag{6}$$

for every $u_+, u_- \in D$. It is strictly convex if the strict inequality holds whenever $u_+ \neq u_-$. In addition, X is concave or strictly concave if $-X$ is convex or strictly convex, respectively.

Furthermore, X is saddle, convex on D_1 and concave on D_2, if

$$\Delta X - \langle X'(u_-), \Delta_1 u\rangle - \langle X'(u_+), \Delta_2 u\rangle \geqslant 0 \tag{7}$$

for every $u_+, u_- \in D$. It is strictly saddle if the inequality is strict whenever $u_+ \neq u_-$.

3 Variational principles

A variational principle is understood to be an assertion stating that the derivative $X'(u)$ of a functional X vanishes at a point $u \in D$ if and only if u is a solution of (3). A sufficient condition for the construction of variational principles is that the operator P be potential, because if $\psi : D \rightarrow \mathscr{F}$ is a potential of P, then the functional $X(u) = \psi(u) - \langle f, u\rangle$ will have the required property.

Theorem 1 *Assume $P : D \rightarrow D^*$ to be potential. Then $u \in D$ is a solution of* (3) *if and only if $X'(u) = 0$. Here, $X : D \rightarrow \mathscr{F}$ is defined as above.*

The theory of dual extremal principles for non-linear functional-valued operators has been developed in complete generality elsewhere [1]. Here, a summary of this theory for the linear case is presented. Therefore, in what follows, the linearity of the operator $P : D \rightarrow D^*$ will be assumed. In addition, the field $\mathscr{F}$ is taken as R^1.

Given a decomposition $D_1 \subset D$, $D_2 \subset D$ of D, every element $u \in D$ can be written as $u = u_1 + u_2$, where u_1, u_2 are the projections of u on D_1 and D_2, respectively. Projections of linear functionals and of functional-valued operators which are linear can be defined similarly. Given $f \in D^*$ and $P : D \rightarrow D^*$, the projections $f_1 \in D^*$ and $f_2 \in D^*$ are such that

$$\langle f_1, u\rangle = \langle f, u_1\rangle; \qquad \langle f_2, u\rangle = \langle f, u_2\rangle \tag{8}$$

for every $u \in D$, while the projections $P_1 : D \rightarrow D^*$ and $P_2 : D \rightarrow D^*$ satisfy the conditions

$$P_1 u = (Pu)_1, \qquad P_2 u = (Pu)_2. \tag{9}$$

From these definitions, it follows that

$$f = f_1 + f_2, \tag{10}$$

$$P = P_1 + P_2. \tag{11}$$

When P is non-negative, the non-negative square root of $\langle Pu, u\rangle$ will be denoted by $\|u\|_P$ whenever $u \in E$. The set

$$N_P = \{u \in D \mid \|u\|_P = 0\} \tag{12}$$

may contain non-zero vectors. However, N_P is always a linear subspace of D [3].

Definition *Let $P: D \to D^*$ be non-negative. Then $u, v \in D$ are said to be* P-equivalent *if and only if $u - v \in N_P$. In this case, one writes*

$$u \approx_P v. \tag{13}$$

This is an equivalence relation because N_P is a linear subspace. For non-positive and negative operators there are definitions and results corresponding to those already given for non-negative and positive operators. It is worth noticing that the relation $\approx_P$ becomes equality when P is either positive or negative.

Definition *Let D_1, D_2 be a decomposition of D and $P: D \to D^*$ be a self-adjoint operator. Then, P is said to be* saddle with respect to D_1, D_2 *if P is non-negative on D_1 and non-positive on D_2. It is* strictly saddle *if P is positive on D_1 and negative on D_2.*

It must be observed that, in this definition, the subspaces can not be interchanged. However, this manner of introducing them simplifies many propositions of the theory. For saddle operators, the bilinear functional $\langle Pu_1, v_1\rangle - \langle Pu_2, v_2\rangle$ is symmetric and possesses a non-negative quadratic form. The non-negative square root of $\langle Pu_1, u_1\rangle - \langle Pu_2, u_2\rangle$ will be denoted by $\|u\|_P$; when P is strictly saddle, $\|u\|_P$ is positive and therefore $\|u\|_P$ is a norm. In this case, the bilinear functional $\langle Pu_1, v_1\rangle - \langle Pu_2, v_2\rangle$ is an inner product.

Definition *When $P: D \to D^*$ is saddle, any pair $u, v \in D$ are said to be* P-equivalent *(i.e., $u \approx_P v$) if and only if $\|u - v\|_P = 0$.*

Again the set (12) is a linear subspace and $\approx_P$ is an equivalence relation.

Let

$$D_I = \{u \in D \mid P_1 u = 0\}, \tag{14a}$$

$$D_{II} = \{u \in D \mid P_2 u = 0\}. \tag{14b}$$

Then, if $P: D \to D^*$ is saddle with respect to the decomposition D_1, D_2, the operator P is non-positive on D_I and non-negative on D_{II}. Furthermore [3],

$$\langle Pu, u\rangle = -\|u\|_P^2 \quad \text{when} \quad u \in D_I, \tag{15a}$$

$$\langle Pu, u\rangle = \|u\|_P \quad \text{when} \quad u \in D_{II} \tag{15b}$$

The dual extremum principles are closely related to Eqns (15). In view of Theorem 1, for symmetric operators, Eqn (3) is equivalent to

$$X,_1(u) = P_1 u - f_1 = 0, \tag{16a}$$

$$X,_2(u) = P_2 u - f_2 = 0. \tag{16b}$$

Theorem 2 *Let $P: D \to D^*$ be saddle with respect to the decomposition D_1, D_2 of D. Define the affine subspaces*

$$\hat{\mathscr{E}}_a = \{u \in D \mid (16a) \text{ holds}\}, \tag{17a}$$

$$\hat{\mathscr{E}}_b = \{u \in D \mid (16b) \text{ holds}\}; \tag{17b}$$

then, for every $u_a \in \hat{\mathscr{E}}_a$ and $u_b \in \hat{\mathscr{E}}_b$,

$$2[X(u_b) - X(u_a)] = \|u_b - u_a\|_P^2 \geqslant 0. \tag{18}$$

When a solution $u_0 \in D$ of (3) *exists,*

$$\|u_b - u_a\|^2 = \|u_b - u_0\|^2 + \|u_0 - u_a\|^2. \tag{19}$$

Proof The proof of (18), when a solution $u_0 \in D$ exists, follows immediately from Eqns (15), because

$$u_b - u_a = (u_b - u_0) + (u_0 - u_a), \tag{20}$$

$u_b - u_0 \in D_{II}$ and $u_0 - u_a \in D_I$. When the existence of u_0 is not assumed, the proof is slightly more complicated and is given in [3].

4 The mirror method

Let $P: D \to D^*$ be a non-negative, but not necessarily symmetric, operator. The mirror method consists in embedding the equation $Pu = f$ into the system

$$Pu = f \quad \text{and} \quad P^*u' = g, \tag{21}$$

where $u, u' \in D$ and $f, g \in D^*$. This system of equations may be written in terms of a symmetric operator $\hat{P}: \hat{D} \to \hat{D}^*$, where $\hat{D} = D \oplus D$ is the outer sum of D with itself; thus, the elements $\hat{u} = (u, u') \in \hat{D}$ are ordered pairs of members $u, u' \in D$. To express (21) in the form $\hat{P}\hat{u} = \hat{f}$ with $\hat{P}$ symmetric, it is enough to define $\hat{P}: \hat{D} \to \hat{D}^*$ and $\hat{f} \in \hat{D}^*$ by

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \langle Pu, v' \rangle + \langle Pv, u' \rangle \tag{22}$$

and

$$\langle \hat{f}, \hat{v} \rangle = \langle f, v' \rangle + \langle g, v \rangle.$$

Due to this fact, it is possible to formulate variational principles for the system (21). In this case, the functional $X: D \to \mathscr{F}$ of Theorem 1 is

$$X(\hat{u}) = X(u, u') = \langle Pu, u' \rangle - \langle f, u' \rangle - \langle g, u \rangle. \tag{24}$$

In general, this functional is not saddle. However, when P is non-negative, it is easy to construct decompositions $\hat{D}_1$, $\hat{D}_2$ of $\hat{D}$ for which $\hat{P}: \hat{D} \to \hat{D}^*$ is saddle. Indeed, let $\alpha, \beta, \gamma, \delta$ be real numbers such that

$\alpha\delta-\beta\gamma\neq 0$, $\alpha\beta<0$ and $\gamma\delta>0$. Then, $\hat{P}$ is saddle with respect to $\hat{D}_1, \hat{D}_2$ if

$$\hat{D}_1=\{\hat{u}\in\hat{D} \mid \alpha u+\beta u'=0\}, \tag{25a}$$

$$\hat{D}_2=\{\hat{u}\in\hat{D} \mid \gamma u+\delta u'=0\} \tag{25b}$$

and Theorem 2 is applicable.

5 Variational formulations of diffraction problems

Consider a linear operator $P: D\rightarrow D^*$. Given such operator, let

$$\mathcal{N}=\{u\in D \mid Au=0\}$$

be the null subspace of the antisymmetric part $A=(P-P^*)/2$ of P, which is well defined because the adjoint P^* always exists. The operator P is said to be formally symmetric if

$$\langle Pu, v\rangle=0 \qquad \forall v\in\mathcal{N} \Rightarrow Pu=0. \tag{27}$$

It is worth noticing that any functional-valued operator $P: D\rightarrow D^*$ satisfies the proposition that is obtained by replacing $\mathcal{N}$ with D in (27). Attention will be restricted in this section to formally symmetric operators.

Definition *A linear subspace $\mathcal{I}\subset D$ is said to be a connectivity condition if*

$$(a)\ \mathcal{N}\subset\mathcal{I}, \tag{29}$$

$$(b)\ \langle Au, v\rangle=0, \qquad \forall u, v\in\mathcal{I}. \tag{30}$$

The connectivity condition is said to be complete *if, in addition,*

(c) for any $w\in D$,

$$\langle Aw, v\rangle=0, \quad \forall v\in\mathcal{I} \Rightarrow w\in\mathcal{I}. \tag{31}$$

The set $\mathcal{E}\subset D$ is defined by

$$\mathcal{E}=\{u\in D \mid Pu=0\}. \tag{32}$$

The relation

$$\{Au, v\rangle=0, \qquad \forall u, v\in\mathcal{E} \tag{33}$$

is clear, because $2\langle Au, v\rangle=\langle Pu, v\rangle-\langle Pv, u\rangle$. In what follows of this section, it will be assumed that there is in D a connectivity condition $\mathcal{I}$; however, the completeness of $\mathcal{I}$ will not be taken for granted.

The problem of diffraction associated with P will now be presented.

Definition *Given $U \in D$ and $V \in D^*$, a motion $u \in D$ is said to be* a solution of the problem of diffraction *if*

$$Pu = f \quad \text{and} \quad V - u = v \in \mathscr{I}, \tag{34}$$

where $f = PU$.

There are several alternative variational formulations of this diffraction problem [9, 10]. Here, only the two which are especially relevant for numerical applications are quoted.

Theorem 3 *Define the functional*

$$X(u, u') = \tfrac{1}{2}\langle Pu, u\rangle + \langle A[u], \bar{u}\rangle + \langle AV, u'\rangle - \langle f, u\rangle \qquad ($$

for every couple $u, u' \in D$. Here,

$$[u] = u' - u, \qquad \bar{u} = \tfrac{1}{2}(u' + u). \tag{36}$$

Then, when $\mathscr{I}$ is complete or, alternatively, when $u' - V \in \mathscr{I}$,

$$\langle X'(u, u'), (v, v')\rangle = 0 \qquad \forall (v, v'), \qquad v \in D \quad \text{and} \quad v' \in \mathscr{I} \qquad ($$

if and only if u is a solution of the diffraction problem and $u' - u \in \mathcal{N}$.

Proof This theorem contains a slight modification of results proved in [9, 10].

6 Characterization of complete connectivity conditions

The main result to be presented in this section is that when the diffraction problem associated with a connectivity condition $\mathscr{I} \subset D$, is well posed, then $\mathscr{I}$ is complete.

For every symmetric operator $S : D \to D^*$, the operator $B = S + A/2$ satisfies

$$A = B - B^*.$$

It will be assumed that operators $B : D \to D^*$ (to be considered in what follows) satisfy (38). When such B is available, two problems are defined.

Definition *Given $U \in D$, the motion $u \in D$ is said to be* a solution of the external boundary-value problem (e.b.v.p.) *if $Bu = BU$ and $u \in \mathscr{I}$. The* internal boundary-value problem (i.b.v.p.) *is defined by replacing $\mathscr{I}$ with $\mathscr{C}$.*

Any of the three problems already introduced are said to satisfy existence if they possess at least one solution for every admissible data; uniqueness when the only solution of the problem with vanishing data is the zero solution; almost uniqueness when any solution of the latter problem belongs to the null set $\mathcal{N}$. For some purposes, it is too restrictive to require uniqueness; owing to this fact, a problem will be said to be well posed if it satisfies existence and almost uniqueness.

It can be proved easily [10], that *the problem of diffraction is well posed if and only if every $u \in D$ can be written as $u = u_1 + u_2$ with $u_1 \in \mathcal{I}$, $u_2 \in \mathcal{E}$ and this representation is almost unique.* In addition, *if for some B satisfying* (38) *the e.b.v.p. satisfies existence, then $\mathcal{I}$ is a complete connectivity condition* [10]. Finally, *if the diffraction problem is well posed, then $\exists B : D \to D^*$, which satisfies* (38), *and the e.b.v.p. satisfies existence.* This latter assertion can be proved defining $Bu = Au_1$, because such B fulfils the required conditions. The above results imply the main property advanced in the introduction of this section.

Theorem 4 *Let $\mathcal{I} \subset D$ be a connectivity condition. If the diffraction problem associated with $\mathcal{I}$ is well posed, then $\mathcal{I}$ is a complete connectivity condition.*

7 Theory of connectivity

In this section, the relation between the problem of diffraction and the problem of connecting is discussed. This latter is basic to the formulation of the finite-element method. It is shown that these two problems are closely connected; furthermore, it is shown that the problem of connecting associated with formally symmetric operators leads to complete connectivity conditions whenever it is well posed.

The problem of connecting consists in constructing solutions in a region such as $R \cup E$ of Fig. 1, by connecting those corresponding to individual subregions such as R and E. The theory to be presented is an abstract

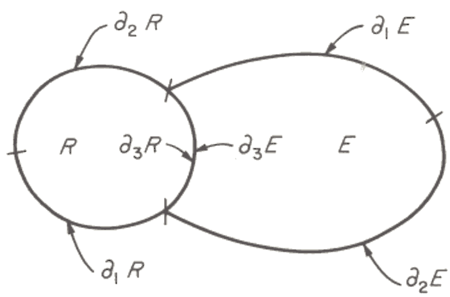

Fig. 1. Regions considered in the application to potential theory and elasticity.

one and can be applied in more general situations as long as the postulates are satisfied.

Let $\hat{P}:\hat{D}\to\hat{D}^*$ be a formally symmetric operator and D, D_E a decomposition of $\hat{D}$, so that every $\hat{u}\in\hat{D}$ can be written in a unique manner as $\hat{u}=u+u_E$, with $u\in D$ and $u_E\in D_E$. It will be assumed that this decomposition is such that

$$\langle\hat{P}\hat{u},\hat{v}\rangle=\langle\hat{P}u,v\rangle+\langle\hat{P}u_E,v_E\rangle \tag{39}$$

for every $\hat{u},\hat{v}\in\hat{D}$. This permits the definition of $P:\hat{D}\to\hat{D}^*$ and $P_E:\hat{D}\to\hat{D}^*$ by the equations

$$\langle P\hat{u},\hat{v}\rangle=\langle\hat{P}u,v\rangle,\qquad\langle P_E\hat{u},\hat{v}\rangle=\langle\hat{P}u_E,v_E\rangle. \tag{40}$$

They satisfy

$$\hat{P}=P+P_E. \tag{41}$$

At the same time, due to (40), they can be thought of as $P:D\to D^*$ and $P_E:D_E\to D_E^*$.

Definition *Assume $\mathcal{S}\subset\hat{D}$ has the following properties:*

(*a*) *$\mathcal{S}$ is a complete connectivity condition for $\hat{P}$;*
(*b*) *for every $u\in D$, $\exists u_E\in D_E\ni u+u_E\in\mathcal{S}$;*
(*c*) *for every $u_E\in D_E$, $\exists u\in D\ni u+u_E\in\mathcal{S}$.*

Then $\mathcal{S}$ will be said to be a smoothness relation *or* condition.

Elements $u\in D$ and $u_E\in D_E$ are said to be smooth extensions of each other when $u+u_E\in\mathcal{S}$. In this section, it is assumed that a smoothness relation $\mathcal{S}$ is given.

Definition *Let $U\in D$ and $U_E\in D_E$ be given. Then $u\in D$ is said to be a* solution of the problem of connecting,† *when*

$$Pu=PU_E \tag{42}$$

and $\hat{u}=u+u_E\in\mathcal{S}$ for some $u_E\in D_E\ni$

$$P_Eu_E=P_EU_E. \tag{43}$$

Formulations of more general problems of connecting which include prescribed jumps have been given [10], as well as associated variational principles.

† Probably, it would be better to call this *problem of connecting in the restricted sense*, because preference is given to the subspace D, while, in a more general treatment, D and D_E play a symmetric role.

To reduce the problem of connecting to a problem of diffraction, it is enough to define the set $\mathcal{I} \subset D$ by

$$\mathcal{I} = \{u \in D \mid \exists \hat{u} = u + u_E \in S, P_E u_E = 0\}. \tag{44}$$

This is because *the set* $\mathcal{I}$ *given by Eqn* (44) *is necessarily a connectivity condition for* $P : D \to D^*$, *which is, in addition, complete when the diffraction problem is well posed,* as has been shown in [10].

8 Applications

Dual variational principles for the heat and wave equations have been given in [1–3] and can be easily extended to elastodynamics and other fields. Here, only applications of the general diffraction problem and of the problem of connecting will be considered.

For applications to elasticity and potential theory, consider functions $u = (u_1, \ldots, u_m)$ defined in the region $R \cup E$ of the n-dimensional Euclidean space (Fig. 1), with boundary $\partial_1 R \cup \partial_2 R \cup \partial_1 E \cup \partial_2 E$; the common boundary between R and E is $\partial_3 R = \partial_3 E$. The unit normal vector $\mathbf{n}$ is taken pointing outwards, from $R \cup E$ on its boundary, and from E on $\partial_3 R = \partial_3 E$. The subspaces D and D_E can be taken as the set of functions which are C^2 on R and on E, respectively. The space $\hat{D}$ is made by couples of such functions. Define $\hat{P} : \hat{D} \to \hat{D}^*$ by

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \int_{R \cup E} v_\alpha \mathcal{L}_\alpha(\mathbf{u})\, \mathrm{d}\mathbf{x} + \int_{\partial_1(R \cup E)} u_\alpha T_\alpha(\mathbf{v})\, \mathrm{d}\mathbf{x}$$

$$\int_{\partial_2(R \cup E)} v_\alpha T_\alpha(\mathbf{u})\, \mathrm{d}\mathbf{x}, \tag{45}$$

where $\partial_1(R \cup E) = \partial_1 R \cup \partial_1 E$, $\partial_2(R \cup E) = \partial_2 R \cup \partial_2 E$, while $\mathcal{L}$ and T are the differential operators:

$$\mathcal{L}_\alpha(\mathbf{u}) = \frac{\partial}{\partial x_j}\left(C_{\alpha j \beta q} \frac{\partial u_\beta}{\partial x_q}\right) + \rho u_\alpha,$$

$$T_\alpha(\mathbf{u}) = C_{\alpha j \beta q} \frac{\partial u_\beta}{\partial x_q} n_j. \tag{46b}$$

With these definitions, Eqns (39) are obviously satisfied. Here, as in what follows, Latin indices run from 1 to n, while Greek ones run from 1 to m. The coefficients ρ and $C_{\alpha j \beta q}$ are smooth functions on R and E separately, such that

$$C_{\alpha j \beta q} = C_{\beta q \alpha j}. \tag{47}$$

By integration by parts, it is seen that

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R} \{\bar{v}_\alpha[T_\alpha(\mathbf{u})] - [u_\alpha]\bar{T}_\alpha(\mathbf{v})\, d\mathbf{x} + \int_{\partial_3 R} \{\bar{T}_\alpha(\mathbf{u})[v_\alpha] - \bar{u}_\alpha[T_\alpha(\mathbf{v})]\}\, d\mathbf{x}, \tag{48}$$

where the square bracket stands for the jumps on $\partial_3 R$ of the function, while the bar over a symbol, e.g., $\bar{v}$, indicates its corresponding average. In view of this equation, a sufficient condition for $\hat{u} = u + u_E \in \hat{D}$ to be in the null set $\hat{\mathcal{N}}$ of $\hat{A}$ is that

$$u_\alpha = u_{E\alpha} = 0 \quad \text{and} \quad T_\alpha(\mathbf{u}) = T_\alpha(\mathbf{u}_E) = 0 \quad \text{on} \quad \partial_3 R. \tag{49}$$

From this fact, it can be seen that (27) is satisfied, and, therefore, $\hat{P}$ is formally symmetric.

It will be assumed in what follows that the coefficients $C_{\alpha j\beta q}$ are such that $\hat{u} = u + u_E \in \hat{\mathcal{N}}$ if and only if (49) are satisfied. When $m = 1$, $n = 3$ and $C_{1j1q} = \delta_{jq}$, $\mathcal{L}_1$ is Laplace's operator if $\rho \equiv 0$ and the reduced wave operator if $\rho \equiv 1$. If $m = n = 3$, $\mathcal{L}_i$ is the operator of static elasticity if $\rho = 0$ and the reduced operator of elastodynamics if $\rho \equiv 1$. In applications to potential theory, Eqn (49) is always satisfied, while the strong ellipticity of C_{ijpq} grants the same condition in applications to elasticity [9, 10].

Let the set $\mathcal{S} \subset \hat{D}$ be defined by the condition that $\hat{u} = u + u_E \in \mathcal{S}$ if and only if

$$[u_\alpha] = [T_\alpha(\mathbf{u})] = 0 \quad \text{on} \quad \partial_3 R. \tag{50}$$

Then, $\hat{\mathcal{N}} \subset \mathcal{S}$. In addition, by inspection of (48), it is seen that $\langle \hat{A}\hat{u}, \hat{v}\rangle$ vanishes whenever $\hat{u}, \hat{v} \in \mathcal{S}$; thus, $\mathcal{S}$ is a connectivity condition. Finally, it can be seen that this connectivity condition is always complete in applications to potential theory and the reduced wave equation, while, again, the same property is enjoyed in applications to elasticity when C_{ijpq} is strongly elliptic. This shows that $\mathcal{S}$ is a smoothness condition, because properties (b) and (c) are obvious, at least if $\partial_3 R$ is assumed to be sufficiently smooth.

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