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SCATTERING OF SH WAVES BY SURFACE CAVITIES OF ARBITRARY SHAPE USING BOUNDARY METHODS

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In this paper a boundary method is used to numerically solve the problem of scattering of SH waves by a bounded surface cavity or arbitrary shape in a half-space. This method reduces the dimension of the problem by one, but avoids the introduction of singular integral equations. A close connection is established between this method and least-squares collocation. Results are obtained using a multipole expansion in terms of Hankel functions about the origin. Comparison with some known exact solutions for SH wave motion yields very good agreement. It is observed that, in the case of a trench with steep walls, local amplification factors can sometimes significantly exceed 100%.

1. Introduction

In earthquake engineering and strong-motion seismology, the surface motion at a given site due to incoming seismic waves is of interest. For rough topography this problem may be approached as one of scattering and diffraction of elastic waves by departures from flatness. Owing to its mathematical complexity the problem has not been completely solved. There are a few known exact solutions (Trifunac, 1973; Wong and Trifunac, 1974). Also, some approximate solutions have been obtained, for example by singular perturbations (Sabina and Willis, 1975, 1977) for scatterers of arbitrary shape, with approximations valid for long wavelengths. Other approximate methods assume the scattering surface to have small slope, and dimensions comparable with the wavelength of the incident wave (Gilbert and Knopoff, 1960; McIvor, 1969; Aki and Larner, 1970; Bouchon, 1973; Bouchon and Aki, 1977a, b; Hudson, 1977).

Therefore, it is of interest to search for other numerical methods, suitable for scatterers of arbitrary shape, and with dimensions similar to the wavelength of the incident wave. Indeed, many such solutions have been produced (Boore, 1972a, b; Smith, 1975; Wong and Jennings, 1975; Ilan, 1977; Singh and Sabina, 1977; Ilan et al., 1979; Sabina et al., 1978; Sánchez-Sesma, 1978; Sánchez-Sesma and Rosenblueth, 1978, 1979; Sills, 1978). In particular, finite difference and finite element methods introduce artificial boundaries and contaminating reflected waves, while some others assume a periodic repetition of the scatterer in space, in order to reduce the unbounded region to a finite one.

In this paper, consideration is given to the problem of scattering of a plane harmonic SH wave, incident upon a bounded surface cavity in an otherwise plane traction-free boundary. In the method of solution, the scattered field is represented as a linear combination of known solutions of the boundary value problem for the unmodified half-space. The coeffi-

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cients are chosen to minimize the mean-square error in the boundary condition on the cavity. The development of this approach has been guided by a recently developed theory of connectivity (Herrera, 1977a, b), which allows a systematic formulation of boundary methods (Herrera, 1978), applicable to many other problems, and which leads naturally to the complete system of functions used here for the half-space (Herrera and Sabina, 1978). However, the corresponding system of functions for problems formulated in the whole space has been used extensively in the null-field method of acoustic and electromagnetism (Bates, 1975). Minimization of the mean-square error on the boundary is a standard technique (Kantorovich and Krylov, 1964; Collatz, 1960) whose implications have not been fully realized until recently. For a similar problem, Millar (1973) has shown that when the mean-square error is minimized on the boundary, the resulting representation converges uniformly to the solution of the problem, provided that a complete set of functions is chosen, as is the case in the present work. The procedure presented here is a boundary method, and as such it has the advantage of reducing the dimensionality of the problem. In connection with electromagnetic scattering problems, this method was proposed by Yasuura (1971), and was used by Meecham (1956) and Ikuno and Yasuura (1973) for a periodic surface and by Yasuura and Ikuno (1971) for a bounded scatterer. A related least-squares boundary method is given by Davies (1973).

2. Statement of the problem

Consider a two-dimensional half-space, $y > 0$ (as in Fig. 1), consisting of two parts: an unbounded region E, and a bounded region R which contains the origin of coordinates O. Let the boundary of E be ∂E and the common boundary between R and E be S. The surface S is assumed smooth in the sense of having a continuously turning unit normal vector $\hat{n}$. The unbounded region E is filled with a linear elastic, homogeneous, isotropic medium of density ρ and rigidity μ; shear wave velocity is $\beta = (\rho/\mu)^{1/2}$. On the other hand, the bounded region R, having a characteristic horizontal linear dimension $2a$, is empty. The surface ∂E is traction-free.

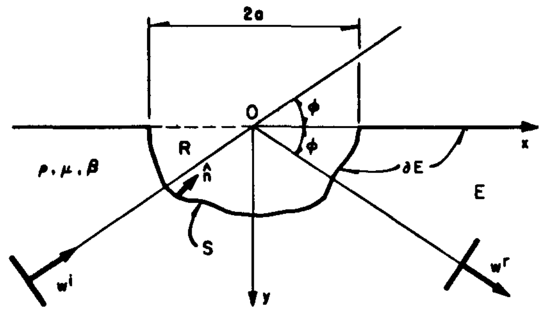

Fig. 1. Illustrative topography of two-dimensional half-space.

A state of antiplane shear deformation and harmonic motions in time is considered, so that the only non-zero component of the displacement is $w = w(x,y) \exp(-i\omega t)$ in the z (out-of-plane) direction, where ω is the circular frequency. Consider a plane harmonic SH wave w^i of unit amplitude, incident upon R at an angle ϕ measured with respect to the x-axis (Fig. 1), where

$$w^i = \exp[ik(-x \cos\phi + y \sin\phi)]$$

with $k = \omega/\beta$. For convenience, the factor $\exp(i\omega t)$ is omitted here and henceforth.

In the absence of the scattering region R, i.e., if the surface were flat, a reflected plane harmonic SH wave

$$w^r = \exp[-ik(x \cos\phi + y \sin\phi)]$$

would arise, so that a total field $w^0 = w^i + w^r$ would be produced in $y > 0$.

In the presence of R, the scattered wave w, produced by R, is sought, such that the total field w^t may be written as

$$w^t = w^0 + w \text{ in } E$$

Thus the scattered field satisfies the following boundary value problem

$$\nabla^2 w + k^2 w = 0 \text{ in } E \tag{1}$$

$$\partial w/\partial n = -\partial w^0/\partial n \text{ on } \partial E \tag{2}$$

and w satisfies Sommerfeld's outward radiation condition at infinity, where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in two dimensions, and $\partial/\partial n$ is the derivative in the direction of the outward normal $\hat{n}$ to ∂E. It may be observed that the right-hand side of eq. 2 is, in general, non-zero only on S, i.e. the common boundary between R and E.

Once w is found, the total field w^t can be determined. In strong-motion seismology it is of interest to find the surface field (i.e. w^t on ∂E), especially on and near S. A method for doing this is described here.

3. Method of solution

Consider a set of linearly independent functions

$v_i(\mathbf{x})$ $(i = 0, 1, 2, ...)$

not necessarily orthogonal, that have the following properties.

(i) Each v_i is a solution of the partial differential equation (1) in E.

(ii) Each v_i fulfils $\partial v_i/\partial n = 0$ on $y = 0$.

(iii) Each v_i satisfies a Sommerfeld outgoing radiation condition at infinity.

Let the Nth approximation to w be

$$w_N(\mathbf{x}) = \sum_{i=0}^{N} \alpha_i^N v_i(\mathbf{x}) \tag{3}$$

where the superindex N on the complex coefficients α_i^N has been introduced to make explicit their dependence on the order of the approximation. Note that, because of the linearity of properties (i) to (iii), w_N also satisfies them for every $N = 0, 1, 2, ...$. However, the boundary condition given by eq. 2 is not necessarily satisfied on S. Hence the coefficients in (3) will be chosen so that the mean-square error on the boundary

$$\epsilon_N^2 = \int_S \left|\frac{\partial w_N}{\partial n} + \frac{\partial w^0}{\partial n}\right|^2 \mathrm{d}s \bigg/ \int_S \left|\frac{\partial w^0}{\partial n}\right|^2 \mathrm{d}s \tag{4}$$

is minimized. This approximating method may be referred to as a boundary method (Collatz, 1960, p. 28). Note that the dimension of the problem has been reduced by one.

By substituting (3) into (4) and minimizing the resulting expression, an $(N + 1) \times (N + 1)$ system of linear algebraic equations is obtained, which may be written as:

$$A\boldsymbol{\alpha} = \boldsymbol{f} \tag{5}$$

where the elements of the matrix A and the vector $\boldsymbol{f}$ are

$$a_{ij} = \int_S (\partial \bar{v}_{i-1}/\partial n)(\partial v_{j-1}/\partial n)\,\mathrm{d}s \tag{6}$$

$$f_i = -\int_S (\partial \bar{v}_{i-1}/\partial n)(\partial w^0/\partial n)\,\mathrm{d}s \tag{7}$$

The bar stands for the complex conjugate. The matrix $A = [a_{ij}]$ is Hermitian and positive definite – a numerically advantageous fact.

In general, it is not possible to find closed forms of the integrals (6) and (7). Thus it is necessary to use numerical integration to obtain values of the coefficients (6) and the inhomogeneous term (7) before solving the system (5). By choosing an appropriate M-point quadrature rule with weights p_k and abscissae s_k, the following approximations are obtained

$$a_{ij} = \sum_{k=1}^{M} p_k(s_k)(\partial \bar{v}_{i-1}/\partial n)(s_k)(\partial v_{j-1}/\partial n)(s_k)$$

$$f_i = -\sum_{k=1}^{M} p_k(s_k)(\partial \bar{v}_{i-1}/\partial n)(s_k)(\partial w^0/\partial n)(s_k)$$

or, in matrix notation

$A = \bar{B}^T P B\,, \quad \boldsymbol{f} = \bar{B}^T P \mathbf{g}\,,$

where B is an $M \times (N + 1)$ matrix, B^T is its transpose, P is an $M \times M$ diagonal matrix, $\mathbf{g}$ is an M vector, and their elements are given by

$$b_{ij} = (\partial v_{j-1}/\partial n)(s_i)\,, \quad p_{ij} = p_i(s_i)\,\delta_{ij}\,,$$

$$g_i = -(\partial w^0/\partial n)(s_i)$$

Here, δ_{ij} are the elements of the unit matrix.

Hence the system (5) may be rewritten as

$$\bar{B}^T P B \boldsymbol{\alpha} = \bar{B}^T P \mathbf{g} \tag{8}$$

For given N, this system is solved for the coefficients α_i^N, and finally (3) gives the scattered field at any desired point in E, or, in particular on ∂E.

Before discussing the numerical details of the solution of (8), a close relationship with the collocation method will be established.

4. The collocation method

As above, it is assumed that the Nth approximation to w is given by (3). In this case, the remaining boundary condition on S is imposed directly on w_N at M points, in order to obtain the coefficients, i.e.

$$\sum_{i=0}^{N} \alpha_i^N \frac{\partial v_i}{\partial n}(s_k) = -\frac{\partial w^0}{\partial n}(s_k), \; s_k \in S \; (k = 1, 2, ..., M),$$

or, in matrix notation

$$B\boldsymbol{\alpha} = \boldsymbol{g}$$

This is pure collocation at M points. It is not clear where these points ought to be chosen, or how many of them should be used, but they are usually taken with a distribution which is as uniform as possible (Collatz, 1960, p. 29). The choice $M = N + 1$, suggested by the need for a determinate $(N + 1) \times (N + 1)$ system of algebraic equations, was used in electromagnetic theory under the name of a point matching method. Its advantages and disadvantages have already been widely discussed (for example, Lewin, 1970). Another possible choice is $M > N$. A least-squares solution of the overdetermined system may be found by minimizing $\bar{\mathbf{r}}^T\mathbf{r}$ where $\mathbf{r} = B\boldsymbol{\alpha} - \mathbf{g}$ is the residual vector of the system. It is well-known (Noble, 1969, p. 143) that $(\bar{B}^T B)^{-1}\bar{B}^T$ is the general ized inverse of B with the minimization property. When the solution thus found is not completely satisfactory, some equations are considered more important than others. It is usually a matter of experience to choose convenient positive weights for each equation. Thus a new system may be obtained

$$QB\boldsymbol{\alpha} = Q\boldsymbol{g} \tag{9}$$

where Q is an $M \times M$ diagonal matrix of positive elements. The least-squares solution of (9) is obtained after premultiplying it by $\bar{B}^T Q$ to obtain the generalized inverse $(\bar{B}^T Q^2 B)^{-1}\, \bar{B}^T Q$ of QB. When $P = Q^2$, the solution is identical with that of (8), which was derived differently. This shows a close connection between the series expansion in non-orthogonal functions, and the collocation method via least-squares. Furthermore, it gives a clearer picture of the meaning of weights which otherwise would need to be guessed. An analogous relation was given by Ikuno and Yasuura (1973) for a periodic surface.

5. Multipole expansion

So far, the set of functions $v_i(\mathbf{x})$ is as general as possible: to proceed further it is necessary to specify them.

Consider the set of cylindrical wave functions

$$v_i = H_i^{(1)}(kr)\cos(i\theta) \; (i = 0, 1, 2, ...) \tag{10}$$

where $H_i^{(1)}(kr)$ is the Hankel function of the first kind, of order i, and (r, θ) are the polar coordinates of the point (x, y). This set satisfies all the conditions (i) to (iii). Thus (3) is the $(N + 1)$th order multipole expansion of w in terms of Hankel functions about the origin.

This particular choice has interesting consequences: (a) the set of functions $\{\partial v_i/\partial n\}$ is complete with respect to all functions that are square-integrable on S; and (b) $\partial w_N/\partial n$ converges uniformly in S to $\partial w/\partial n$ in the mean-square sense. These facts can be proved by adapting the work of Millar (1973), who dealt with the full two-dimensional problem. Furthermore, w_N converges uniformly in E to the solution w of (1) and (2) in the mean-square sense.

It is worthwhile observing that when S is a semicircle of radius a, the functions $H_i^{(1)}(kr)$ and their derivatives are constant on S, and the set of basis functions $\{v_i\}$ or $\{\partial v_i/\partial n\}$ is orthogonal on S in the sense that the matrix A is diagonal. The coefficients α_i^N are then independent of N, and eq. 3 represents simply $N + 1$ terms of the trigonometric Fourier series for w at any fixed value of r.

When S departs slightly from a circle, it seems reasonable to expect that the set $\{\partial v_i/\partial n\}$ departs very little from the orthogonality condition on S in the sense that the matrix A is "numerically" diagonal, i.e. the magnitude of the diagonal elements is much larger than that of the off-diagonal elements.

6. Other expansions

Other sets of functions could be chosen. For instance

$$v_i(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_i) \;\; (i = 0, 1, 2, ...)$$

where each $\mathbf{x}_i$ is a distinct point in R and $G(\mathbf{x}, \mathbf{x}_i)$ is the Green's function for the Neumann problem corresponding to the Helmholtz equation in the half-space.

Such a set was used by Sánchez-Sesma and Rosenblueth (1978, 1979), in what is essentially a collocation method and least-squares solution.

A multipole expansion in terms of Mathieu functions is another possible set. Its completeness and uniform convergence in the mean-square sense may be established along similar lines to those indicated here. Many other sets are possible, but a complete study of all possible sets is beyond the scope of this paper.

7. Numerical considerations

Since the displacement w is a complex function, it is necessary to calculate its modulus and phase, as functions of the normalized abscissa x/a

$$|w| = [(Re\ w)^2 + (Im\ w)^2]^{1/2}$$

$$ph(w) = \tan^{-1}(Im\ w/Re\ w)$$

The latter is arbitrarily divided by a normalizing factor 2π, mainly to facilitate comparison with other known solutions. In addition, the appropriate value of $\tan^{-1}$ is chosen to make $ph(w)$ a continuous function, and a constant is subtracted to make its value zero at $x/a = 0$.

To deal with incident waves of different frequencies for fixed topography, a normalized frequency (or wavenumber), η is introduced, where

$$\eta = \omega a/\pi\beta = ka/\pi\ .$$

Evaluation of (4) provides a criterion for the accuracy achieved as a function of the order of the approximation N. Also, by repeating the method for different values of N, it may be concluded that sufficient accuracy has been obtained when two solutions differ on S by less than the required error bound.

In addition to the value of N, it is necessary to fix some integration rule for the evaluation of eqs. 6 and 7. In view of the factor $\cos i\theta$ in v_i, there are integrands in eq. 6 with up to N complete oscillations on S. Numerical experiments have given very poor results when the integration rule used fewer than $2N$ points, confirming a result from Yasuura and Ikuno (1971). Further, by varying the distribution of points, unsatisfactory results have been obtained, with fewer than two integration points in each cycle of the factor $\cos^2(N\theta)$ in b_{NN}. This suggests that θ is the most suitable parameter to use in the equation of the curve S.

A compound trapezoidal rule, with equal intervals in θ, clearly showed convergence to a single value for each of the α_i^N as the number of points increased above $2N$. However, to obtain values of α_i^N with 1% accuracy, as many as $10N$ points were needed, using that rule. Nevertheless, the same precision is attainable with fewer points using integration rules of higher order, provided the curve S, $r = r(\theta)$, is sufficiently smooth. In particular, with a compound nine-point Lobatto rule – coefficients from Stroud and Secrest (1966) – approximately $3N$ points were sufficient. A further increase in order did not appear productive, presumably because, with increasing irregularity of the distribution of integration points, more than $3N$ points are required to maintain two in each cycle for $\cos^2(N\theta)$.

Having evaluated eqs. 6 and 7, it remains to solve eq. 5. Since the matrix A is Hermitian and positive definite, it should be possible to factorize it into triangular factors $L \times U$ without row interchanges, with all the pivots (diagonal elements of the factors) positive if rounding errors do not accumulate seriously. The larger the diagonal elements of A compared to the off-diagonal elements, the smaller should be the accumulation of rounding error. Further, it is possible to choose a Cholesky type factorization, $U = L^*$, the asterisk indicating the transposed complex conjugate, or to factorize A in the form LDL^*, where L is lower triangular with a unit diagonal, and D is real diagonal. This last form has in fact been used.

Probably the most costly part of the computation is the calculation of the Hankel functions, $H_i^{(1)}(kr)$. The imaginary parts – Bessel functions of the second kind, $Y_i(kr)$ – have been evaluated using a routine from the IBM (1970) Scientific Subroutine Package, slightly modified to provide values simultaneously for $i = 0, 1, \ldots, N$. The real parts – Bessel functions of the first kind, $J_i(kr)$ – have been evaluated by a backwards recurrence algorithm, obtained by simplifying Algorithm 236 (Gautschi, 1964) for the special case of a positive, integer index (real argument). However, Bessel functions of the second kind, and, even more so, their derivatives, increase rapidly as the index increases:

$$\frac{\mathrm{d}Y_i(kr)}{\mathrm{d}r} \sim \frac{k}{2\pi}\left(\frac{2}{kr}\right)^{i+1} i! \text{ as } i \to \infty$$

These functions quite rapidly exceed the capacity of most computers. Therefore, the routines have been further modified to provide directly a normalized set of functions $H_i^{(1)}(kr)/i!$, for which eq. 10 becomes

$$v_i = [H_i^{(1)}(kr)/i!] \cos(i\theta) \quad (i = 0, 1, 2, \dots)$$

Nevertheless, if many of these basis functions are to be calculated, it would seem that very high precision is needed – possibly using double precision. Even with seven significant figures correct in the calculation of the Bessel functions $J_i(kr)/i!$, the matrix A was sometimes found to be ill-conditioned, or even to have negative pivots, for N greater than about 40.

It should be noted that the matrix A does not depend upon the angle of incidence ϕ. Thus a considerable economy of computation is achieved if several angles of incidence are analysed simultaneously for a given geometry and wavenumber k.

8. Results

In order to assess the method, it has been applied to two cases with known exact solutions: semi-cylindrical canyons (Trifunac, 1973); and semi-elliptical canyons (Wong and Trifunac, 1974).

In the case of the semi-cylindrical canyon, it was found that precision better than the 1% quoted by Trifunac (1973) could be obtained with as few functions (N + 1) as four (for η = 0.25) to 11 (for η = 2.0). Typically in these cases, the value of ϵ_N^2 was less than 5×10^{-4}, and $|\partial w_N/\partial n + \partial w^0/\partial n|/2k$ attained point values of the order of 10^{-2}.

For semi-elliptical canyons, Wong and Trifunac (1974) give the solution as a series involving Mathieu functions, and as graphs of surface displacement amplitude and normalized phase vs. normalized distance (x/a) from –3 to 3, with four angles of incidence, 0, 30, 60 and 90°. Using the method described here, and multipole expansions in terms of Hankel functions, semi-elliptical canyons of aspect ratio b = 0.70 and 0.50 have been analysed. (b = ratio of minor to major axis. For each value of b, there is both a shallow and a deep canyon.)

For b = 0.70, using various values of N in the range 20 to 40, results coincided in general to within 0.05 of those given by Wong and Trifunac (1974), which is about 2.5%, and as good as the precision with which it is possible to read their graphs. Exceptions occurred for the larger values of η and $|x/a|$, where differences reached 0.31. (For details see Sabina et al., 1978.) However, by repeating the same cases with other values of N, it was concluded that errors were less than 0.005, and 0.0005 for the larger values in $|x/a|$, and that Wong and Trifunac had not taken sufficient terms in their series solution. Indeed, it appears that a 1% precision can be obtained with as few functions (N + 1) as 15 (for η = 0.50 (shallow) or η/b = 0.50 (deep)) to 21 (for η = 2.0 (shallow) or η/b = 2.0 (deep)). Typically, once more, $\epsilon_N^2 < 10^{-3}$ and $|\partial w_N/\partial n + \partial w^0/\partial n|/2k < 0.05$.

For b = 0.5, it appears that more than 40 basis functions are needed to obtain similar precision, but

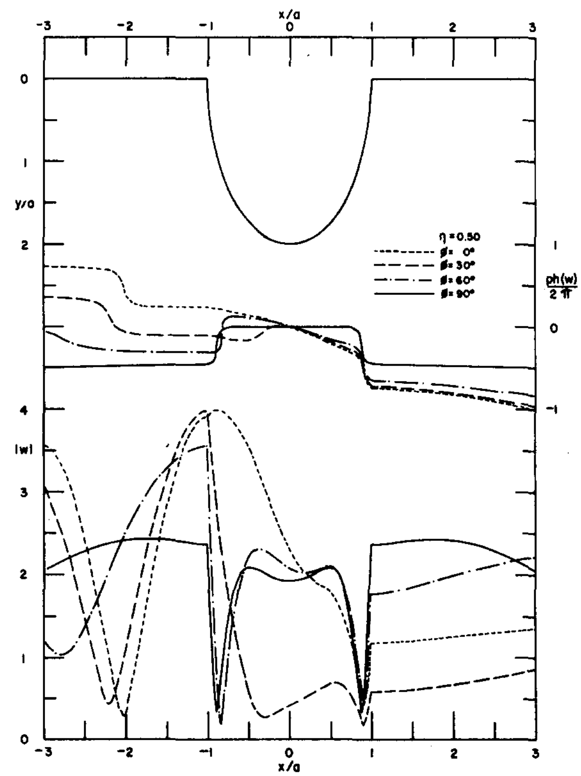

Fig. 2. Cavity shape (top), normalized displacement phase $ph(w)/2\pi$ (centre) and normalized displacement amplitude $|w|$ (bottom) against normalized distance x/a. Note that the units of shape and amplitude are shown on the left, and units of the phase on the right. The cavity is a deep semi-ellipse of b = 0.50 with η = 0.50, and four angles of incidence ϕ = 0, 30, 60 and 90°.

with $N = 40$ in the deep case, results are still indistinguishable from the graphs of Wong and Trifunac (1974), giving $\epsilon_N^2 < 2\times10^{-2}$ and $|\partial w_N/\partial n + \partial w^0/\partial n|/2k$ of the order of 10^{-1}. Judging from the values of ϵ_N^2, the results are even better than this in the shallow case. However, there are no published results for comparison.

It may be concluded that the basis used is very good for semi-elliptical cavities of aspect ratio $b \geqslant 0.50$ for the range of frequencies examined $\eta \leqslant 2.0$ ($\eta/b \leqslant 2.0$ in deep cases), and probably also for somewhat larger values of η. By way of example, graphs of cavity shape, displacement amplitude, and normalized displacement phase, plotted against normalized distance x/a, for $b = 0.50$ and $\phi = 0$, 30, 60 and 90°, are given in Fig. 2 (deep cavity: $\eta = 0.50$, $N = 40$), Fig. 3 (shallow cavity: $\eta = 0.50, N = 38$) and Fig. 4 (shallow cavity: $\eta = 2.0, N = 40$).

As the usefulness of the basis given by (10) has been demonstrated, it was further applied to a family of trenches, with vertical walls of depth d at $x = \pm a$

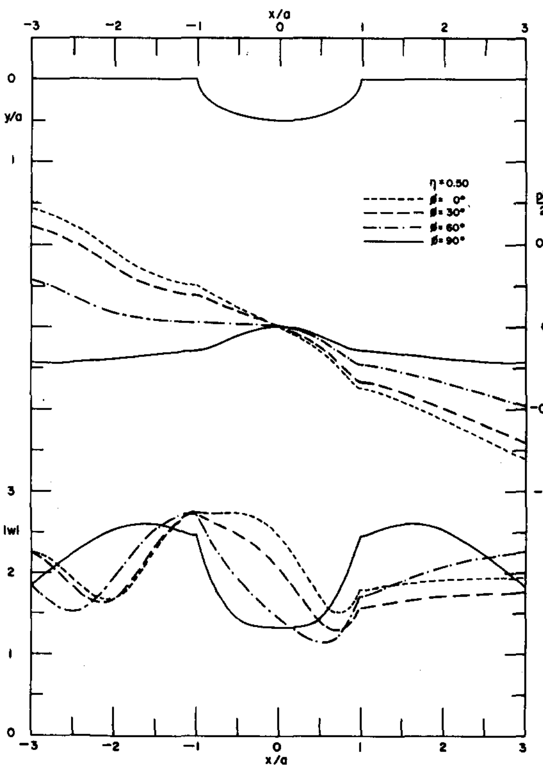

Fig. 3. As Fig. 2, but for a shallow semi-ellipse.

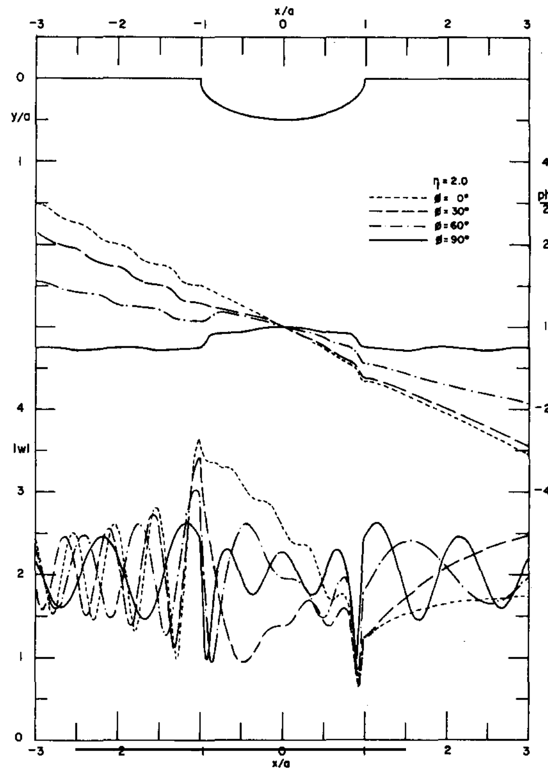

Fig. 4. As Fig. 3, but for $\eta = 2.0$.

and curved floors, using

$$r = a[\cos^2\theta + (d^2\cos^2\theta - \sin^2\theta)^2]^{-1/2}, \quad |\tan\theta| > d$$

When $d = 0.0$, there is no vertical wall as such, but the steep slope is more abrupt than for the semi-cylindrical cavity (see Fig. 5).

The results obtained for two trenches and different frequencies are displayed in Fig. 5 ($d = 0.0$, $\eta = 1.0, N = 16$), Fig. 6 ($d = \tan 30°, \eta = 1.0, N = 36$) and Fig. 7 ($d = \tan 30°, \eta = 1.50, N = 36$). Since the trench with $d = \tan 30°$ has vertical walls at $|x/a| = 1$, the plots of surface displacement (or normalized phase) against normalized distance x/a show intervals in the abscissae, marked as thick lines, which correspond to the vertical wall precisely at $|x/a| = 1$. The values of N used were such as to give $\epsilon_N^2 < 6\times10^{-3}$, and $|\partial w_N/\partial n + \partial w^0/\partial n|/2k < 0.05$.

Qualitatively similar results to those for semi-cylindrical and semi-elliptical cavities were observed. However, it is of interest to note that, for waves propagating towards the right-hand side, the amplitude of

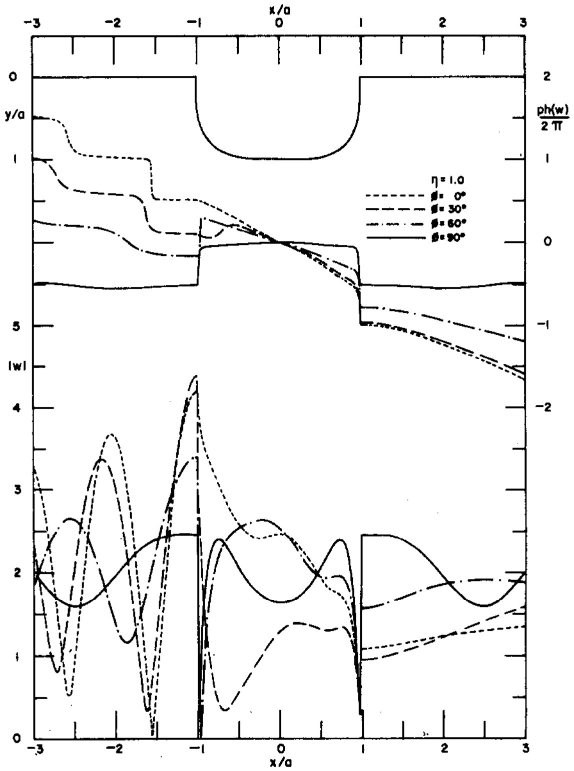

Fig. 5. As Fig. 2, but for $\eta = 1.0$ and a trench with $d = 0.0$.

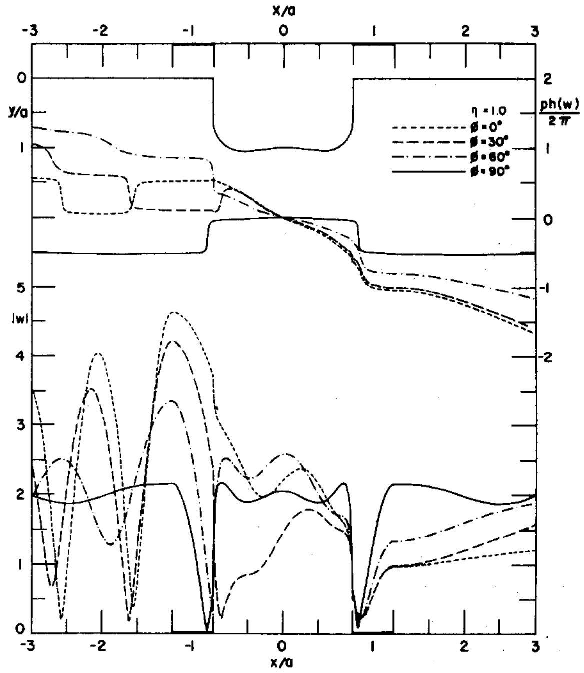

Fig. 6. As Fig. 5, but for $d = 0.58$ and $\eta = 1.0$. Note that the thick lines in the abscissae correspond to the vertical wall at precisely $|x/a| = 1$.

the scattered wave near the left-hand wall can considerably exceed that of w^0, the sum of the incident and reflected plane waves.

This effect is also observed in some phase curves, which do not tend to the linear relationship

$$ph(w) = -\pi\eta(x/a) \cos \phi$$

which holds in the absence of the cavity. However, the phase diagrams for both the semi-cylindrical and semi-elliptical cavities do tend to that linear form for large $|x/a|$.

Finally, it should be noted that this method converges to the correct results, without any limitation on the topographic slope, such as that implicit in methods involving the Rayleigh hypothesis (Aki and Larner, 1970; Bouchon, 1973; Bouchon and Aki, 1977 a,b). Lack of accuracy of such a method for steep slopes was reported by Boore (1972 b).

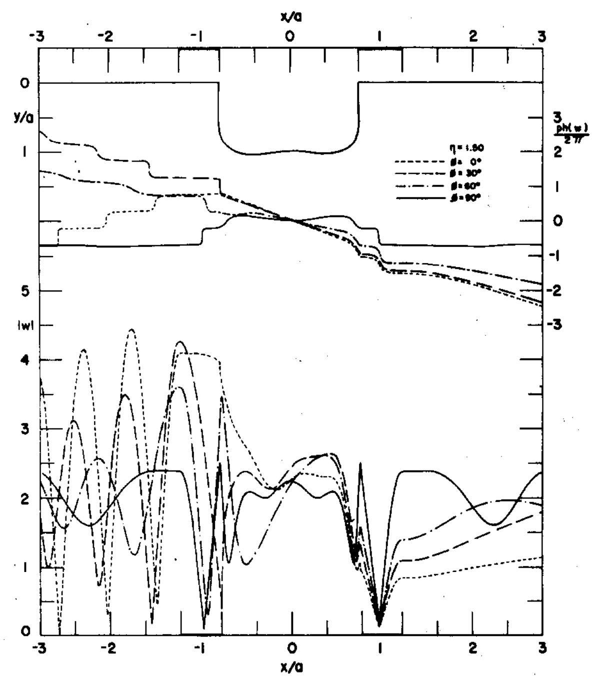

Fig. 7. As Fig. 6, but for $\eta = 1.50$.

9. Conclusions

The problem of the scattering of SH waves by bounded surface cavities in a half-space with an otherwise plane traction-free surface is solved numerically using a boundary method. The scattered field is represented as a finite linear combination of a given complete non-orthogonal basis: each basis function is a solution of the boundary value problem in the absence of the cavity. The coefficients are chosen by minimizing the mean-square error in the boundary condition on the cavity.

A relation between the least-squares collocation method and the convergence in the mean of a series expansion in non-orthogonal basis functions is established.

The range of usefulness of the basis chosen – multipole expansion in terms of Hankel functions about the origin – has been established. It owes its suitability to the numerical quasi-orthogonality of the basis when the cavity is not very different from a circle.

Other bases may be more convenient for elongated shapes. For example, a multipole expansion in terms of Mathieu functions, but their numerical advantages have not yet been demonstrated,

Finally, it is shown that trenches with near-vertical walls can produce local amplification factors of significantly more than 100%.

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