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# Theory of Connectivity. A Unified Approach to Boundary Methods


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ABSTRACT

A theory of connectivity recently developed by the author, is described briefly, and a unified formulation of boundary methods is presented. Boundary integral equations and series expansions in terms of a basic set of functions are among approaches included in this unified formulation. The theory of connectivity therefore appears to be a useful tool for discussing questions of completeness of the basic set of functions and convergence of the approximating procedures; in addition, it supplies a systematic formulation of variational principles for this kind of problem.


## 1. INTRODUCTION

In recent years boundary methods are being used extensively in applied mathematics [1], because they reduce the dimensionality of problems and also because, when they are used in conjunction with finite elements, they permit reducing the size of the regions treated numerically.

At present the method most extensively used is the boundary integral equation derived from Maxwell-Betti's formula [2].

Another approach is the singularity method [3]. In this procedure singular solutions are replaced by integral representations. The method can be subdivided into two, depending on whether boundary values and the sought solution are defined on the same curve or in different curves. Integral equations of the latter type are gaining favor among engineers [4,5].

A related but different approach discussed by Kantorovich and Krylov [6] depends on the use of a basic denumerable set of solutions to approximate the sought functions.

To apply boundary methods efficiently, it is important to settle questions of completeness of the basic set of solutions and questions of convergence of the approximating procedure. Such matters may have important practical implications; in radio science [7], for example, problems of convergence have complicated and unnecessarily restricted the applicability of these methods [8].

The present author has recently developed a theory of connectivity [2, 9-12] which can be used to settle these matters in specific applications. Some features of this theory are: (i) it supplies a unified formulation of boundary methods; (ii) it answers questions of completeness; (iii) it establishes conditions for convergence; and (iv) it provides a systematic formulation of variational principles for such problems. In this paper part of the theory is described briefly and illustrations of the kind of results it yields are given. Some of this material has already been published, but a more detailed and complete exposition is being prepared.

The presentation is divided into three parts: the general diffraction problem, the problem of connecting, and general variational principles.

The problem of connecting constitutes a particular example, although a very general one, of the problem of diffraction; the main concern is to connect solutions defined in two different but neighboring regions such as R and E in Fig. 1.

Among the variational principles, three kinds can be distinguished: (i) principles involving the region $R \cup E$ applicable to problems with discontinuous fields such as those recently surveyed by Nemat-Nasser [13]; (ii) principles involving one of the regions only (R, for example);

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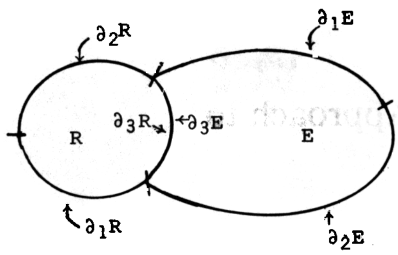


Fig. 1. Regions R and E.

and (iii) principles involving the common boundary $\partial_3 R = \partial_3 E$ between the two regions.

## 2. GENERAL DIFFRACTION PROBLEM

The theory has been developed using extensively functional valued operators $P: D \to D^*$, where D is an arbitrary linear space with no additional algebraic structure assumed to be defined, and $D^*$ its algebraic dual; i.e. the linear space made of the linear functional defined on D. The use of these operators in the treatment of partial differential equations, permit achieving generality and simplicity; their application to partial differential equations, which is not standard, has been discussed previously by the author [14, 15, 16]. Linear operators of this type always possess an adjoint $P^*: D \to D^*$ of the same kind [14], and it is therefore possible to define $A = P - P^*$, which is the antisymmetric part of P, except for a 1/2 factor. As an example, D could be Soblev space $H^s(R)$ $(s \geq 2)$, where R is the region illustrated in Fig. 1.

For every $u, v \in D$, let

$$\langle Pu, v\rangle = \int_R v \frac{\partial^2 u}{\partial x_i \partial x_i}\, d\underset{\sim}{x} + \int_{\partial_1 R} u \frac{\partial v}{\partial n}\, d\underset{\sim}{x} - \int_{\partial_2 R} v \frac{\partial u}{\partial n}\, d\underset{\sim}{x}\,, \qquad (2.1)$$

where the boundary of R is assumed to be decomposed into three parts $\partial_i R$ $(i = 1, 2, 3)$. Integrating by parts, it is seen that $A = P - P^*$ is given by

$$\langle Au, v\rangle = \int_{\partial_3 R} \left\{v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right\} d\underset{\sim}{x} \qquad (2.2)$$

when R is bounded. Equation (2.2) also holds when R is unbounded, if elements of D are restricted to satisfy suitable radiation conditions.

The null subspace, $N = \{u \in D \mid Au = 0\}$ of A, plays a special role in the theory, because it defines the set of boundary values which are relevant for the problems considered. For example, when A is given by (2.2), $N = \{u \in D \mid u = \partial u/\partial n = 0, \text{ on } \partial_3 R\}$, and the equivalence relation, defined by the condition $u - v \in N$, is tantamount to $u = v$ and $\partial u/\partial n = \partial v/\partial n$, on $\partial_3 R$.

*A linear subspace* $I \subset D$ *is said to be a connectivity condition when*

(1) $N \subset I$,

(2) $\langle Au, v\rangle = 0 \quad \forall u, v \in I$

*The connectivity* I *is said to be complete when*

(3) *for every* $u \in D$, *one has*

$$\langle Au, v\rangle = 0 \quad \forall v \in I \to u \in I\,. \qquad (2.5)$$

The use of the notion of completeness to describe property (3) is natural because this property implies that I is largest as a commutative class; indeed, any element $u \in D$ that commutes with every element $v \in I$, necessarily belongs already to I.

As an example, it is recalled that $I = \{u \in D \mid u = 0, \text{ on } \partial_3 R\}$ is a complete connectivity condition when A is given by (2.2). A more general example of connectivity condition is the set $\mathcal{E} = N + N_P$, where $N_P$ is the null subspace of P and $P: D \to D^*$ is any linear operator.

When $P: D \to D^*$ is given by (2.1), the linear subspace $\mathcal{E} \in D$ just defined, is characterized by the fact that the boundary values u, $\partial u/\partial n$ on $\partial_3 R$ of any function $u \in \mathcal{E}$, can be extended into a function $u' \in D$, such that

$$\nabla^2 u' = 0, \quad \text{in } R\,, \qquad (2.6a)$$

$$u' = 0, \quad \text{on } \partial_1 R\,, \qquad (2.6b)$$

$$\partial u'/\partial n = 0, \quad \text{on } \partial_2 R\,. \qquad (2.6c)$$

More precisely, $u \in E$ if and only if $\exists u' \in D_\ni$ satisfies Eqs. (2.6), $u = u'$, and $\partial u/\partial n = \partial u'/\partial n$, on $\partial_3 R$.

The definition of the general diffraction problem to be considered is given next. *Given* $U \in D$ *and* $V \in D$, *an element* $u \in D$ *is solution of the diffraction problem when*

$$Pu = PU \quad \textit{and} \quad u - V \in I \tag{2.7}$$

Here, I is assumed to be a connectivity condition for P

It will be said that the problem of diffraction satisfies existence, when such problem possesses at least one solution for every $U \in D$ and every $V \in D$. Using this nomenclature, it is possible to state two interesting properties associated with the general diffraction problem; they are given under the assumption that $I \subset D$ is a connectivity condition (not necessarily complete) for P.

Theorem 1: *If the problem of diffraction satisfies existence, then* I *and* E *are complete. In addition every* $u \in D$ *can be written in an almost unique manner as*

$$u = u_1 + u_2, \; u_1 \in I, \; u_2 \in E. \tag{2.8}$$

Here, almost uniqueness is used in the sense that $u_1$ and $u_2$ are unique except for elements belonging to the null subspace N.

As an example let P be given by (2.1), with $\partial_1 R = \partial_2 R$ void, so that R is given as in Fig. 2. In this case $E \subset D$ is the set of functions whose boundary values u, $\partial u/\partial n$ on $\partial R$, can be extended into a harmonic function on R. Let $I = \{u \in D \mid u = 0 \;, \text{ on } \partial R\}$ be the given connectivity condition. Given any $U \in D$ and $V \in D$, the corresponding diffraction problem is

$$\nabla^2 u = \nabla^2 U; \quad \text{in} \quad R \,, \tag{2.9a}$$

$$u = V; \quad \text{on} \quad \partial R \,. \tag{2.9b}$$

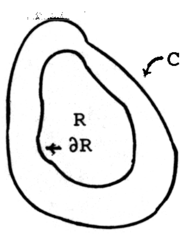


Fig. 2. Curve C enclosing region R.

This is a boundary value problem. Theorem 1 implies that E is complete, and therefore that the condition

$$\int_{\partial R} \{u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\} d\underset{\sim}{x} = 0 \quad \forall v \in D_\ni \; \nabla^2 v = 0 \text{ on } R \tag{2.10}$$

is necessary and sufficient, when problem (2.9) satisfies existence, in order for the boundary values u, $\partial u/\partial n$ on $\partial R$, to coincide with the corresponding values of some function which is harmonic on R.

A subset $B \subset I$ of a connectivity condition I is said to be complete when for every $u \in D$, has

$$\langle Au, w\rangle = 0 \quad \forall w \in B \rightarrow u \in I \,. \tag{2.11}$$

A complete denumerable subset is said to be a connectivity basis when every finite collection $\{Aw_\alpha \mid w_\alpha \in B, \alpha = 1, \ldots, N\}$ is linearly independent.

As an example, let $G(\underset{\sim}{x},\underset{\sim}{y})$ be a fundamental solution of Laplace's equation in the whole space with singularity in $\underset{\sim}{y}$, and define for every $\underset{\sim}{y} \notin R$ a function $w_{\underset{\sim}{y}}(\underset{\sim}{x})$ on R, by $w_{\underset{\sim}{y}}(\underset{\sim}{x}) = G(\underset{\sim}{x},\underset{\sim}{y})$. Take the set $B_0 = \{w_{\underset{\sim}{y}} \mid \underset{\sim}{y} \notin R\}$. Then, in view of (2.10) and well-known results of potential theory, the set $B_0$ is a complete subset of the connectivity condition E. Even more, a procedure presented previously by Herrera and Sabina [11], can be used to show that $\{w_{\underset{\sim}{y}_\alpha} \mid \alpha = 1, 2, \ldots\}$ is a denumerable connectivity basis whenever $\{\underset{\sim}{y}_\alpha \mid \alpha = 1, 2, \ldots\}$ is taken as a denumerable dense subset of a curve C (or a surface if the dimension of the space is greater than 2) enclosing the region R (Fig. 2).

A relation between Hilbert space bases and connectivity bases can be given, at least for some special cases which are, however, widely applicable. Assume there is a mapping $A: \mathcal{D} \rightarrow \mathcal{D}$, where $\mathcal{D} = D/N$, such that: (i) $A^2 u = -u$, and (ii) $(u,v) = \langle Au, Av\rangle$ is an inner product with the property that $\mathcal{D}$ is a Hilbert space with respect to this inner product. Then, it can be shown that when $B \subset I$ is a connectivity basis, then $B$ necessarily is a basis of I/N, as a Hilbert subspace of $\mathcal{D}$.

Going back to the example illustrated in Fig. 2, it can be observed that the elements of the quotient space $\mathcal{D} = D/N$ are pairs of functions $(u, \partial u/\partial n)$ defined on $\partial R$ and corresponding to the values of the function and its normal derivative. If the mapping $A: \mathcal{D} \to \mathcal{D}$ is defined so that $A(u, \partial u/\partial n) = (\overline{\partial u/\partial n}, -\overline{u})$, where the bar stands for the complex conjugate, then

$$(u,v) = \langle Au,Av\rangle = \int_{\partial R} \{\frac{\partial u}{\partial n}\frac{\overline{\partial v}}{\partial n} + u\overline{v}\}\, d\underset{\sim}{x} \qquad (2.12)$$

is an inner product and $\mathcal{D}$ is a Hilbert space with respect to this inner product. Taking the denumerable dense subset $\{y_\alpha \mid \alpha = 1, 2, \ldots\}$ of the curve C as before, the set $\{w_{y_\alpha} \mid \alpha = 1,2,\ldots\}$ is a denumerable basis for the functions which are harmonic on R. We would like to choose the norm in region R so that convergence in the norm associated with the inner product (2.14), implies convergence on R. Millar [8] has given related results. However, the definition of the mapping $A$ presents complications that will be discussed in more detail in a complete exposition that is being prepared.

## 3. PROBLEM OF CONNECTING

In this section an abstract problem motivated by the problem of connecting solutions of partial differential equations defined on neighboring regions such as R and E in Fig. 1, will be formulated.

Let $D_R$ and $D_E$ be two linear spaces and define $\hat{D} = D_R \oplus D_E$ where $\oplus$ stands for the outer sum operation. Thus, elements $\hat{u} \in \hat{D}$ are pairs $(u_R, u_E)$ such that $u_R \in D_R$ and $u_E \in D_E$. Consider an operator $\hat{P}: \hat{D} \to \hat{D}^*$ with the additive property

$$\langle\hat{P}\hat{u},\hat{v}\rangle = \langle\hat{P}u_R,v_R\rangle + \langle\hat{P}u_E,v_E\rangle \ . \qquad (3.1)$$

Let $\hat{P}_R: \hat{D} \to \hat{D}^*$ and $\hat{P}_E: \hat{D} \to \hat{D}^*$ be defined by

$$\langle\hat{P}_R\hat{u},\hat{v}\rangle = \langle\hat{P}u_R,v_R\rangle; \ \langle\hat{P}_E\hat{u},\hat{v}\rangle = \langle\hat{P}u_E,v_E\rangle \ . \qquad (3.2)$$

Then

$$\hat{P} = \hat{P}_R + \hat{P}_E \text{ and } \hat{A} = \hat{A}_R + \hat{A}_E \ , \qquad (3.3)$$

where $\hat{A}_R = \hat{P}_R - \hat{P}^*_R$ and $\hat{A}_E = \hat{P}_E - \hat{P}^*_E$ .

As an example, take the spaces $D_R$ and $D_E$ in a manner similar to the example in Section 2, and define

$$\langle\hat{P}_R\hat{u},\hat{v}\rangle = \int_R v\nabla^2 u\, d\underset{\sim}{x} + \int_{\partial_1 R} u\frac{\partial v}{\partial n}\, d\underset{\sim}{x} - \int_{\partial_2 R} v\frac{\partial u}{\partial n}\, d\underset{\sim}{x}$$

and $\hat{P}_E$ replacing R by E in (3.4). Then

$$\langle\hat{A}\hat{u},\hat{v}\rangle = \int_{\partial_3 R} \{[u - \frac{\partial v}{\partial n}] - [v\frac{\partial u}{\partial n}]\}\, d\underset{\sim}{x}$$

where $\partial_3 R = \partial_3 E$ is the common boundary between R and E (Fig. 1), and the square brackets stand for the difference of the limiting values on E and on R; e.g., $[u] = u_E - u_R$.

In order to be able to formulate the problem of connecting, it is necessary to have a criterion of smoothness across the connecting boundary. General properties of criteria considered in the theory are given next.

*Smooth elements will be characterized by a subset* $S \subset \hat{D}$. *Elements* $\hat{u} = (u_R,u_E) \in S$ *will be said to be smooth. When* $\hat{u} = (u_R,u_E)$ *is smooth,* $u_R \in D_R$ *is said to be a smooth extension of* $u_E \in D_E$ *and conversely. It will be assumed that* (i) $S$ *is a complete connectivity condition for* $\hat{P}$; *and* (ii) *every* $u_R \in D_R$ *possesses at least one smooth extension* $u_E \in D_E$ *and conversely.*

In the example considered previously, the set $S = \{\hat{u} \in \hat{D} \mid u_R = u_E,\ \partial u_R \mid \partial_n = \partial u_E/\partial n, \text{ on } \partial_3 R\}$ defines a smoothness condition possessing the above mentioned properties.

When a smoothness condition $S$ is given, it is possible to define the problem of connecting. *Given* $\hat{U} \in \hat{D}$ *and* $\hat{V} \in \hat{D}$, *an element* $\hat{u} \in \hat{D}$ *is said to be a solution of this problem, if* $\hat{u}$ *is* *solution of the problem of diffraction, with S as connectivity condition.* Therefore is solution of the problem of connecting when

$$\hat{P}\hat{u} = \hat{P}\hat{U} \quad \text{and} \quad \hat{u} - \hat{V} \in S \ . \tag{3.6}$$

Applying (3.6) to our example, it is seen that the first equation there is tantamount to

$$\nabla^2 u_R = \nabla^2 U_R, \text{ on } R \ ; \ \nabla^2 u_E = \nabla^2 U_E \ , \tag{3.7a}$$

$$u_R = U_R, \text{ on } \partial_1 R; \quad u_E = U_E, \text{ on } \partial_1 E \ , \tag{3.7b}$$

$$\partial u_R/\partial_n = \partial U_R/\partial n, \text{ on } \partial_2 R; \ \partial u_E/\partial_n = \partial U_E/\partial n, \text{ on } \partial_2 E, \tag{3.7c}$$

while the second condition holds if and only if

$$[u] = [V]; \ [\partial u/\partial n] = [\partial V/\partial n], \text{ on } \partial_3 R \ . \tag{3.8}$$

## 4. VARIATIONAL PRINCIPLES

A few examples of general variational principles that can be obtained for the diffraction problem and the problem of connecting, are given in this section. Any pretense of exhaustivity will be left aside; among the variational principles that will not be discussed here, extremal and dual extremal principles deserve to be mentioned. However, those given in this section can readily be applied to problems with discontinuous fields. Alternative forms were presented previously [9, 10], and a more systematic discussion is being prepared.

It can be shown that when the problem of diffraction satisfies existence, there exists an operator B: $D \to D^*$ such that

(1) $Bu = 0 \leftrightarrow u \in I$ ;

(2) $A = B - B^*$ (4.2

(3) B and B* can be varied independently; more precisely, given any $U \in D$ and $V \in D$, $\exists\, u \in D \ni Bu = BU$ and $B^*u = BV$ ;

(4) $u \in D$ is solution of the problem of diffraction, if and only if

$(P-B)u = PU - BV$ ;

(5) $P - B$ is symmetric;

(6) $\Omega'(u) = 0$ if and only if u is solution of the problem of diffraction, where

$$\Omega(u) = \frac{1}{2} \langle (P-B)u,u\rangle - \langle PU-BV,u\rangle \ . \tag{4.4}$$

This last result follows from (4) and (5). Indeed (4) and (5) together show that the problem of diffraction can be formulated in terms of a symmetric operator. Equation (4.4) follows from a general result given by Herrera [14] which is essentially Ritz formula for this kind of operator.

For the problem of connecting the above results imply that when this problem satisfies existence, there exists $\hat{J}: \hat{D} \to \hat{D}^*$ such that

(1) $\hat{J}\hat{u} = 0 \leftrightarrow \hat{u} \in S$ ; (4.5)

(2) $\hat{A} = \hat{J} - \hat{J}^*$ ; (4.6)

(3) $\hat{J}$ and $\hat{J}^*$ can be varied independently;

(4) $\hat{u} \in \hat{D}$ is a solution of the problem of connecting, if and only if

$(\hat{P} - \hat{J})\hat{u} = \hat{P}\hat{U} - \hat{J}\hat{V}$ ; (4.7)

(5) $\hat{P} - \hat{J}$ is symmetric;

(6) $\Omega'(u) = 0$ if and only if $\hat{u}$ is a solution of the problem of connecting; here

$$\Omega(\hat{u}) = \frac{1}{2} \langle (\hat{P}-\hat{J})\hat{u},\hat{u}\rangle - \langle \hat{P}\hat{U}-\hat{J}\hat{V},\hat{u}\rangle \ .$$

Property (1) shows that it is appropriate to call $\hat{J}: \hat{D} \to \hat{D}^*$ the jump operator. Indeed, it is natural to say that two elements $\hat{u} \in \hat{D}$ and $\hat{v} \in \hat{D}$ have the same jump when $\hat{u} - \hat{v} \in S$. Thus, property (1) shows that two elements $\hat{u}, \hat{v}$ have the same jump if and only if $\hat{J}\hat{u} = \hat{J}\hat{v}$.

As an example, let us obtain variational principles for linear static elasticity with discontinuous fields in the region $R \cup E$ of Fig. 1. Assuming that the only admissible jumps are on $\partial_3 R = \partial_3 E$, take

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \int_{R \cup E} v_i \frac{\partial \tau_{ij}}{\partial x_j}(\underset{\sim}{u}) d\underset{\sim}{x} + \int_{\partial_1 (R \cup E)} u_i T_i(\underset{\sim}{v}) d\underset{\sim}{x} - \int_{\partial_2 (R \cup E)} v_i T_i(\underset{\sim}{u}) d\underset{\sim}{x}$$

Then $\hat{J}: \hat{D} \to \hat{D}^*$, given by

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{[u_i] \overline{T_i(\underset{\sim}{v})} - \overline{v}_i [T_i(\underset{\sim}{u})]\} dx \ , \tag{4.10}$$

has the properties (1) to (6). The functional in the variational principle is given by Eq. (4.8). Here

$$\tau_{ij}(\underset{\sim}{u}) = C_{ijpq} \frac{\partial u_p}{\partial x_q} ; \ T_i(\underset{\sim}{u}) = \tau_{ij}(\underset{\sim}{u}) n_j \ , \tag{4.11}$$

where $C_{ijpq}$ is the elastic tensor, n is the unit normal to $\partial_3 R = \partial_3 E$ points outwards from R, the brackets [ ] stand for the jumps (taken as before), and the bar is used for the average across the boundary.

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