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Theory of connectivity: a systematic formulation of boundary element methods

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A theory of connectivity recently developed by the author is applied to construct a systematic formulation of boundary element methods. The concept of complete connectivity condition is shown to supply an alternative to boundary integral equations. The general problem of connecting solutions defined in neighbouring regions R and E is shown to lead to complete connectivity conditions which permit the formulation of three kinds of variational principles; they involve, respectively, $R \cup E$, R and the common boundary between R and E, only.

Introduction

In a very general and loose sense, the purpose of boundary element methods is to construct solutions to specific problems by choosing properly the boundary values of general solutions which are known in subregions. Often the general solutions are given by means of Green's functions and in such cases, boundary element formulations are established starting from boundary integral equations.[1]

The general problem of connecting solutions defined in neighbouring regions (*Figure 1*) has been formulated recently by the author[2–5] in an abstract manner. For such general problem variational principles have been obtained; for diffraction problems posed in unbounded regions such principles lead to functionals involving a bounded region only.[6,2] The problem of connecting is characterized by the set of functions that satisfy homogeneous conditions on E and by the set of functions that satisfy homogeneous conditions on R. Each of these sets separately constitutes what in this theory is called 'a connectivity condition' and it is essential to establish necessary and sufficient conditions for a function to be a member of such connectivity conditions. One such necessary and sufficient condition is supplied by a suitable boundary integral equation. However, the theory of connectivity shows that in most cases connectivity conditions are complete[3–5]; i.e. there is a bilinear form with the property that a function satisfies such boundary integral equation if and only if the bilinear form commutes when evaluated at the function and every element of the connectivity condition. Specially relevant are denumerable bases of connectivity conditions; i.e. denumerable subsets of connectivity conditions with the property that a function belongs to the connectivity condition, if and only if the bilinear form commutes when evaluated at the function and every one of the elements of the basis. In this paper the general theory is explained briefly and it is then applied to Laplace, reduced wave, wave and heat equations. General applications to static, quasi-static and dynamic elasticity are also given. A procedure[5] for constructing bases that are independent of the regions considered is illustrated by exhibiting such bases for Laplace and reduced wave equations. The possibility of constructing bases applicable to a general class of regions is remarkable and yields important numerical and analytical advantages. The results presented here are related to what in other fields[7] is called 'null field approach' but the theory of connectivity described here allows greater flexibility and generality. Indeed, there are many possible ways of constructing bases, including the use of boundary integral equations as one such possibility. On the other hand, to our knowledge this is the first time that such methods have been extended to time dependent problems.

In the last part of the paper the theory of connectivity is used to develop the basis for a systematic formulation of boundary element methods. It is shown that for linear problems associated with formally symmetric operators one is always led to consider two complete connectivity conditions $\mathscr{I}$ (internally generated motions) and $\mathscr{E}$ (externally generated motions). Using them a systematic formulation is given which includes general variational principles of three types: principles for solutions on $R \cup E$ with prescribed jumps which include as particular cases those reported by Prager[8] and Nemat-Nasser[9] for elasticity

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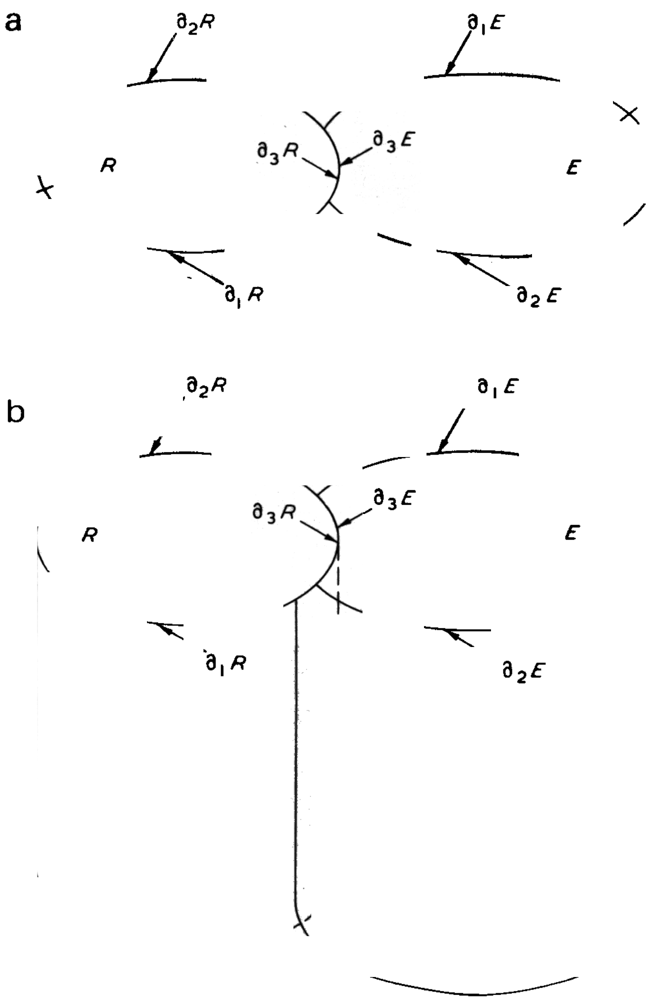

Figure 1 Regions considered in application. (a), time independent problems; (b), time dependent problems

principles involving the solution on R only, which are similar to those reported by Mei and Chen[6] for the linearized theory of free surface flows; and a class of principles which to our knowledge had not been reported previously which only involve values on the common boundary of the two regions ($\partial_3 R = \partial_3 E$ in *Figure 1*).

The paper is based on a general formulation of problems[10,4] using functional valued operators developed by the author.[11,12] This approach simplifies and permits greater flexibility in the treatment of continuum mechanics and its partial differential equations because it is applicable in any linear space, not necessarily normed, or with an inner product, or complete. As an example, heat and wave equations or those of dynamic elasticity do not satisfy the conventional notion of formal symmetry, however, they can be associated with formally symmetric operators in the sense introduced by the author[3] and it is due to this fact that the theory here presented is applicable to those equations.

Preliminary results

For operators[10,2,3,11,12] $P:D\to D^*$ defined on a linear space D and with values in the space D^* of all the linear functionals defined on D, let $N\subset D$ be the null set of A, i.e.:

$$N = \{u\in D \mid Au = 0\} \tag{1}$$

where $A = P - P^*$ and P^* is the adjoint operator of P. At this stage P will not be assumed to be formally symmetric.[5]

A linear subspace $\mathscr{I} \subset D$ is said to be a connectivity condition for A if:

(i) $N\subset \mathscr{I}$ (2)

(ii) $\langle Au, v\rangle = 0\ \forall u, v\in \mathscr{I}$ (3)

The connectivity condition is said to be complete when:

(iii) For any $w\in D$, one has:

$$\langle Aw, v\rangle = 0\ \forall\, v\in \mathscr{I} \Rightarrow w\in \mathscr{I} \tag{4}$$

The operator A is necessarily antisymmetric. Given such A, there are many operators for which $A = P - P^*$ because one can always add any symmetric operator S to obtain $P+S$ which also has that property. In applications we will be specially interested in a class of operators that will be called boundary operators and denoted by B, for which $A = B - B^*$.

Given a complete connectivity condition for $\mathscr{I}\subset D$, a subset $B\subset \mathscr{I}$ is said to be a complete system for $\mathscr{I}$ if, for every $u\in D$, one has:

$$\langle Au, w\rangle = 0\ \forall\, w\in B \Rightarrow u\in \mathscr{I} \tag{5}$$

B is a basis of $\mathscr{I}$ if, in addition, every finite set of functionals $\{Aw_\alpha \mid w_\alpha\in B, \alpha = 1,\ldots,N\}$ is linearly independent. In this case equation (5) can be replaced by:

$$\langle Au, w_\alpha\rangle = 0\ \forall\, \alpha = 1, 2, \ldots, \Rightarrow u\in \mathscr{I} \tag{6}$$

Examples of complete connectivity conditions

When diffraction problems, boundary element methods and many other applications are considered, one looks for functions which satisfy a set of conditions, such as partial differential equations on a region R. These functions are further restricted by the condition that they can be extended smoothly into solutions on a neighbouring region E, of some other partial differential equations; such condition imposes restrictions on the values which are admissibles on the common boundary $\partial_3 E$ between R and E. Such restrictions are frequently characterized by means of boundary integral equations.[1]

An alternative more flexible procedure is supplied by the theory of connectivity developed recently by the author.[2,3,5] In this section the applicability of the method is illustrated by giving examples of complete connectivity conditions; these include: potential theory, reduced wave, heat and wave equations and elasticity (statical, reduced equation and dynamical).

In applications to time independent problems, E will be a region of the n-dimensional Euclidean space (*Figure 1a*) with boundary $\partial E = \partial_1 E \cup \partial_2 E \cup \partial_3 E$, where $\partial_\alpha E (\alpha = 1, 2, 3)$ are mutually disjoint sets and $\partial_3 E = \partial_3 R$ is that part of the boundary of E which is shared with R. For applications to time-dependent problems (*Figure 1b*), a region that satisfies the same conditions as region E above will be called E_x while the region E will be taken as $E_x X[0, T]$; therefore, the common part of E with R will be:

$$\partial_3 E = \partial_3 E_x X[0, T]$$

as shown in *Figure 1b*. The unit normal vector $\boldsymbol{n}$ will be taken pointing outwards from the regions considered and outwards from R in the common part $\partial_3 R$ of the boundary.

For applications to potential theory, reduced wave, heat and wave equations, an auxiliary linear space D_E will be considered, and will be taken to be made of functions u (real or, alternatively, complex valued), which are C^2 in E

such that u and $\partial u/\partial n$ are continuous on $\partial_3 E$. In applications to elasticity, the elastic tensor C_{ijpq}, defined on E, will be assumed to satisfy the usual symmetry conditions and to be strongly elliptic.[2,3] Functions:

$$u = (u_1, u_2, \ldots, u_n) \in D_E$$

will be vector valued and C^2 on E and such that u as well as:

$$T_i(u) = \tau_{ij}(u) n_j \qquad \text{on } \partial_3 E \tag{7}$$

are continuous. Here, n is the unit normal vector to $\partial_3 E$ and:

$$\tau_{ij}(u) = C_{ijpq} \frac{\partial u_p}{\partial x_q} \tag{8}$$

Latin indices run from 1 to n and sum over the range of repeated indices is understood.

For illustration purposes the regions R and E shown in *Figure 1* were chosen bounded; however, the results presented in this paper are not restricted to bounded regions.[5] When E is unbounded, it is only necessary that the functions of D_E satisfy suitable radiation conditions.

The linear space D is arbitrary, except for the fact that certain boundary operators are well defined. For potential theory, reduced wave, heat and wave equations this requires that with every element $u \in D$ there is defined a pair of functions u, $\partial u/\partial n$ on $\partial_3 R = \partial_3 E$. When elasticity is considered, it is required that associated with every $u \in D$ there is defined a pair of vector valued functions u, $T(u)$ on $\partial_3 R = \partial_3 E$.

Potential theory and reduced wave equation

In applications to potential theory and the reduced wave equation let $\mathscr{I} \subset D$ be made by functions $v \in D$ for which there is at least a function $v_E \in D_E$, such that:

$$\frac{\partial^2 v_E}{\partial x_i \partial x_i} + k^2 v_E = 0 \qquad \text{on } E \tag{9a}$$

$$v_E = 0 \quad \text{on } \partial_1 E; \quad \frac{\partial v_E}{\partial n} = 0 \quad \text{on } \partial_2 E \tag{9b}$$

ınd:

$$v_E = v, \quad \frac{\partial v_E}{\partial n} = \frac{\partial v}{\partial n}; \qquad \text{on } \partial_3 R = \partial_3 E \tag{10}$$

If $k \neq 0$, equation (9a) is the reduced wave equation and Laplace equation if $k = 0$. When E is unbounded the functions v_E are required in addition to satisfy a radiation condition.[5]

The set of functions $\mathscr{I}$ is characterized by the fact that $u \in \mathscr{I}$, if and only if:

$$\int_{\partial_3 R} u \frac{\partial v}{\partial n} \mathrm{d}x = \int_{\partial_3 R} v \frac{\partial u}{\partial n} \mathrm{d}x, \quad \forall\, v \in \mathscr{I} \tag{1}$$

If $\{w_\alpha \in D; \alpha = 1, 2, \ldots\}$ is a denumerable basis of $\mathscr{I}$, then (11) can be replaced by:

$$\int_{\partial_3 R} u \frac{\partial w_\alpha}{\partial n} \mathrm{d}x = \int_{\partial_3 R} w_\alpha \frac{\partial u}{\partial n} \mathrm{d}x, \quad \forall\, \alpha = \ ,2, \tag{12}$$

Generally, when considering problems of connecting, bases associated with different regions are also different. However, in most cases it is possible to develop bases which are to a large extent independent of the regions considered and a method to construct them has been given by Herrera and Sabina.[5] This remarkable fact is very useful in applications.

Examples of such bases are given in *Table 1*. Two alternative possibilities for the region E are considered, one for which E is a bounded region (*Figure 2b*) and the other one for which E is the exterior of a bounded region (*Figure 2a*). Two- and three-dimensional problems are included. In all these applications $\partial_1 E$ and $\partial_2 E$ are void; hence $\partial E = \partial_3 E$. When E is bounded bases for Laplace equation are harmonic polynomials. In a similar manner bases for problems formulated in a half-space have been obtained[5]; they are the odd or even functions given in *Table 1* depending on whether a Dirichlet or a Neuman zero condition is satisfied on the free boundary of the half-space.

In *Table 1* (r, ϕ) and (r, θ, ϕ) are polar and spherical coordinates, respectively. In each case ϕ is the polar angle. When E is unbounded, the coordinate origin is assumed to be in the interior of R. The Bessel and Hankel functions of the first class[13,14] and order n are denoted by $J_n(x)$ and $H_n^{(1)}(x)$, respectively, while $Y_{ln}(\theta, \phi)$ are normalized spherical harmonics defined by:

$$Y_{ln}(\theta, \phi) = \left[\frac{2l+1}{4\pi} \frac{(l-n)!}{(l+n)!}\right]^{1/2} P_l^n(\cos\theta) e^{in\phi}$$

Here $P_l^n(x)$ is the associated Legendre function:

$$Y_{ln}^*(\theta, \phi) = (-1)^n Y_{l,-n}(\theta, \phi) \tag{14}$$

and $j_l(x)$ and $h_l^{(1)}(x)$ are the spherical Bessel and Hankel function of the first class[13] and order l.

Heat and wave equations

In applications to heat equation, let $\mathscr{I} \subset D$ be made by functions $v \in D$ for which there is at least a function $v_E \in D_E$, such that:

$$\frac{\partial^2 v_E}{\partial x_i \partial x_i} - \frac{\partial v_E}{\partial t} = 0 \qquad \text{on } E \tag{15a}$$

$$v_E = 0 \quad \text{on } \partial_1 E; \quad \frac{\partial v_E}{\partial n} = 0 \quad \text{on } \partial_2 E \tag{15b}$$

$$v_E(x, 0) = 0 \qquad \text{on } E_x \tag{15c}$$

and with the property that the boundary values (equations (10)) of v and v_E are the same on $\partial_3 E$. In applications to wave equation the definition of $\mathscr{I}$ is modified supplementing equations (15) with:

$$\partial v_E/\partial t = 0 \qquad \text{on } E_x, t = 0 \tag{15d}$$

and replacing (15a) by:

$$\frac{\partial^2 v_E}{\partial x_i \partial x_i} - \frac{\partial^2 v_E}{\partial t^2} = 0 \qquad \text{on } E \tag{15e}$$

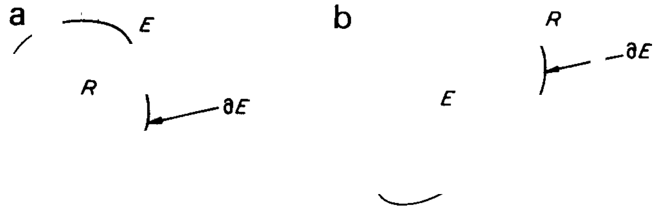

Figure 2 Regions for which bases of *Table 1* are applicable (a), E unbounded; (b), E bounded

In both cases, the set of functions $\mathscr{I}$ is characterized by the fact that $u\in\mathscr{I}$, if and only if:

$$\int_{\partial_3 E_x} u * \frac{\partial v}{\partial n}\,\mathrm{d}x = \int_{\partial_3 E_x} v * \frac{\partial u}{\partial n}\,\mathrm{d}x, \ \forall\ v\in\mathscr{I} \tag{16}$$

This condition can be replaced by:

$$\int_{\partial_3 E_x} u * \frac{\partial w_\alpha}{\partial n}\,\mathrm{d}x = \int_{\partial_3 E_x} w_\alpha * \frac{\partial u}{\partial n}\,\mathrm{d}x, \ \forall\ \alpha = 1, 2, \tag{17}$$

when $\{w_\alpha \in D; \alpha = 1, 2, \ldots\}$ is a denumerable basis of $\mathscr{I}$. In these equations the following notation has been adopted:

$$u * \frac{\partial v}{\partial n} = \int_0^T u(t\;\; T)\frac{\partial v}{\partial n}(t)\,\mathrm{d}t \tag{18}$$

Elastostatics and reduced equations of elasticity

For this kind of application the elements $v\in\mathscr{I}\subset D$ are taken so that there is at least a function $v_E\in D_E$ satisfying:

$$\frac{\partial}{\partial x_j}(\tau_{ij}(v_E)) + k^2 v_{Ei} = 0 \quad \text{on } E \tag{19a}$$

$$v_E = 0 \ \text{ on } \partial_1 E; \ \ T(v_E) = 0 \ \text{ on } \partial_2 E \tag{19b}$$

with the property that:

$$v_E = v, \ T(v_E) = T(v) \qquad \text{on } \partial_3 E \tag{19c}$$

The class $\mathscr{I}$ of functions so defined, is characterized by the fact that $u\in\mathscr{I}$, if and only if:

$$\int_{\partial_3 E} u_i T_i(v)\,\mathrm{d}x = \int_{\partial_3 E} v_i T_i(u)\,\mathrm{d}x, \ \ \forall\ v\in\mathscr{I} \tag{20}$$

This condition can be replaced by:

$$\int_{\partial_3 E} u_i T_i(w_\alpha)\,\mathrm{d}x = \int_{\partial_3 E} w_{\alpha i} T_i(u)\,\mathrm{d}x, \ \ \forall\ \alpha = \tag{21}$$

Table 1 Bases for Laplace and reduced wave equations

			Bounded E	Unbounded E
Laplace equation	Two dimension	$w_0(r,\phi)$	1	$-\ln r$
		$w_{n\alpha}(r,\phi)$*	$r^n T_\alpha(n\phi)$†	$r^{-n}T_\alpha(n\phi)$
	Three dimensions	$w_{nq}(r,\theta,\phi)$‡	$r^n Y^*_{nq}(\theta,\phi)$	$r^{-n-1}Y_{nq}(\theta,\phi)$
Reduced wave equation	Two dimensions	$w_0(r,\phi)$	$J_0(kr)$	$H_0^{(1)}(kr)$
		$w_{n\alpha}(r,\phi)$*	$J_n(kr)T_\alpha(n\phi)$†	$H_n^{(1)}(kr)T_\alpha(n\phi)$
	Three dimensions	$w_{nq}(r,\theta,\phi)$‡	$j_n(kr)Y^*_{nq}(\theta,\phi)$	$h_n^{(1)}(kr)Y_{nq}(\theta,\phi)$

* $\alpha = 1, 2; n = 1, 2, \ldots$

† $T_\alpha(x) = \frac{\sin}{\cos} x$ if $\alpha = \begin{matrix}1\\2\end{matrix}$

‡ $n = 0, 1, 2, \ldots; -n \leqslant q \leqslant n$

when $\{w_\alpha \in \mathscr{I}\}$ $(\alpha = 1, 2, \quad)$ is a denumerable basis for $\mathscr{I}$

Elastodynamics

In this case equations (19) are supplemented with:

$$v_E = \partial v_E/\partial t = 0, \qquad \text{on } E_x, t = 0$$

and equation (19a) replaced by:

$$\frac{\partial}{\partial x_j}(\tau_{ij}(v_E)) - \frac{\partial^2 v_{Ei}}{\partial t^2} = 0 \quad \text{on } E$$

At the same time the substitution:

$$\int_{\partial_3 E} u_i T_i(v)\,\mathrm{d}x \longleftrightarrow \int_{\partial_3 E_x} u_i * T_i(v)\,\mathrm{d}x$$

has to be carried out everywhere in equations (20) and (21). Here, the notation given in equation (18) is used.

Problem of connecting

The concept of formal symmetry was introduced previously[3] for functional valued operators $P: D\to D^*$. Such an operator is said to be formally symmetric when:

$$\langle Pu, v\rangle = 0 \ \ \forall\ v\in N \Rightarrow Pu = 0 \tag{25}$$

The problem of connecting for formally symmetric operators is defined and discussed in this section. It deals with constructing solutions in a region such as $R\cup E$ of *Figure 1*, by connecting those corresponding to individual subregions such as R and E. The theory has been, however, formulated in an abstract manner and can be applied in more general situations, as long as its postulates are satisfied.

Let $\hat{P}: \hat{D}\to\hat{D}^*$ be a formally symmetric operator and D, D_E a decomposition of $\hat{D}$, so that every $\hat{u}\in\hat{D}$ can be written in a unique manner as $\hat{u} = u + u_E$, with $u\in D$ and $u_E\in D_E$. It will be assumed that this decomposition is such that:

$$\langle\hat{P}\hat{u}, \hat{v}\rangle = \langle\hat{P}u, v\rangle + \langle\hat{P}u_E, v_E\rangle \tag{26}$$

for every $\hat{u}, \hat{v}\in\hat{D}$. This permits defining $P: \hat{D}\to\hat{D}^*$ and $P_E: \hat{D}\to\hat{D}^*$ by:

$$\langle P\hat{u}, \hat{v}\rangle = \langle\hat{P}u, v\rangle; \ \ \langle P_E\hat{u}, \hat{v}\rangle = \langle\hat{P}u_E, v_E\rangle$$

They satisfy:

$$\hat{P} = P + P_E \tag{28}$$

In view of equation (27) they can be thought of as $P: D\to D^*$ and $P_E: D_E\to D_E^*$.

Assume $S\subset\hat{D}$ has the following properties:

(a) S is a complete connectivity condition for $\hat{P}$
(b) For every $u\in D$, $\exists u_E\in D_E \ni u + u_E\in S$
(c) For every $u_E\in D_E$, $\exists u\in D \ni u + u_E\in S$

Then S will be said to be a smoothness relation or condition. Elements $u\in D$ and $u_E\in D_E$ are said to be smooth extensions of each other when $u + u_E\in S$. In what follows it will be assumed that S is a smoothness relation.

Given any $\hat{u} = (u, u_E)\in\hat{D}$, the following notation will be adopted:

$$\hat{u}\approx\hat{v} \Leftrightarrow \hat{u} - \hat{v}\in S \tag{29a}$$

$$[\hat{u}] = u' - u, \ |\hat{u}| = \tfrac{1}{2}(u' + u) \tag{29b}$$

$$[\hat{u}]_E = u'_E - u_E, \ \ |\hat{u}|_E = \tfrac{1}{2}(u'_E + u_E) \tag{29c}$$

where $u' \in D$ and $u'_E \in D_E$ are smooth extensions of $u_E \in D_E$ and $u \in D$, respectively. Three connectivity conditions will be defined:

$$\mathcal{J} = \{u_E \in D_E \mid \exists\, v_E \approx u_E,\ P_E v_E = 0\} \tag{30a}$$

$$\mathcal{E} = \{u \in D \mid \exists\, v \approx u,\ Pv = 0\} \tag{30b}$$

$$\hat{\mathcal{J}} = \mathcal{J} + \mathcal{E} \tag{30c}$$

They are connectivity conditions for A_E, A and $\hat{A}$, respectively. Here, $A_E = P_E - P_E^*$, while $\hat{A} = \hat{P} - \hat{P}^*$.

Given $U \in D$, $U_E \in D_E$ and $\hat{W} \in \hat{D}$, an element $\hat{u} = (u, u_E) \in \hat{D}$ is said to be a solution of the problem of connecting (with prescribed jumps), if:

$$Pu = PU, P_E u_E = P_E U_E, \hat{u} \approx \hat{W} \tag{31}$$

An element $u \in D$ is said to be an R-solution of the problem of connecting if there exists a solution $\hat{u}_0 = (u_0, u_{EO})$ of the problem of connecting such that $u = u_0$. Finally, $u \in \hat{D}$ is said to be a boundary solution if there is a solution $\hat{u}_0 = (u_0, u_{EO})$ of the problem of connecting such that $u \approx u_0$ and $u_E \approx u_{EO}$.

Define for every $\hat{u} \in \hat{D}$, the functionals:

$$X(\hat{u}) = \langle \hat{P}(\hat{u} - 2\hat{U}), \hat{u} \rangle + \langle A([\hat{u}] - 2[\hat{W}]), |\hat{u}| \rangle \tag{32a}$$

$$X_R(\hat{u}) = \langle P(u - 2U), u \rangle + \langle AU', u' \rangle + \langle A([u] - 2[W]), |u| \rangle \tag{32b}$$

$$X_B(\hat{u}) = \langle A[\hat{u} + \hat{U} - 2\hat{W}], |\hat{u}| \rangle + \langle A|U|, [\hat{u}] \rangle \tag{32c}$$

It can be shown[3] that each one of the functionals (32) is well defined in the sense that its value is independent of the particular smooth extension chosen. Then the following three general variational principles hold for corresponding problems of connecting.

(a) $\hat{u} \in \hat{D}$ is a solution of the problem of connecting if and only if $X'(\hat{u}) = 0$. Here, X' is the derivative of X.
(b) $u \in D$ is an R-solution of the problem of connecting if and only if $\delta X_R(\hat{u}) = 0$ for some $\hat{u} = (u, u_E)$, where the variation δX_R is taken on the subspace $D + \mathcal{J} \subset D$.
(c) Let $\hat{u} \in \hat{D}$ be such that $\hat{u} \approx \hat{W}$, then $\hat{u}$ is a boundary solution of the problem of connecting if and only if $\delta X_B(\hat{u}) = 0$ where the variation is taken on the subspace $\mathcal{J} + \mathcal{E} \subset D$.

For numerical applications the explicit form of the latter variation is relevant. When $\hat{u} \sim \hat{W}$, from (c) it follows that a necessary and sufficient condition for $\hat{u}$ to be a boundary solution of the problem of connecting is that:

$$2\langle A|u|, v' \rangle = -\langle A_E([W]_E + 2U_E), v_E \rangle,\ \forall\, v_E \in \mathcal{J} \tag{33a}$$

$$2\langle A|u|, v \rangle = \langle A(2U + [\hat{W}]), v \rangle,\ \forall\, v \in \mathcal{E} \tag{33b}$$

Here $v' \in D$ is any continuous extension of v_E. Equations (33) can also be deduced from the fact that $u_E - U_E \in \mathcal{J}$ while $u - U \in \mathcal{E}$. If $w_{E\alpha} \in \mathcal{J}$, $w_\alpha \in \mathcal{S}, \alpha = 1, 2, \ldots$ are denumerable bases of the connectivity conditions $\mathcal{J}$ and $\mathcal{E}$, respectively, then equations (33) can be replaced by the denumerable set of equations obtained substituting v_E by $w_{E\alpha}$ in equation (33a), and v and w_α in equation (33b).

An illustrative example

Given functions U, W, continuous on R and on E separately (*Figure 1a*), consider the problem of finding a function u on R and E, such that:

$$\nabla^2 u = \ ^2U\nabla \text{ on } R \text{ and on } E \tag{34a}$$

$$u = U, \text{ on } \partial_1 R \cup \partial_1 E; \quad \frac{\partial u}{\partial n} = \frac{\partial U}{\partial n} \text{ on } \partial_2 R \cup \partial_2 E \tag{34b}$$

and such that the following jump conditions:

$$u_E - u_R = W_R - W_E \quad \text{on } \partial_3 R = \partial_3 E$$

$$k_E \frac{\partial u_E}{\partial n} - k_R \frac{\partial u_R}{\partial n} = k_E \frac{\partial W_E}{\partial n} - k_R \frac{\partial W_R}{\partial n} \quad \text{on } \partial_3 R$$

are satisfied. Here k_E and k_R are two constants; k will be the function which is identical to k_E on E and to k_R on R.

To deal with this problem it is convenient to take the linear spaces D and D_E as sets of functions which are C^2 on R and E respectively. In addition the functions of D together with their first-order derivatives, possess extensions which are continuous on $\partial_1 R$, $\partial_2 R$ and $\partial_3 R$, separately; similar conditions are satisfied by members of D_E that can be obtained replacing R and E. The operator P of equation (27) is defined by:

$$\langle P\hat{u}, \hat{v} \rangle = k_R \left\{ \int_R v \nabla^2 u \, dx + \int_{\partial_1 R} u \frac{\partial v}{\partial n} dx - \int_{\partial_2 R} v \frac{\partial u}{\partial n} dx \right\}$$

The definition of P_E is obtained replacing R by E everywhere in this equation.

In this case:

$$\langle A\hat{u}, \hat{v} \rangle = k_R \int_{\partial_3 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} dx$$

The value of $\langle A_E \hat{u}, \hat{v} \rangle$ is minus the expression obtained replacing R by E everywhere in equation (37). In addition $A = A + A_E$. Functions $(u_R, u_E) \in D$ will be said to be smooth if:

$$u_R = u_E \text{ and } k_R \partial u_R / \partial n = k_E \partial u_E / \partial n, \quad \text{on } \partial_3 R \tag{38}$$

The set $\mathcal{S}$ of smooth functions can be shown to constitute a smoothness relation.

With these definitions, the associated problem of connecting is the same as the one defined by equations (34) and (35). R-solutions and boundary solutions are restrictions of solutions of the problem of connecting to region R and to the boundary $\partial_3 R = \partial_3 E$, respectively. The three general variational principles given previously are applicable to these problems. Equations (32) yield the following functionals.

For the problem of connecting:

$$X(\hat{u}) = \int_{R \cup E} ku(\nabla^2 u - \nabla^2 U) dx + \int_{\partial_1 R \cup \partial_1 E} k(u - 2U) \frac{\partial u}{\partial n} dx + \int_{\partial_2 R \cup \partial_2 E} ku \left(2 \frac{\partial U}{\partial n} - \frac{\partial u}{\partial n}\right) dx + \int_{\partial_3 R} \left\{ |u| \left[k \left(\frac{\partial u}{\partial n} - 2 \frac{\partial W}{\partial n} \right) \right] - [u - 2W] \left| k \frac{\partial u}{\partial n} \right| \right\} dx$$

For the R-solution:

$$X_R(\hat{u}) = k_R \left\{ \int_R u(\nabla^2 u - \nabla^2 U)\,\mathrm{d}x + \int_{\partial_1 R} (u \quad 2U) \frac{\partial u}{\partial n}\,\mathrm{d}x + \int_{\partial_2 R} u\left(2\frac{\partial U}{\partial n} - \frac{\partial u}{\partial n}\right)\mathrm{d}x \right\}$$
$$+ k_E \int_{\partial_3 R} \left\{ u_E \frac{\partial U_E}{\partial n} - U_E \frac{\partial u_E}{\partial n} \right\} \mathrm{d}x$$
$$+ \int_{\partial_3 R} \left\{ |u| \left[k\left(\frac{\partial u}{\partial n} - 2\frac{\partial W}{\partial n}\right)\right] - [u - 2W] \left| k\frac{\partial u}{\partial n} \right| \right\} \mathrm{d}x \tag{40}$$

For the boundary solution, when u satisfies the jump conditions (35) from the start:

$$X_B(\hat{u}) = \int_{\partial_3 R} \left\{ |u| \left[k\frac{\partial(u+U-2W)}{\partial n} \right] - [u+U-2W] \left| k\frac{\partial u}{\partial n} \right| \right\} \mathrm{d}x$$
$$\int_{\partial_3 R} \left\{ [u] \left| k\frac{\partial U}{\partial n} \right| - |U| \left[k\frac{\partial u}{\partial n} \right] \right\} \mathrm{d}x \tag{41}$$

Here as in what follows | | and [ ] stand for average and jumps across $\partial_3 R$; thus, for example:

$$|u| = (u_E + u)/2; \quad [u] = u_E - u \quad \text{on } \partial_3 R \tag{42}$$

Assume $\{w_{E\alpha}\}$ and $\{w_\alpha\}$ are bases for the connectivity conditions $\mathscr{I}$ and $\mathscr{E}$, respectively, then equations (33) imply that the average values $|u|$ and $|k\,\partial u/\partial n|$ of any boundary solution u of the problem of connecting are characterized by:

$$2\int_{\partial_3 R} \left\{ w_{E\alpha} \left| k\frac{\partial u}{\partial n} \right| - k_E |u| \frac{\partial w_{E\alpha}}{\partial n} \right\} \mathrm{d}x = 2k_E \int_{\partial_3 R} \left\{ w_{E\alpha} \frac{\partial U_E}{\partial n} - U_E \frac{\partial w_E}{\partial n} \right\} \mathrm{d}x$$
$$- \int_{\partial_3 R} \left\{ w_{E\alpha} \left[k\frac{\partial W}{\partial n} \right] - k_E [W] \frac{\partial w_{E\alpha}}{\partial n} \right\} \mathrm{d}x$$
$$\forall\, \alpha = 1, 2, \ldots \tag{43a}$$

and:

$$2\int_{\partial_3 R} \left\{ w_\alpha \left| k\frac{\partial u}{\partial n} \right| - k_R |u| \frac{\partial w_\alpha}{\partial n} \right\} \mathrm{d}x = 2k_R \int_{\partial_3 R} \left\{ w_\alpha \frac{\partial U}{\partial n} - U \frac{\partial w_\alpha}{\partial n} \right\} \mathrm{d}x$$
$$+ \int_{\partial_3 R} \left\{ w_\alpha \left[k\frac{\partial W}{\partial n} \right] - k_R [W] \frac{\partial w_\alpha}{\partial n} \right\} \mathrm{d}x,$$
$$\forall\, \alpha = 1, 2, \ldots \tag{43b}$$

This is a denumerable set of conditions which is suitable for boundary element formulations. For two- and three-dimensional problems, the bases given in *Table 1* can be used.

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