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Connectivity as an alternative to boundary integral equations: Construction of bases

(numerical methods/finite element method/variational principles/partial differential equations/scattering problems)

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ABSTRACT In previous papers Herrera developed a theory of connectivity that is applicable to the problem of connecting solutions defined in different regions, which occurs when solving partial differential equations and many problems of mechanics. In this paper we explain how complete connectivity conditions can be used to replace boundary integral equations in many situations. We show that completeness is satisfied not only in steady-state problems such as potential, reduced wave equation and static and quasi-static elasticity, but also in time-dependent problems such as heat and wave equations and dynamical elasticity. A method to obtain bases of connectivity conditions, which are independent of the regions considered, is also presented.

When numerical methods are applied to mechanical problems, usually the general analytical solutions are known in subregions. In order to profit from this knowledge, in many situations, e.g., in diffraction problems, boundary element methods are applied; most frequently, they are formulated by using boundary integral equations (1). A general theory of connectivity has been developed recently (2, 3) which offers an alternative to integral equations and can be used with advantage in many problems. This theory considers a class of functions called a connectivity condition. If one uses a boundary integral equation to characterize this set, one is led to standard formulations; however, the theory of connectivity (2, 3) shows that such a set can also be characterized by the fact that a certain bilinear functional is symmetric when evaluated on its members. When a denumerable basis of the connectivity condition is available, it is enough to impose the denumerable set of conditions, corresponding to the requirement that this bilinear functional, when evaluated on the trial solution, vanishes for every member of the basis. The problem is simplified further because in very general situations it is possible to construct bases of connectivity conditions that are independent of the regions considered.

To illustrate the method, we apply it to both steady-state and time-dependent problems; more specifically, potential theory, wave and heat equations, and elasticity are considered. Bases of connectivity conditions for a region that is the exterior or, alternatively, the interior of a bounded region in two- or three-dimensional Euclidean space are constructed. They possess the remarkable property of remaining the same regardless of the specific region considered. This fact is very useful in numerical and other applications; e.g., in potential theory, when the region is bounded, these bases are harmonic polynomials.

In previous work, the theory was formulated for formally symmetric operators only (2, 3). Although this assumption is essential for the variational principles to hold, many results remain valid even if the operators are not formally symmetric. Application of the theory is simplified by the use of boundary operators that are not usually formally symmetric; therefore, a proof is supplied of the results required in this more general form. As in preceding papers (2, 3), systematic use is made of functional-valued operators defined on arbitrary linear spaces without any additional structure (i.e., not necessarily normed, or with an inner product, or complete).

Preliminary results

For operators (2–6) $B{:}D \rightarrow D^*$ defined on a linear space D and with values in the space D^* of all the linear functionals defined on D, let $\mathcal{N} \subset D$ be the null set of A; i.e.,

$$\mathcal{N} = \{u \in D \,|\, Au = 0\} \qquad [1]$$

where $A = B - B^*$ and B^* is the adjoint operator to B. Contrary to previous work (2, 3) no formally symmetric operators will be considered in this paper.

Definition 1. A linear subspace $\mathcal{I} \subset D$ is said to be a connectivity condition if

$$(i) \quad \mathcal{N} \subset \mathcal{I}, \qquad [2]$$

$$(ii) \quad \langle Au,v \rangle = 0 \quad \forall u,v \in \mathcal{I} \qquad [3]$$

The connectivity condition is said to be complete when,

$$(iii) \quad \text{For any } w \in D, \text{ one has}$$
$$\langle Aw,v \rangle = 0 \quad \forall v \in \mathcal{I} \Rightarrow w \in \mathcal{I} \qquad [4]$$

Definition 2. Let $B{:}D \rightarrow D^*$ be such that

$$A = B - B^*. \qquad [5]$$

Given $U \in D$, the element $u \in D$ is said to be a solution of the external boundary value problem (ebvp) if $Bu = BU$ and $u \in \mathcal{I}$. This problem is said to satisfy existence if it has at least one solution for every $U \in D$.

The main result to be used in what follows is given next.

THEOREM 1. *Let $\mathcal{I} \subset$ D be a connectivity condition. If, for some* B *fulfilling Eq.* 5, *the ebvp satisfies existence, then $\mathcal{I}$ is complete.*

Proof: Assume $U \in D$ is such that $\langle AU,v \rangle = 0, \forall v \in \mathcal{I}$, and take $u \in \mathcal{I}$ as a solution of the ebvp corresponding to U. Define $w = U - u$; clearly, $Bw = 0$. Given any $V \in D$, take $v \in \mathcal{I}$ as a solution of the ebvp corresponding to V. Then

$$\langle Aw,V \rangle = \langle Bw,V \rangle - \langle BV,w \rangle = -\langle Bv,w \rangle$$
$$= \langle Aw,v \rangle = \langle Au,v \rangle - \langle AU,v \rangle = 0.$$

Thus, $w \in \mathcal{N} \subset \mathcal{I}$ and $U = u + w \in \mathcal{I}$.

Definition 3. Let $\mathcal{I} \subset D$ be a complete connectivity condition for an operator $B{:}D \rightarrow D^*$. A subset $B \subset \mathcal{I}$ is said to be a

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Abbreviation: ebvp, external boundary value problem.

complete system for $\mathcal{I}$ if, for every $u \in D$, one has

$$\langle Au,w\rangle = 0 \quad \forall w \in \mathcal{B} \Rightarrow u \in \mathcal{I}. \qquad [6]$$

$\mathcal{B}$ is a basis of $\mathcal{I}$ if, in addition, every finite set of functionals $\{Aw_\alpha | w_\alpha \in \mathcal{B}, \alpha = 1, \ldots, N\}$ is linearly independent.

When $\mathcal{B} = \{w_\alpha \in \mathcal{I}, \alpha = 1,2, \ldots\}$ is a denumerable basis of $\mathcal{I}$, Eq. 6 can be replaced by

$$\langle Au,w_\alpha\rangle = 0 \quad \forall \alpha = 1,2, \ldots \Rightarrow u \in \mathcal{I}. \qquad [7]$$

Example of complete connectivity conditions

When diffraction problems, boundary element methods, and many other applications are considered, one looks for functions that satisfy a set of conditions, such as partial differential equations on a region R. These functions are further restricted by the condition that they can be extended smoothly into solutions on a neighboring region E of some other partial differential equations; such condition imposes restrictions on the values that are admissible on the common boundary $\partial_3 E$ between R and E. Such restrictions are frequently characterized by means of boundary integral equations (1).

An alternative, more flexible, procedure is supplied by the theory of connectivity (2, 3). If a denumerable basis of the connectivity condition is available, one can replace the boundary integral equation by the denumerable set of conditions 7. In this section the applicability of the method is illustrated by examples of complete connectivity conditions; these include potential theory, reduced wave, heat and wave equations, and elasticity (statical, reduced equation, and dynamical). The next section is devoted to obtaining denumerable bases of connectivity conditions for potential theory and the reduced wave equation; those bases have the remarkable property that they are the same for a wide class of regions E.

In applications to time-independent problems, E will be a region of the n-dimensional Euclidean space (Fig. 1) with boundary $\partial E = \partial_1 E \cup \partial_2 E \cup \partial_3 E$, where $\partial_\alpha E$ $(\alpha = 1,2,3)$ are mutually disjoint sets and $\partial_3 E = \partial_3 R$ is the part of the boundary of E that is shared with R. For applications to time-dependent problems (Fig. 2), a region that satisfies the same conditions as region E above will be called E_x, while the region E will be taken as $E_x \times [0,T]$. Therefore, the common part of E with R will be $\partial_3 E = \partial_3 E_x \times [0,T]$ as shown in Fig. 2.

For applications to potential theory, reduced wave, and heat and wave equations, D_E will be the linear space of functions u (real or, alternatively, complex valued) that are C^2 in E such that u and $\partial u/\partial n$ are continuous on $\partial_3 E$. The space D will be taken as the linear space of couples $\hat{u} = (u, \partial u/\partial n)$, where u and $\partial u/\partial n$ are continuous functions on $\partial_3 E$.

In applications to elasticity, the elastic tensor C_{ijpq}, defined on E, will be assumed to satisfy the usual symmetry conditions and to be strongly elliptic (2, 3). Functions $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in D_E$ will be vector valued, C^2 on E, and such that $\mathbf{u}$ as well

$$T_i(\mathbf{u}) = \tau_{ij}(\mathbf{u})n_j \quad \text{on } \partial_3 E \qquad [8]$$

are continuous. Here, $\mathbf{n}$ is the unit normal vector to $\partial_3 E$ and

$$\tau_{ij}(\mathbf{u}) = C_{ijpq}\,(\partial u_p/\partial x_q \qquad [9]$$

Latin indices run from 1 to n, and sum over the range of repeated indices is understood. The space D will be made by all the couples $\hat{u} = (\mathbf{u},\mathbf{T})$ such that $\mathbf{u}$ and $\mathbf{T}$ are continuous functions on $\partial_3 E$.

(a) Potential Theory and Reduced Wave Equation. LEMMA 1. *Let $\mathcal{I} \subset$ D be a linear subspace of* D. *Assume further:*

(i) The couple $\hat{u} = (0,0) \subset \mathcal{I}$

$$(ii) \quad \int_{\partial_3 E} u \frac{\partial v}{\partial n}\, dx = \int_{\partial_3 E} v \frac{\partial u}{\partial n}\, dx \quad \forall \hat{u},\hat{v} \in \mathcal{I}. \qquad [10]$$

(iii) Given any $(U, \partial U/\partial n) \in D$, *there is at least one* $(u, \partial u/\partial n) \in \mathcal{I}$ *such that* $u = U$ *on* $\partial_3 E$.

Then, for every $\hat{u} = (u, \partial u/\partial n) \in D$, *one has*

$$\int_{\partial_3 E} u \frac{\partial v}{\partial n}\, dx = \int_{\partial_3 E} v \frac{\partial u}{\partial n}\, dx \quad \forall \hat{v} \in \mathcal{I} \Rightarrow \hat{u} \in \mathcal{I}. \qquad [11]$$

Proof: Let $B{:}D \to D^*$ be defined by

$$\langle B\hat{u},\hat{v}\rangle = \int_{\partial_3 E} u \frac{\partial v}{\partial n}\, dx. \qquad [12]$$

Then $A = B - B^*$ is given by

$$\langle A\hat{u},\hat{v}\rangle = \int_{\partial_3 E} \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} dx. \qquad [13]$$

Clearly, the null set $\mathcal{N}$ of A is $\mathcal{N} = \{(0,0)\}$. This fact, together with assumptions (i) and (ii), imply in view of Eq. 13 that $\mathcal{I}$ is a connectivity condition for B. Due to assumption (iii), the ebvp satisfies existence; hence, Eq. 11 follows from *Theorem 1.*

THEOREM 2. *Let region* E *be bounded and* $\mathcal{I} \subset$ D *be made by the couples* $\hat{v} = (v, \partial v/\partial n) \in D$, *which are boundary values for some* $v \in D_E$ *satisfying*

$$(\partial^2 v/\partial x_i \partial x_i) + k^2 v = 0 \quad \text{on } E, \qquad [14a]$$

$$v = 0 \text{ on } \partial_1 E; \quad (\partial v/\partial n) = 0 \quad \text{on } \partial_2 E. \qquad [14b]$$

Assume that for every continuous function V *defined on* $\partial_3 E$ *the boundary value problem*

$$v = V \text{ on } \partial_3 E = \partial_3 R \qquad [15]$$

subjected to Eqs. 14 *possesses at least one solution* $v \in D_E$.

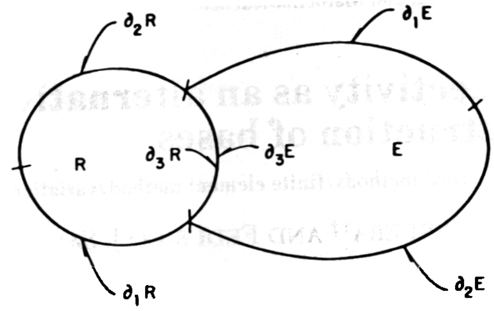

FIG. 1 Regions considered in time-independent problems.

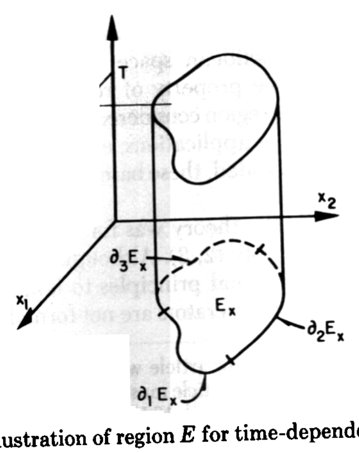

FIG. 2. Illustration of region E for time-dependent problems.

Then, for every couple $\hat{u} = (u, \partial u/\partial n) \in D$, *one has*

$$\int_{\partial_3 E} u \frac{\partial v}{\partial n} dx = \int_{\partial_3 E} v \frac{\partial u}{\partial n} dx \quad \forall \hat{v} \in \mathcal{J} \Rightarrow \hat{u} \in \mathcal{J}. \quad [16]$$

Proof: $\mathcal{J}$ is a linear subspace of D fulfilling assumptions (*i*)-(*iii*) of *Lemma 1*. To obtain (*i*), observe that the zero function satisfies Eqs. **14**. The existence of solution of the boundary value problem **15**, implies (*iii*). To prove assumption (*ii*), Green's formula of integration by parts is applied on E; this is the only point that requires a special treatment in order to extend these results to unbounded regions. This can be achieved by the introduction of suitable radiation conditions (7, 8). Frequently, such conditions are implied by assumptions on the asymptotic behavior at infinity, but this is not always the case. Therefore, in what follows it will be assumed that the radiation conditions to be considered are such that Green's formulas can be applied to any couple of functions satisfying them.

COROLLARY 1. *Let* E *in* Theorem 2, *be unbounded. Then, if functions are required to satisfy a radiation condition in the definition of* $\mathcal{J}$ *and the boundary value problem,* Theorem 2 *remains valid.*

Proof: This corollary is clear because Eq. **10** still holds.

(b) Heat and Wave Equations. It has already been explained how the region E and the spaces D_E and D are chosen for this kind of application. In addition, for any couple of functions α, β defined in the interval $[0,T]$, we adopt the notation

$$\alpha * \beta = \int_0^T \alpha(T-t)\beta(t)dt. \quad [17]$$

LEMMA 2. *In applications to heat and wave equations,* Lemma 1 *remains valid if the substitution*

$$\int_{\partial_3 E} u \frac{\partial v}{\partial n} dx \leftrightarrow \int_{\partial_3 E_x} u * \frac{\partial v}{\partial n} dx \quad [18]$$

is made for every couple $\hat{u}, \hat{v} \in D$.

Proof: Apply the arguments used to prove *Lemma 1* to the operator $B{:}D \rightarrow D^*$ given by

$$\langle B\hat{u}, \hat{v} \rangle = \int_{\partial_3 E_x} u * \frac{\partial v}{\partial n} dx. \quad [19]$$

THEOREM 3. *If Eqs.* 14 *are replaced by*

$$(\partial u/\partial t) - (\partial^2 u/\partial x_i \partial x_i) = 0 \quad on\ E, \quad [20a]$$

$$u = 0 \quad on\ \partial_1 E; \quad (\partial u/\partial n) = 0 \quad on\ \partial_2 E, \quad [20b]$$

$$u(x,0) = 0 \quad on\ E_x \quad [20c]$$

or, alternatively, by

$$(\partial^2 u/\partial t^2) - (\partial^2 u/\partial x_i \partial x_i)\ ,\ 0 \quad on\ E, \quad [21a]$$

$$u = 0 \quad on\ \partial_1 E; \quad (\partial u/\partial n) = 0 \quad on\ \partial_2 E \quad [21b]$$

$$u(x,0) = (\partial u/\partial t)(x,0) = 0 \quad on\ E_x \quad [21c]$$

and the substitution **18** *is carried out everywhere, then* Theorem 2 *and* Corollary 1 *remain valid.*

Proof: This theorem follows from *Lemma 2*. The arguments are similar to those used to prove *Theorem 2;* however, a special argument is required to prove that functions $\hat{u}, \hat{v} \in \mathcal{J}$ satisfy the equation

$$\int_{\partial_3 E_x} u * \frac{\partial v}{\partial n} dx = \int_{\partial_3 E_x} v * \frac{\partial u}{\partial n} dx \quad [22]$$

This relation can be obtained by considering the integral

$$\int_{E_x} v * \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_i \partial x_i}\right) dx = 0 \quad [23a]$$

for the heat equation or, alternatively,

$$\int_{E_x} v * \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_i \partial x_i}\right) dx = 0 \quad [23b]$$

for the wave equation. Then, integration by parts on space and time (see, e.g., ref. 5) yields Eq. **22**.

(c) Elastostatics and Reduced Equations of Elasticity. In the present subsection the function spaces D and D_E will be taken as explained previously for this kind of application.

LEMMA 3. *Let* $\mathcal{J} \subset D$ *be a linear subspace of* D. *Assume that:*

(*i*) *The couple* $\hat{u} = (0,0) \in \mathcal{J}$

(*ii*) $\displaystyle\int_{\partial_3 E} u_i T_i(v) dx = \int_{\partial_3 E} v_i T_i(u) dx \quad \forall u,v \in \mathcal{J}$ [24]

(*iii*) *Given any* $(U,T(U)) \in D$, *there is at least one*

$(u,T(u)) \in \mathcal{J}$ *such that* $u = U$.

Then, for every $\hat{u} = (u,T(u)) \in D$, *one has*

$$\int_{\partial_3 E} u_i T_i(v) dx = \int_{\partial_3 E} v_i T_i(u) dx \quad \forall \hat{v} \in \mathcal{J} \Rightarrow \hat{u} \in \mathcal{J}$$

Proof: The proof is very similar to that of *Lemma 1* and can be obtained by considering the operator $B{:}D \rightarrow D^*$ given by

$$\langle B\hat{u}, \hat{v} \rangle = \int_{\partial_3 E} u_i T_i(v) dx. \quad [26]$$

Corresponding to *Theorem 2*, one has

THEOREM 4. *Let region* E *be bounded and* $\mathcal{J} \subset D$ *be made by the couples* $\hat{v} = (v,T(v)) \in D$, *which are the boundary values for some* $v \in D_E$, *such that*

$$(\partial(\tau_{ij}(v))/\partial x_j) + k^2 v_i = 0 \quad on\ E, \quad [27a]$$

$$v = 0 \quad on\ \partial_1 E; \ T(v) = 0 \quad on\ \partial_2 E. \quad [27b]$$

Assume, further, that for every continuous function U *defined on* $\partial_3 E$, *the boundary value problem*

$$v = U \quad on\ \partial_3 E \quad [28]$$

subjected to Eqs. 27 *possesses at least one solution* $v \in D_E$. *Then, for each couple* $\hat{u} = (u,T(u)) \in D$, *one has*

$$\int_{\partial_3 R} u_i T_i(v) dx = \int_{\partial_3 R} v_i T_i(u) dx \quad \forall \hat{v} \in \mathcal{J} \Rightarrow \hat{u} \in \mathcal{J}. \quad [29]$$

The same results hold when region E *is unbounded if, in the definition of* $\mathcal{J}$ *and of the boundary value problem, functions are required to satisfy a radiation condition.*

Proof: It is very similar to those of *Theorem 3* and *Corollary 1;* the details will be left out.

(d) Elastodynamics. The region E and the spaces D and D_E will be taken as explained previously for this kind of application. In addition, the notation given in Eq. **17** will be used.

LEMMA 6. *In applications to elastodynamics,* Lemma 3 *remains valid if the substitution*

$$\int_{\partial_3 E} u_i T_i(v) dx \leftrightarrow \int_{\partial_3 E_x} u_i * T_i(v) dx \quad [30]$$

is made for every couple $\hat{u}, \hat{v} \in D$.

Proof: The proof is similar to that of *Lemma 1* and can be obtained by considering the operator $B{:}D \rightarrow D^*$ given by

$$\langle B\hat{u}, \hat{v} \rangle = \int_{\partial_3 E} u_i * T_i(v) dx. \quad [31]$$

Corresponding to *Theorem 4*, one has

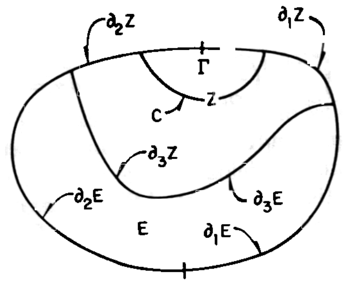

FIG. 3. Regions and auxiliary curves for the construction of bases for connectivity conditions.

THEOREM 5. *If Eqs.* 27 *are replaced by*

$$(\partial^2 u_i/\partial t^2) - (\partial(\tau_{ij}(\mathbf{u}))/\partial x_j) = 0 \quad on\ E, \qquad [32a]$$

$$\mathbf{u} = 0 \quad on\ \partial_1 E;\ \mathbf{T}(\mathbf{u}) = 0 \quad on\ \partial_2 E, \qquad [32b]$$

$$\mathbf{u} = \mathbf{T}(\mathbf{u}) = 0 \quad when\ t = 0,\ \mathbf{x} \in E_x \qquad [32c]$$

and substitution 30 *is carried out everywhere, then* Theorem 4 *remains valid.*

Proof: The only point that requires a special treatment is the proof that

$$\int_{\partial_3 E_x} u_i * T_i(\mathbf{v}) dx = \int_{\partial_3 E_x} v_i * T_i(\mathbf{u}) dx \qquad [33]$$

whenever $\hat{\mathbf{u}},\hat{\mathbf{v}} \in \mathcal{I}$. This is obtained by integration by parts of (see, e.g., ref. 9)

$$\int_{E_x} v_i * \left[\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\ \tau_{ij}(\mathbf{u}))\right] \qquad [34]$$

Method for obtaining bases of connectivity conditions

The method will be illustrated in connection with applications to potential theory and the reduced wave equation. Keeping the notation introduced in previous sections, let Z be a neighboring region to E such that $\partial_3 E = \partial E \cap \partial Z$ (Fig. 3). Let $\mathcal{I} \subset D$ be defined as in *Theorem* 2 and assume there is available in $Z \cup E$, a Green's function $G(\mathbf{x},\mathbf{y})$ such that

(*i*) $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}) \quad \forall \mathbf{x},\mathbf{y} \in Z \cup E$.

(*ii*) For every $\hat{u} = (u,\partial u/\partial n) \in D$

$$\int_{\partial_3 E} \left\{u \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) - G(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial n}\right\} d\mathbf{x} = 0 \quad \forall \mathbf{y} \in Z \qquad [35]$$

if and only if $\hat{u} \in \mathcal{I}$.

(*iii*) For every fixed $\mathbf{x}_0 \in \partial_3 E$, the Green function $G(\mathbf{x}_0,\mathbf{x})$ satisfies Eqs. 14a and b on Z, $\partial_1 Z$ and $\partial_2 Z$, respectively.

Define, for every $\mathbf{y} \in Z$, the function

$$w_y(\mathbf{x}) = G(\mathbf{x},\mathbf{y}) \quad \mathbf{x} \in E. \qquad [36]$$

Then, assumption (*ii*) is tantamount to saying that the set

$$\mathcal{B}_Z = \{\hat{w}_y = (w_y,\partial w_y/\partial n) \in D \,|\, y \in Z\} \qquad [37]$$

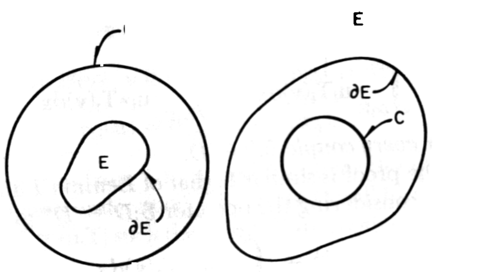

FIG. 4. Regions for which bases of Tables 1 and 2 are applicable Left) Bounded *E*; (*Right*) *E* is the exterior of a bounded region.

Table 1. Bases for two-dimensional problems

	Bounded *E*	Unbounded *E*
	Laplace equation	
$p_0(r,\phi)$	$-(r\ln r)^{-1}$	r^{-1}
$p_{n\alpha}(r,\phi)$*	$2nr^{n-1}T_\alpha(n\phi)$†	$2nr^{-n-1}T_\alpha(n\phi)$
$w_0(r,\phi)$	1	$-\ln r$
$w_{n\alpha}(r,\phi)$	$r^n T_\alpha(n\phi)$	$r^{-n}T_\alpha(n\phi)$
	Reduced wave equation	
$p_0(r,\phi)$	$2i[\pi r H_0^{(1)}(kr)]^{-1}$	$2i[\pi r J_0(kr)]^{-1}$
$p_{n\alpha}(r,\phi)$	$i[\pi r H_n^{(1)}(kr)]^{-1} \times T_\alpha(n\phi)$	$i[\pi r J_n(kr)]^{-1} \times T_\alpha(n\phi)$
$w_0(r,\phi)$	$J_0(kr)$	$H_0^{(1)}(kr)$
$w_{n\alpha}(r,\phi)$	$J_n(kr)T_\alpha(n\phi)$	$H_n^{(1)}(kr)T_\alpha(n\phi)$

* $\alpha = 1,2$; [illegible] 1

† $T_\alpha(x) = \begin{matrix}\sin \\ \cos\end{matrix}\ x$ if $\alpha = \begin{cases}1 \\ 2\end{cases}$

is a complete system for the connectivity condition $\mathcal{I}$ of *Theorem* 2. This system can be further restricted. Indeed, let $\Gamma \subset Z$ be a subregion of Z (Fig. 3) and write $C = \partial\Gamma - (\partial_1 Z \cup \partial_2 Z)$. Define

$$\mathcal{B}_C = \{\hat{w}_y \in \mathcal{B}_Z \,|\, y \in C\}. \qquad [38]$$

Under very general conditions given in the next theorem, $\mathcal{B}_C$ is also complete.

THEOREM 6. *Assume*

(*a*) *The boundary value problem*

$$(\partial^2 u/\partial x_i \partial x_i) + k^2 u = 0 \quad on\ \Gamma, \qquad [39a]$$

$$u = 0 \quad on\ C \cup \partial_1\Gamma; \quad (\partial u/\partial n) = 0 \quad on\ \partial_2\Gamma, \qquad [39b]$$

where $\partial_1\Gamma = \partial\Gamma \cap \partial_1 Z$ *and* $\partial_2\Gamma = \partial\Gamma \cap \partial_2 Z$, *has a unique solution.*

(*b*) *Any function* u *that satisfies Eq.* 39a *on* Z *and vanishes identically on an open subregion of* Z *necessarily vanishes identically on* Z.

Then $\mathcal{B}_C$, *given by Eq.* 38, *is a complete system for the connectivity condition I.*

Proof: Given $\hat{u} = (u,\partial u/\partial n) \in D$, define

$$W(\mathbf{y}) = \int_{\partial_3 E} \left\{u \frac{\partial w_y}{\partial n} - w_y \frac{\partial u}{\partial n}\right\} d\mathbf{x} \quad \mathbf{y} \in Z. \qquad [40]$$

In view of condition (*iii*), $W(y)$ satisfies Eq. 39a and the boundary conditions 39b, except possibly on C. However, if *u* is such that the condition

$$\int_{\partial_3 E} u \frac{\partial w_y}{\partial n} d\mathbf{x} = \int_{\partial_3 E} w_y \frac{\partial u}{\partial n} d\mathbf{x} \quad \forall \mathbf{y} \in C, \qquad [41]$$

then $W(y)$ vanishes identically on C and *W* is the unique so-

Table 2. Bases for three-dimensional problems

	Bounded *E*	Unbounded *E*
	Laplace equation	
$p_{nq}(r,\theta,\phi)$†	$(2n+1)r^{n-1} \times Y^*_{nq}(\theta,\phi)$	$(2n+1)r^{-n-2} \times Y_{nq}(\theta,\phi)$
$w_{nq}(r,\theta,\phi)$	$r^n Y^*_{nq}(\theta,\phi)$	$r^{-n-1}Y_{nq}(\theta,\phi)$
	Reduced wave equation	
$p_{nq}(r,\theta,\phi)$	$[ikr^2 h_n^{(1)}(kr)]^{-1} \times Y^*_{nq}(\theta,\phi)$	$[ikr^2 j_n(kr)]^{-1} \times Y_{nq}(\theta,\phi)$
$w_{nq}(r,\theta,\phi)$	$j_n(kr)Y^*_{nq}(\theta,\phi)$	$h_n^{(1)}(kr)Y_{nq}(\theta,\phi)$

† $n = 0,1,2,$ $-n \le q \le n$.

lution of the boundary value problem [39]. Thus, W vanishes identically in Γ. Hence, W vanishes identically on Z. This shows, by condition (*ii*), that Eq. **41** implies $\hat{u} \in \mathcal{J}$. Thus, $\mathcal{B}_C$ as given by Eq. **38** is complete.

COROLLARY 2. *Let the system of functions* $p_\alpha(y)$, $\alpha = 1,2,\ldots$, *be a basis for the continuous functions on* C. *For every* α, *define*

$$w_\alpha(x) = \int_C G(x,y)p_\alpha(y)dy \qquad [42]$$

and $\hat{w}_\alpha = (w_\alpha, \partial w_\alpha/\partial n) \in \mathcal{J}$. *Then,*

$$\mathcal{B} = \{\hat{w}_\alpha, \alpha = 1,2,\ldots\} \qquad [43]$$

is a complete system for the connectivity condition $\mathcal{J}$.

Proof: It follows immediately from the fact that

$$\int_C W(y)p_\alpha(y)dy = \int_{\partial_3 E}\left\{u\frac{\partial w_\alpha}{\partial n} - w_\alpha\frac{\partial u}{\partial n}\right\}dx, \qquad [44]$$

where $W(y)$ is given by Eq. **40.** This expression vanishes for every $\alpha = 1,2\ldots$ if and only if $W(y)$ vanishes identically on C. In this case $\hat{u} \in \mathcal{J}$ by virtue of *Theorem 6*.

Under very general conditions the procedure of *Corollary* 2, yields a basis of $\mathcal{J}$, as will be seen in the applications given in the next section.

Bases of connectivity conditions

Generally, when considering problems of connecting, the complete system associated with different regions will also be different. However, in most cases it is possible to develop complete systems that are to a large extent independent of the regions considered, as will be seen in the examples presented in this section. This remarkable fact can be very useful in numerical and other applications.

To illustrate the method, *Corollary* 2 has been applied to Laplace's and the reduced wave equations in two and three dimensions. Two alternative possibilities for the region E are considered, one for which E is a bounded region (Fig. 4 *left*) and the other one for which E is the exterior of a bounded region (Fig. 4 *right*). The curve C is taken as a circle (or a sphere in three-dimensional applications) with center in the origin. The results are given in Tables 1 and 2. In all these applications $\partial_1 E = \partial_2 E$ are void; hence, $\partial E = \partial_3 E$.

A symmetric Green's function for Laplace's equation in two dimensions is (10):

$$G(x,x_0) = \frac{1}{2\pi}\left[\ln(1/r) + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^n \cos n(\phi - \phi_0)\right] \qquad [45]$$

for $r > r_0$. Symmetry condition for the Green function implies that r and r_0 have to be interchanged when $r < r_0$. If E is a bounded region, the circle C must enclose E (Fig. 4 *left*). When E is the exterior of a bounded region with the origin of the coordinate system in its interior, the circle C must be contained in the complement of E (Fig. 4 *right*). To derive the results given in Table 1 for Laplace's equations, it is only necessary to choose the basis $p_n(r,\phi)$, $n = 0,1,2,\ldots$, for the continuous functions defined on the circle C of radius r in *Corollary* 2 (see Table 2). The bases $w_n(r,\phi)$, $n = 1,2,\ldots$, so derived, are given in the corresponding line in Table 1.

The other results in Tables 1 and 2 were obtained by treating similarly suitable Green's functions. For the two-dimensional reduced wave equation satisfying Sommerfeld's outgoing radiation condition, it is (11):

$$G(x,x_0) = \frac{i}{4}\sum_{n=0}^{\infty} \epsilon_n J_n(kr_0)H_n^{(1)}(kr)\cos n(\phi - \phi_0)$$

for $r > r_0$. When $r < r_0$, the variables r and r_0 have to be interchanged. Here $\epsilon_0 = 2$ and $\epsilon_n = 1$, $n = 1,2,\ldots$.

For the three-dimensional Laplace's equation, the symmetric Green function used was (10):

$$G(x,x_0) = \sum_{l=0}^{\infty}\sum_{n=-l}^{l}\frac{1}{2l+1}\frac{r_0^l}{r^{l+1}}Y_{ln}^*(\theta_0,\phi_0)Y_{ln}(\theta,\phi) \qquad [47]$$

for $r > r_0$. Here, again, the variables with a zero subindex must be interchanged with those without subindex when $r < r_0$.

A symmetric Green's function for the three-dimensional reduced wave equation which satisfies Sommerfeld's outgoing radiation condition is (10):

$$G(x,x_0) = ik\sum_{l=0}^{\infty}\sum_{n=-l}^{l} j_l(kr_0)h_l^{(1)}(kr)Y_{ln}^*(\theta_0,\phi_0)Y_{ln}(\theta,\phi) \qquad [48]$$

for $r > r_0$ and a corresponding expression for the case $r < r_0$.

In Eqs. **46–48**, $J_n(x)$ and $H_n^{(1)}(x)$ are the Bessel and Hankel functions of the first class (10–11), of order n; $Y_{ln}(\theta,\phi)$ are normalized spherical harmonics defined by

$$Y_{ln}(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-n)!}{(l+n)!}\right]^{1/2} P_l^n(\cos\theta)e^{in\phi}. \qquad [49]$$

Here $P_l^n(x)$ is the associated Legendre function, and

$$Y_{ln}^*(\theta,\phi) = (-1)^n Y_{l,-n}(\theta,\phi), \qquad [50]$$

and $j_l(x)$ and $h_l^{(1)}(x)$ are the spherical Bessel and Hankel functions of the first class (10), of order l. (r,ϕ) and (r,θ,ϕ) are polar and spherical coordinates, respectively. In each case ϕ is the polar angle.

When $E \cup Z$ is a half-space whose boundary is the plane $\phi = 0, \pi$, bases can be constructed in a similar fashion. Note, however, that they can be easily deduced from the tables. For instance, if a Dirichlet (Neumann) condition is satisfied on $\phi = 0,\pi$, the Green function for this problem is twice the odd (even) part of the free space. Green functions and the integration around C, which now is a semicircle (hemisphere for the three-dimensional case), is half the other one. Thus, the corresponding bases are the odd (even) functions given in the table. Notice, however, that the region E must be entirely contained in the half-space.

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