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# Variational Principles for Problems with Linear Constraints : Prescribed Jumps and Continuation Type Restrictions

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## Variational Principles for Problems with Linear Constraints: Prescribed Jumps and Continuation Type Restrictions<sup>†</sup>

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An abstract theory of problems subjected to linear constraints is developed. It supplies a general framework for boundary methods that are the subject of extensive research at present, as a tool to study numerically partial differential equations associated with many problems of science and engineering. Two kinds of general problem are considered, one for which the solutions are required to satisfy prescribed jumps and the other for which solutions can be continued smoothly into neighbouring regions as functions that satisfy given equations. The general theory is developed systematically, but only applications to variational principles are reported here. In previous papers the possibility of using this theory to discuss more general questions has been suggested; such applications will be discussed thoroughly in a further paper that is being prepared.

#### 1. Introduction

BOUNDARY METHODS for treating numerically partial differential equations associated with many problems of science and engineering are currently receiving attention. Most frequently, boundary methods have been formulated by means of integral equations based on Maxwell Betti's formula (Brebbia, 1978; Cruse & Rizzo, 1975; Cruse, 1974 and Rizzo, 1967). Alternatives have been considered by some authors such as Heise (1978), Sabina, Herrera & England (1978), Sanchez-Sesma & Rosenblueth (1978) and Kupradze, Gegelia, Baschelejschwili & Burtschuladge (1976). The main advantages of boundary methods stem from a reduction of the dimensions involved in the problems.

In very general terms, one can say that the general solution which is used for the formulation of boundary methods may depend on a continuous or, alternatively, on a discrete parameter. In the first case the family is usually prescribed by means of singular solutions and the sought solution is constructed using integral representations. Usually the boundary of the region represents the domain of definition for the kernel in the integral but other approaches are possible (Heise, 1978; Rieder, 1962, 1968; Kupradze, 1965; Kupradze *et al.*, 1976 and Oliveira, 1968).

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When a denumerable family of solutions is used one is led to series representations, or more generally to a sequence of least-squares approximations (Millar, 1973). The series expansion method has been used most extensively in acoustics and electromagnetic field computations (Bates, 1975). Applications of this procedure have also been made in other fields such as free-surface flows (Mei & Chen, 1976) and seismology (Sabina, Herrera & England, 1978). A technique recently applied by Sanchez-Sesma & Rosenblueth (1978), may be thought as a transition between the singularity and the series expansion methods.

Some of the alternative formulations have clear numerical advantages such as the avoidance of singular integral equations. However, there has been a lack of clarity in the application of these methods and many questions are not well understood. For a class of integral representations, Oliveira (1968) showed that severe restrictions which apparently are required for the applicability of the method, are not actually needed in order to solve problems successfully. The "Rayleigh hypothesis" restricts drastically the applicability of the series expansion method in acoustics and electromagnetic field computations (Bates, 1975). However, this assumption can be avoided altogether if a different point of view is adopted (Millar, 1973).

When boundary methods are used to reduce the size of the region to be treated numerically, it is important to match this part with the rest of the space efficiently and this can be done using variational principles. For cases such as diffraction problems, in which the regions considered are unbounded, the associated variational principles have the interesting property in that the corresponding functionals involve a bounded region only (Mei & Chen, 1976).

A general theory of problems subjected to linear restrictions or constraints, recently developed (Herrera, 1977*a*, *b*, *c*; Herrera, 1978*a*, *b*; and Herrera & Sabina, 1978) is presented in this paper. This theory supplies a unified approach to boundary methods.

In Section 2, valued functional operators and the general problem with linear constraints, are introduced; regular and completely regular constraints are also defined.

In Section 3, canonical decompositions of a linear space D, are defined and their relation with problems with linear restrictions is exhibited.

In Section 4, the concept of an operator B that decomposes A is introduced. A oneto-one correspondence between operators that decompose A and canonical decompositions of D is established.

In Section 5, the problem of connecting is introduced. This is an abstract version of a problem posed on a region such as  $R \cup E$  in Fig. 1, where R and E are neighbouring sub-regions, subjected to a prescribed smoothness criterion across the common boundary. In application, such a problem corresponds to a problem formulated in discontinuous fields and with prescribed jump conditions. It is shown that the existence of a solution for this problem grants that the set of functions that can be extended smoothly into solutions of the homogeneous equations on E (this set is here called a continuation type restriction), constitutes a linear sub-space that is completely regular for the equations on R. A survey of variational principles for problems with prescribed jump has been given by Nemat-Nasser (1972a, b).

In Section 6, two general variational principles for problems with linear restrictions

are formulated; one is relevant for problems with prescribed jumps and the other one for problems in which the boundary method is used to reduce the size of the region treated numerically.

Finally, in Section 7, applications are made to Laplace, reduced wave and heat equation. Applications to Elasticity are explained for static, periodic and dynamical problems. Also, an application to a two-phase problem is considered, in which region R (Fig. 1) is occupied by an inviscid liquid, while there is in E an elastic solid, as when a dam is filled. An application to free surface flows was given previously (Herrera, 1977a).

There are two theoretical questions which acquire great practical importance in specific applications; conditions under which a basic set of functions is complete and conditions which assure the convergence of the approximating procedure. The theory presented here can be used to discuss these matters. Indeed, completely regular constraints can be characterized by connectivity bases that were introduced in a previous paper, where a general method for constructing such bases was also developed (Herrera & Sabina, 1978). Furthermore, the notion of connectivity basis can be related with that of Hilbert space basis (Herrera, 1978b). When this is possible, a connectivity basis becomes a Hilbert space basis and the completeness of the basic set of functions is established. Once this has been shown, a procedure similar to one applied by Kantarovich & Krylov (1964, pp. 44–68) to Laplace equation, can be used to choose the coefficients of the linear combinations in a manner that assures the uniform convergence of the approximating sequence.

In this paper applications of the theory have been restricted to variational principles, leaving the discussion of the questions of completeness and convergence for a further paper now being prepared.

As in previous work by the author (Herrera, 1974; Herrera & Bielak, 1976 and Herrera & Sewell, 1978), functional valued operators are used systematically, because they have been demonstrated to be suitable for the discussion of questions related to differential and integral equations. Indeed, functional valued operators supply a very flexible language which permits treating problems with generality, simplicity, clarity and rigor. In this respect, the author hopes that this article will estimulate more extensive use of Functional Analysis to treat questions relevant in specific applications, because it shows that notions of a relatively elementary nature, and therefore within the grasp of a larger audience, can be used to achieve those desired features.

Some of the theorems take as an assumption, the existence of solution of the abstract problems considered. In specific applications this hypothesis requires taking



the linear space on which the operators are defined, so as to satisfy it. There are treatises available which discuss thoroughly, questions of existence of solutions for partial differential equations (Lions & Magenes, 1968; see also, Babuska & Aziz, 1972).

The terminology of the theory has been revised; the problem with linear restrictions had been called in previous papers, problem of diffraction. Regular and completely regular sub-spaces, were called before, connectivity and complete connectivity conditions, respectively. It was felt that these changes were necessary because the former terminology had been suggested by specific applications, and apparently, was misleading at the more general level that the theory has achieved.

## 2. Problems with Linear Restrictions

In what follows F is the field of real or, alternatively, of complex numbers. Let D be a linear space and D\* its algebraic dual; i.e. D\* is the set of linear functionals defined on D. With the usual algebraic structure, D\* is itself a linear space. In this paper attention is restricted to operators  $P: D \to D^*$  which are linear. The value  $P(u) \in D^*$ of P at  $u \in D$ , is a linear functional. Write  $\langle P(u), v \rangle \in F$  for the value of the functional  $P(u) \in D^*$  at  $v \in D$ . When P is linear, it is customary to drop the parenthesis in P(u), and in this case the operator  $P: D \to D^*$  is uniquely determined by the bi-linear functional  $\langle Pu, v \rangle$ . In this case, the adjoint operator  $P^*: D \to D^*$  always exists and it is defined by means of the transposed bi-linear functional  $\langle Pv, u \rangle$ . Attention will be restricted to linear operators  $P: D \to D^*$ .

There are many problems that can be cast in the following framework.

Definition 2.1. Consider  $P: D \to D^*$  and a subspace  $I \subset D$ . Given  $U \in D$  and  $V \in D$ , an element  $u \in D$  is said to be a solution of the problem with linear restrictions or constraints, when

$$Pu = PU$$
 and  $u - V \in I$ . (2.1)

As an example, consider the operator  $P: D \to D^*$  defined by

$$\langle Pu, v \rangle = \int_{R} v \nabla^2 u \, d\mathbf{x}$$
 (2.2)

where region R is illustrated in Fig. 2. There are many ways of choosing D, since it is only required to be a linear space without any further structure. For definitiveness, one may think of D as being the Sobolev space  $H^{s}(R)$ ;  $s \ge 2$  (Babuska & Aziz, 1972). Define the linear sub-space  $I \subset D$  by

$$I = \{ u \in D | u = 0, \text{ on } \partial R \}$$

$$(2.3)$$

Then, problem (2.1) is Poisson's equation

 $\nabla^2 u = \nabla^2 U = f_R; \quad \text{on } R$ 

subjected to boundary conditions of Dirichlet type

$$u = V = f_{\partial R}$$
; on  $\partial R$ .



In the formulation given here, the functions  $f_R$  and  $f_{\partial R}$  may be defined by means of equations (2.4) when  $U \in D$  and  $V \in D$  are given. Although it is more common to give  $f_R$  and  $f_{\partial R}$  as data of the problems, the use of  $U \in D$  and  $V \in D$  gives notational advantages when carrying out the general development of the theory.

Given  $P: D \to D^*$  it is possible to define  $A: D \to D^*$  by

$$A = P - P^*, \tag{2.5}$$

because  $P^*$  always exists. The null sub-space  $N_A$  of A will be denoted by

$$N = \{ u \in D | Au = 0 \}.$$
(2.6)

Definition 2.2. A sub-space  $I \subset D$  is said to be regular for P, when

(a)  $I \subset D$  is a commutative subspace of P; i.e.

$$\langle Au, v \rangle = 0 \quad \forall \ u, v \in I.$$
  
 $N \subset I.$ 

Regular sub-spaces frequently have the following additional property

(c) For every  $u \in D$ , one has

$$\langle Au, v \rangle = 0 \quad \forall \ v \in I \Rightarrow u \in I.$$
 (2.9)

A regular sub-space possessing property (c) will be said to be completely regular for P.

To illustrate this notion, it can be seen that in the previous example  $A: D \to D^*$  is given by

$$\langle Au, v \rangle = \int_{\partial R} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} d\mathbf{x}$$
 (2.10a)

and

(b)

$$N = \{ u \in D | u = \partial u / \partial n = 0; \text{ on } \partial R \}.$$
 (2.10b)

Therefore,  $I \subset D$  as defined by equation (2.3) is a regular sub-space for P; even more, it is completely regular.

Sub-spaces that are completely regular for P, can be characterized in a simple manner.

LEMMA 2.1. Let  $I \subset D$  be a linear sub-space. Then I is completely regular for the operator P, if and only if, for every  $u \in D$  one has

$$\langle Au, v \rangle = 0 \quad \forall \ v \in I \Leftrightarrow u \in I.$$

$$(2.11)$$

*Proof.* Observe that condition (2.11) is the conjunction of properties (a) and (c). Thus, it is enough to prove that when  $I \subset D$  satisfies (2.11),  $N \subset I$ . This is immediate, because any  $u \in N$  satisfies the premise in (2.11).

With every linear operator  $P: D \rightarrow D^*$ , it is possible to associate a sub-space  $I_P$  that is regular for P. It is defined by

$$I_P = N + N_P$$

where  $N_P$  is the null sub-space of P. The corresponding result follows.

LEMMA 2.2. The linear space  $I_P$  defined by equation (2.12) is a regular sub-space for P. Proof. Condition (2.8) is clearly satisfied by  $I_P$ . In order to show that (2.7) is also satisfied, given any  $u \in I_P$  and  $v \in I_P$ , write  $u = u_P + u_N$  and  $v = v_P + v_N$ , where  $u_P, v_P \in N_P$  while  $u_N, v_N \in N$ . Then

$$\langle Au, v \rangle = \langle Au_P, v_P \rangle = \langle Pu_P, v_P \rangle - \langle Pv_P, u_P \rangle = 0.$$
(2.13)

In view of the fact that N is a linear sub-space of D, it is possible to consider the quotient spaces  $\mathbf{D} = D/N$ ,  $\mathbf{I} = I/N$  and  $\mathbf{I}_P = I_P/N$ . The elements of these spaces are cosets. The space D will be referred to as the reduced space; in applications to boundary value problems the elements of D are characterized by boundary values of the functions of the corresponding cosets.

For the operator  $P: D \to D^*$  given by (2.2), N is given by (2.10b) and therefore, each coset of  $\mathbf{D} = D/N$  is characterized by a pair of functions  $\{u, \partial u/\partial n\}$  defined on  $\partial R$ .

Definition 2.3. The problem with linear restrictions (2.1), is said to satisfy

(a) Existence, when there is at least one solution for every  $U \in D$  and  $V \in D$ ;

- (b) Uniqueness, when U = 0 and  $V = 0 \Rightarrow u = 0$ ;
- (c) Almost uniqueness, when

$$U = 0$$
 and  $V = 0 \Rightarrow u \in N$ 

By a reduced solution or boundary solution, it is meant an element  $u \in D = D/N$  such that  $u - U^c \in I_P$  while  $u - V^c \in I$  where  $U^c$ ,  $V^c \in D$  stand for the cosets associated with U and V, respectively.

In applications to boundary problems almost uniqueness corresponds to uniqueness of suitable boundary values. For example, when N is given by (2.10b), the boundary values u and  $\partial u/\partial n$  are unique if almost uniqueness is satisfied.

The case when V = 0 in problem (2.1), will be called the basic problem. The properties given in Definition 2.3 depend on corresponding properties of the basic problem, only.

LEMMA 2.3. The problem with linear restrictions (2.1) satisfies existence, uniqueness or almost uniqueness, respectively, if and only if, the basic problem enjoys corresponding properties.

*Proof.* The proof follows from the fact that if  $w \in D$  is defined by w = u - V, with  $V \in D$  fixed, then

$$Pu = PU$$
 and  $u - V \in I \Leftrightarrow Pw = P(U - V)$  and  $w \in I$ .

There is a very straightforward result that will be used when formulating variational principles in Section 6. Let  $S: D \to D^*$  be symmetric and  $f \in D^*$ ; then,

$$Su = f \Leftrightarrow \Omega'(u) = 0 \tag{2.15}$$

where

$$\Omega(u) = \frac{1}{2} \langle Su, u \rangle - \langle f, u \rangle.$$
(2.16)

Here the derivative  $\Omega'$  of  $\Omega: D \to \mathscr{F}$  is taken in the sense of additive Gateaux variation (Nashed, 1971), which is probably the weakest definition of derivative. Relation (2.15) is essentially Ritz formula, it follows from the fact that when S is symmetric

$$\Omega'(u) = Su - f. \tag{2.17}$$

### 3. On the Occurrence of Canonical Decompositions

In this section it will be seen that there is frequently associated a pair of completely regular sub-spaces with the problem with linear restrictions (2.1).

Definition 3.1. Let  $I_1 \subset D$  and  $I_2 \subset D$  be two completely regular sub-spaces for P. Then the ordered pair  $(I_1, I_2)$  is said to constitute a canonical decomposition of D, with respect to P, when

$$I_1 + I_2 = D$$
 and  $I_1 \cap I_2 = N$ .

Clearly, a pair  $(I_1, I_2)$  of completely regular sub-spaces for P, is a canonical decomposition of D, if and only if, every  $u \in D$  can be written as

$$u = u_1 + u_2; \quad u_1 \in I_1, \quad u_2 \in I_2$$
 (3.2)

and this representation is almost unique in the sense that  $u_1 - u'_1 \in N$  and  $u_2 - u'_2 \in N$ whenever  $u'_1, u'_2$  is any other pair satisfying (3.2).

Going back to the example considered in Section 2, a canonical decomposition  $(I_1, I_2)$  of D, can be constructed by taking  $I_1$  as the sub-space given by equation (2.3) and

$$I_2 = \{ u \in D | \partial u / \partial n = 0, \text{ on } \partial R \}.$$
(3.3)

The interest of canonical decompositions springs from the fact that given a subspace  $I \subset D$ , which is regular for P, under very general assumptions, the pair  $(I, I_P)$ constitutes a canonical decomposition of D. The following discussion will be oriented to prove this fact. LEMMA 3.1. Let  $I \subset D$  be a regular sub-space for P. Assume the basic problem satisfies existence. Then, for every  $u \in I_P$ , we have

$$\langle Au, w \rangle = 0 \quad \forall \ w \in I \Rightarrow u \in N.$$
 (3.4)

*Proof.* In view of (2.12), every  $u \in I_P$ , can be written as  $u = u_P + u_N$ , with  $u_P \in N_P$  and  $u_N \in N$ . Given any  $W \in D$ , take  $w \in I$  such that Pw = PW; this is possible because the basic problem satisfies existence. Then

$$\langle Au, W \rangle = \langle Au_P, W \rangle = -\langle PW, u_P \rangle = -\langle Pw, u_P \rangle = \langle Au_P, w \rangle = \langle Au, w \rangle.$$
 (3.5)

Therefore,

$$\langle Au, w \rangle = 0 \quad \forall \ w \in I \Rightarrow \langle Au, W \rangle = 0 \quad \forall \ W \in D$$
  
$$\Rightarrow u \in N.$$
 (3.6)

Corollary 3.1. If  $I \subset D$  is regular for P and the basic problem satisfies existence, then

$$I \cap I_P = N \tag{3.7}$$

and the solution of the problem with linear constraints is almost unique.

*Proof.* I and  $I_P$  are regular for P, so that  $N \subset I \cap I_P$  by Definition 2.2 of regular subspace. Conversely,  $N \supset I \cap I_P$ , because the hypotheses of Lemma 3.1 are satisfied whenever  $u \in I \cap I_P$ . The second part of this corollary follows from the first part.

The dual of Lemma 3.1, which is obtained by interchanging the roles of I and  $I_P$ , is also true.

LEMMA 3.2. Let  $I \subset D$  be a regular sub-space for P. Assume the basic problem satisfies existence. Then, for every  $u \in I$ , we have

$$\langle Au, v \rangle = 0 \quad \forall \ v \in I_P \Rightarrow u \in N.$$
 (3.8)

*Proof.* The proof is similar to that of Lemma 3.1, but use has to be made of Lemma 2.3.

THEOREM 3.1. Let  $I \subset D$  be a regular sub-space for P. If the problem with linear restrictions satisfies existence, then the pair  $(I, I_P)$  constitutes a canonical decomposition of D. In particular  $I \subset D$  and  $I_P \subset D$  are completely regular sub-spaces for P.

*Proof.* Assume  $U \in D$ , is such that

$$\langle AU, v \rangle = 0 \quad \forall \ v \in I$$

Define w = U - u, where  $u \in I$  is such that Pu = PU Therefore,  $w \in I_P$  and simultaneously

$$\langle Aw, v \rangle = \langle AU, v \rangle - \langle Au, v \rangle = 0 \quad \forall \in I$$
(3.10)

This shows by Lemma 3.1, that  $w \in N \subset I$ . Hence  $U = u + w \in I$  and I is completely regular. Making use of Lemma 3.2, dual of Lemma 3.1, it is possible to prove in a similar fashion, that  $I_P$  is also completely regular. Corollary 3.1 shows that  $I \cap I_P = N$ , thus, by Definition 3.1, it remains only to prove  $I + I_P = D$ . This is immediate, because given  $U \in D$ , choose  $U_1 \in I$  such that  $PU_1 = PU$ , which is possible because existence for the problem with linear constraints is assumed. Define  $U_2 = U - U_1$ , then  $U = U_1 + U_2$  and  $U_1 \in I$  while  $U_2 \in N_P \subset I_P$ .

## 4. Decompositions of A and Canonical Decompositions

There is a close connection between canonical decompositions and certain classes of decompositions of the operator A. This section is devoted to establish such relations.

Definition 4.1. An operator  $B: D \to D^*$  is said to be a boundary operator when

$$N_B \supset N. \tag{4.1}$$

Here  $N_B$  is the null sub-space of B.

As an example,  $B: D \to D^*$  given by

$$\langle Bu, v \rangle = \int_{\partial R} v \frac{\partial u}{\partial n} d\mathbf{x}$$
 (4.2)

is a boundary operator when A is given by (2.10a).

Definition 4.2. Given operators  $P: D \to D^*$  and  $Q: D \to D^*$ , one says that P and Q can be varied independently when for every  $U \in D$  and  $V \in D$ , there exists  $u \in D$  such that

Pu = PU and Qu = QV. (4.3)

The proof of the following lemma is straightforward. LEMMA 4.1. Let  $P: D \to D^*$  and  $Q: D \to D^*$  be linear operators. Then the following assertions are equivalent:

- (a) P and Q can be varied independently.
- (b) For every  $U \in D$ ,  $\exists u \in D_{\exists}$  Pu = PU; Qu = 0. (4.4)
- (c) For every  $V \in D$ ,  $\exists u \in D_{\exists}$  Pu = 0; Qu = QV. (4.5)

As an example, the operator  $B: D \to D^*$  as given by (4.2) and  $B^*: D \to D^*$ 

$$\langle B^*u, v \rangle = \int_{\partial R} u \frac{\partial v}{\partial n} d\mathbf{x}$$
 (4.6)

can be varied independently.

Definition 4.3. An operator  $B: D \to D^*$  is said to decompose A, when B and  $B^*$  can be varied independently and

$$A = B - B^*.$$

Applying Definition 4.3, we can see that the operator  $B: D \to D^*$  defined by (4.2), decomposes A.

LEMMA 4.2. Assume  $B: D \to D^*$  decomposes A. Then B and  $B^*$  are boundary operators. Proof. In view of Definition 4.1, it is necessary to prove, that when B decomposes A, one has

$$Au=0\Rightarrow Bu=0.$$

If B decomposes A,

$$Au = 0 \Rightarrow Bu = B^*u. \tag{4.9}$$

Given any  $V \in D$ , choose  $v \in D$  such that Bv = 0 and  $B^*v = B^*V$ . Then if Au = 0,

$$\langle Bu, V \rangle = \langle Bu, v \rangle = \langle B^*u, v \rangle = \langle Bv, u \rangle = 0.$$
 (4.10)

This shows that Bu = 0, because  $V \in D$  is arbitrary. Hence, B is a boundary operator. The fact that  $B^*$  is also a boundary operator follows from the above result when it is observed that  $-B^*$  decomposes A whenever B does.

It is possible to establish a one-to-one correspondence between operators  $B: D \to D^*$  that decompose A and canonical decompositions of D.

**THEOREM 4.1.** Assume  $B: D \to D^*$  decomposes A, then the pair of linear sub-spaces  $(I_1, I_2)$  given by

$$I_1 = \{ u \in D | Bu = 0 \} = N_B \tag{4.11a}$$

and

$$I_2 = \{ u \in D | B^* u = 0 \} = N_{B^*}$$
(4.11b)

constitutes a canonical decomposition of D with respect to P.

Conversely, given any canonical decomposition  $(I_1, I_2)$ , define  $B: D \to D^*$  by

$$\langle Bu, v \rangle = \langle Au_2, v_1 \rangle. \tag{4.12}$$

where  $u = u_1 + u_2$ ,  $u_1 \in I_1$ ,  $u_2 \in I_2$ , and similarly for v. Then B decomposes A and satisfies (4.11). Even more, this is the only operator with these properties.

*Proof.* To prove this Theorem, it will be first shown that when B decomposes A,  $I_1$  and  $I_2$  as given by (4.11), are completely regular. This can be seen by showing that condition (2.11) of Lemma 2.1 is satisfied by  $I_1$  and  $I_2$ . Now

$$\langle Au, v \rangle = \langle Bu, v \rangle - \langle Bv, u \rangle = 0, \quad \forall \ u, v \in I_1.$$
 (4.13)

To prove the converse implication in (2.11), observe that given any  $V \in D$ , it is possible to choose  $v \in D$  such that Bv = 0 (i.e.  $v \in I_1$ ) and simultaneously  $B^*v = B^*V$ , because B and  $B^*$  can be varied independently. With this choice of  $v \in I_1$ 

$$\langle Bu, V \rangle = \langle B^*v, u \rangle = -\langle Au, v \rangle. \tag{4.14}$$

This shows that  $\langle Au, v \rangle = 0 \,\forall v \in I_1$  implies  $u \in I_1$  because  $V \in D$  is arbitrary in (4.14). Hence,  $I_1$  is completely regular. A similar argument proves the corresponding result for  $I_2$ .

In order to show that  $(I_1, I_2)$  is a canonical decomposition of D, it remains to prove that  $I_1 \cap I_2 = N$  and  $I_1 + I_2 = D$ . Clearly,  $I_1 \cap I_2 \supset N$  in view of Lemma 4.2. Conversely,  $N \supset I_1 \cap I_2 = N_B \cap N_{B^*}$ , because  $A = B - B^*$ . Given  $u \in D$  choose  $u_1 \in D$ so that  $Bu_1 = 0$  while  $B^*u_1 = B^*u$ , which is possible because B and  $B^*$  can be varied independently. Define  $u_2 = u - u_1$ , then  $B^*u_2 = 0$  and  $u = u_1 + u_2$ ; this shows that  $D = I_1 + I_2$  because  $u_1 \in I_1$  while  $u_2 \in I_2$ . The proof of the first part of Theorem 4.1 is now complete.

To prove the second part, let  $(I_1, I_2)$  be an arbitrary canonical decomposition of D. Given any  $u, v \in D$ , take  $u_1, v_1 \in I_1$  and  $u_2, v_2 \in I_2$  as the components of the almost unique representations of u and v, corresponding to the canonical decomposition  $(I_1, I_2)$  of D. Then, the operator  $B: D \to D^*$  given by (4.12) is unambiguously defined. The commutative property (2.7) of regular sub-spaces implies that

$$\langle Au, v \rangle = \langle Au_2, v_1 \rangle - \langle Av_2, u_1 \rangle. \tag{4.15}$$

This shows  $A = B - B^*$ . To prove that B and  $B^*$  can be varied independently, let  $U \in D$  and  $V \in D$  be given, then  $u = V_1 + U_2$  satisfies Bu = BU and  $B^*u = B^*V$ . Thus, B decomposes A. To see that equation (4.11) is satisfied, observe that Bu = 0 implies

$$\langle Bu, v \rangle = \langle Au_2, v \rangle = 0 \quad \forall v \in D.$$
 (4.16)

Hence  $u_2 \in N$  and therefore  $u = u_1 + u_2 \in I_1$ . Conversely, if  $u \in I_1$ , then  $u_2 \in N$  and Bu = 0 by virtue of (4.12). This completes the proof of (4.11a); the proof of (4.11b) is similar.

To prove uniqueness, it will be shown that equation (4.12) is necessarily satisfied by any such B. Assume  $B: D \to D^*$  is such that  $A = B - B^*$  and it satisfies (4.11). Then  $Bu_1 = 0, \forall u_1 \in I_1$  and  $B^*u_2 = 0, \forall u_2 \in I_2$ ; therefore

$$\langle Au_2, v_1 \rangle = \langle Au_2, v \rangle = \langle Bu_2, v \rangle = \langle Bu, v \rangle. \tag{4.17}$$

Observe that the one-to-one correspondence between operators that decompose A and canonical decompositions would not be true, if canonical decompositions had not been introduced as ordered pairs in Definition 3.1.

#### 5. The Problem of Connecting

There are many problems that can be formulated as problems with linear restrictions; a very general example is the problem of connecting.

Although the formulation to be presented is an abstract one, it is motivated by a specific situation. Assume there are two neighbouring regions R and E (Fig. 1) with boundaries  $\partial R$  and  $\partial E$ , respectively. By reasons that will become apparent in some of the examples to be given, the common boundary between R and E will be denoted  $\partial_3 R = \partial_3 E$ . The general problem is to find solutions to specific partial differential equations on  $R \cup E$  subjected to a given smoothness criterion across the connecting boundary  $\partial_3 R = \partial_3 E$ . Problems of this kind occur frequently in applications; the smoothness criterion may be in potential theory, for example, that u and  $\partial u/\partial n$  be continuous across  $\partial_3 R$ , or in Elasticity, that displacements and tractions be continuous across that part of the boundary, but more complicated criteria may be included in the theory.

Let  $\hat{D}$  be a linear space and  $\hat{P}: \hat{D} \to \hat{D}^*$  a functional valued operator defined on that space. Here again,  $\hat{P}$  is assumed to be linear; in addition,  $\hat{D} = D_R \oplus D_E$  where  $D_R$  and  $D_E$  are two linear spaces. Elements  $\hat{u} \in \hat{D}$  will be thought as pairs  $(u_R, u_E)$ , where  $u_R \in D_R$  and  $u_E \in D_E$ . The space  $\hat{D}^*$  is the algebraic dual of  $\hat{D}$  and the operator  $\hat{P}$  is assumed to have the additive property

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \langle \hat{P}u_R, v_R \rangle + \langle \hat{P}u_E, v_E \rangle \tag{5.1}$$

for every  $\hat{u} = (u_R, u_E)$ ,  $\hat{v} = (v_R, v_E)$ . If the operators  $\hat{P}_R : \hat{D} \to \hat{D}^*$  and  $\hat{P}_E : \hat{D} \to \hat{D}^*$  are defined by

$$\langle \hat{P}_R \hat{u}, \hat{v} \rangle = \langle \hat{P} u_R, v_R \rangle; \quad \langle \hat{P}_E \hat{u}, \hat{v} \rangle = \langle \hat{P} u_E, v_E \rangle$$

then

$$\hat{P} = \hat{P}_R + \hat{P}_E.$$

Operators  $P_R: D_R \to D_R^*$  and  $P_E: D_E \to D_E^*$  can also be defined; they are given by

$$\langle P_R u_R, v_R \rangle = \langle \hat{P} u_R, v_R \rangle; \quad \langle P_E u_E, v_E \rangle = \langle \hat{P} u_E, v_E \rangle.$$
(5)

Then

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \langle P_{R}u_{R}, v_{R} \rangle + \langle P_{E}u_{E}, v_{E} \rangle$$

Using these operators, the following can be defined

$$\hat{A} = \hat{P} - \hat{P}^{*}, \qquad \hat{A}_{R} = \hat{P}_{R} - \hat{P}_{R}^{*}; \qquad A_{R} = P_{R} - P_{R}^{*}; \\ \hat{A}_{E} = \hat{P}_{E} - \hat{P}_{E}^{*}; \qquad A_{E} = P_{E} - P_{E}^{*}.$$

They satisfy

$$\hat{A} = \hat{A}_R + \hat{A}_E \tag{5.7}$$

and

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle.$$
(5.8)

The null sub-spaces of  $\hat{A}$ ,  $\hat{A}_R$ ,  $\hat{A}_E$ ,  $A_R$  and  $A_E$  will be denoted by  $\hat{N}$ ,  $\hat{N}_R$ ,  $\hat{A}_E$ ,  $N_R$  and  $\hat{N}_E$ , respectively. The relation

$$\hat{N} = N_R \oplus N_E$$

will be used later; it is equivalent to

$$\hat{u} = (u_R, u_E) \in \hat{N} \Leftrightarrow u_R \in N_R \text{ and } u_E \in N_E.$$
 (5.10)

This latter relation follows from (5.8).

The general problem to be considered will be one with linear restrictions, where the linear sub-space  $\hat{\mathbf{S}} \subset \hat{D}$  specifying the linear restriction will be assumed to satisfy special conditions. Elements  $\hat{u} = (u_E, u_R) \in \hat{\mathbf{S}}$  will be called smooth; when  $\hat{u} = (u_E, u_R)$  is smooth,  $u_E \in D_E$  and  $u_R \in D_R$  will be said to be smooth extensions of each other. Definition 5.1. Let  $\hat{\mathbf{S}} \subset \hat{D} = D_R \oplus D_E$  be a linear sub-space. Then  $\hat{\mathbf{S}}$  will be said to be a smoothness condition or relation if every  $u_R \in D_R$  possesses at least one smooth extension  $u_E \in D_E$  and conversely.

Definition 5.2. Given a smoothness relation  $\hat{S} \subset \hat{D}$  and elements  $\hat{U} \in \hat{D}$ ,  $\hat{V} \in \hat{D}$ , the problem of connecting consists in finding an element  $\hat{u} \in \hat{D}$  such that

$$\hat{P}\hat{u} = \hat{P}\hat{U}$$
 and  $\hat{u} - \hat{V} \in \hat{S}$ . (5.11)

Clearly, the problem of connecting is a problem with linear restrictions in the sense of Definition 2.1 and the results of previous sections are applicable. The smoothness relation  $\hat{\mathbf{S}}$  will be said to be regular and completely regular for  $\hat{P}$ , when as a sub-space, it is regular and completely regular for  $\hat{P}$ , respectively.

LEMMA 5.1. A smoothness condition  $\hat{\mathbf{S}} \subset \hat{D}$  is completely regular for  $\hat{P}$ , if and only if

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle = 0, \quad \forall \ \hat{v} \in \hat{\mathbf{S}} \Leftrightarrow \hat{u} \in \hat{\mathbf{S}}.$$
(5.12)

*Proof.* This lemma follows from (2.11) and (5.8).

As an example, take  $D_R = H^s(R)$  and  $D_E = H^s(E)$ , with  $s \ge 2$ . Assume each of the boundaries  $\partial R$  and  $\partial E$  of regions R and E (Fig. 1) is divided into three parts  $\partial_i R$  and  $\partial_i E$  (i = 1, 2, 3), where  $\partial_3 R = \partial_3 E$  is the common boundary between R and E. Let **n** be the unit normal vector on these boundaries, which will be taken pointing outwards from R and from E. On the common boundary  $\partial_3 R = \partial_3 E$ , there are defined two unit normal vectors which have opposite senses, one associated with R and the other one with E. Some times they will be represented by  $\mathbf{n}_R$  and  $\mathbf{n}_E$ ; more often, however, the ambiguity will be resolved by the suffix used under the integral sign.

Define  $P_R: D_R \to D_R^*$  by

$$\langle P_R u_R, v_R \rangle = \int_R v_R \nabla^2 u_R \, d\mathbf{x} + \int_{\partial_1 R} u_R \, \frac{\partial v_R}{\partial n} \, d\mathbf{x} - \int_{\partial_2 R} v_R \, \frac{\partial u_R}{\partial n} \, d\mathbf{x}$$
(5.13)

and let  $P_E: D_E \to D_E^*$  satisfy the equation that is obtained when R is replaced by E in (5.13). Then

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_3 E} \left\{ v_E \frac{\partial u_E}{\partial n} - u_E \frac{\partial v_E}{\partial n} \right\} d\mathbf{x} + \int_{\partial_3 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} d\mathbf{x}$$

while

$$\hat{N} = \{ \hat{u} \in \hat{D} | u_R = u_E = \partial u_R / \partial n = \partial u_E / \partial n = 0, \text{ on } \partial_3 R \}.$$

Let

$$\hat{\mathbf{S}} = \{ \hat{u} \in \hat{D} | u_R = u_E; \ \partial u_R / \partial n_R = \partial u_E / \partial n_R, \text{ on } \partial_3 R \}.$$
(5.16)

Functions  $u_R \in D_R = H^s(R)$  ( $s \ge 2$ ) are such that their boundary values  $u_R$ ,  $\partial u_R/\partial n$  belong to  $H^{s-1/2}(\partial_3 R)$  and  $H^{s-3/2}(\partial_3 R)$ , respectively (see, for example, Babuska & Aziz, 1972). A corresponding result holds for functions  $u_E \in D_E = H^s(E)$ . This shows that every  $u_R \in D_R$  can be extended smoothly into a function  $u_E \in D_E$ , and conversely. Thus  $\hat{S}$  is a smoothness relation.

In this case the problem of connecting is

$$\nabla^2 \hat{u} = \nabla^2 \hat{U}, \quad \text{on } R \cup E \tag{5.17a}$$

$$\hat{u} = \hat{U}, \quad \text{on } \partial_1(R \cup E)$$
 (5.17b)

$$\frac{\partial \hat{u}}{\partial n} = \frac{\partial U}{\partial n}, \quad \text{on } \partial_2(R \cup E)$$
 (5.17c)

subjected to

$$u_E - u_R = V_E - V_R, \qquad \partial (u_E - u_R) / \partial n = \partial (V_E - V_R) / \partial n, \quad \text{on } \partial_3 R. \tag{5.18}$$

When  $\hat{v} = (v_R, v_E) \in \hat{\mathbf{S}}$ .

$$\langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle = \int_{\partial_3 R} \left\{ v_R \left( \frac{\partial u_R}{\partial n} - \frac{\partial u_E}{\partial n} \right) - (u_R - u_E) \frac{\partial v_R}{\partial n} \right\} d\mathbf{x}$$
 (5.19)

for arbitrary  $\hat{u} = (u_R, u_E) \in \hat{D}$ . Using (5.19) it can be seen that condition (5.12) is satisfied by  $\hat{S} \subset \hat{D}$ ; this shows that  $\hat{S}$  is completely regular for  $\hat{P}$ .

Well known results about the existence of solution for boundary value problems of elliptic equations (Babuska & Aziz, 1972) can be used to show that the problem of connecting corresponding to equations (5.17) and (5.18), satisfies existence when  $D_R = H^s(R)$ ,  $D_E = H^s(E)$  and  $s \ge 2$ , when the boundaries of R and E satisfy suitable regularity assumptions.

When  $\hat{S}$  is completely regular, it is easy to construct a completely regular sub-space which together with  $\hat{S}$  constitutes a canonical decomposition of  $\hat{D}$ , for the operator  $\hat{P}$ . Definition 5.2. An element  $\hat{u} = (u_R, u_E) \in \hat{D}$  is said to have zero mean when  $(u_R, -u_E) \in \hat{S}$ . The collection of elements of  $\hat{D}$  with zero mean will be denoted by  $\hat{M}$ . THEOREM 5.1. When the smoothness relation  $\hat{S}$  is completely regular, the pair  $(\hat{S}, \hat{M})$ constitutes a canonical decomposition of  $\hat{D}$ .

*Proof.* In view of Definition 3.1, it is required to prove that  $\hat{\mathbf{M}}$  is completely regular for P and that

$$\hat{\mathbf{S}} \cap \hat{\mathbf{M}} = \hat{N}; \qquad \hat{\mathbf{S}} + \hat{\mathbf{M}} = \hat{D}. \tag{5.20}$$

Clearly,  $\hat{\mathbf{M}}$  is a linear sub-space of  $\hat{D}$ . In addition, Lemma 5.1 and the fact that  $\hat{\mathbf{S}}$  is completely regular imply that (5.12) holds. In view of Definition 5.2,  $\hat{\mathbf{S}}$  can be replaced by  $\hat{\mathbf{M}}$  in (5.12) without altering its validity. This shows that  $\hat{M}$  is completely regular for  $\hat{P}$ .

Assume  $\hat{u} = (u_R, u_E) \in \hat{S} \cap \hat{M}$ ; i.e.  $(u_R, u_E) \in \hat{S}$  and  $(u_R, -u_E) \in \hat{S}$ . Then  $(u_R, 0) \in \hat{S}$ , which implies

$$\langle A_R u_R, v_R \rangle = 0, \quad \forall \ v_R \in D_R \tag{5.21}$$

by virtue of (5.12) and the fact that any  $v_R$  has a smooth extension. Hence,  $u_R \in N_R$ . In a similar manner, it is seen that  $u_E \in N_E$ . Therefore,  $\hat{u} \in N_R \oplus N_E = \hat{N}$ , by (5.9), and the first equation in (5.20) is established. To show the second of those equations, given any  $\hat{u} = (u_R, u_E) \in \hat{D}$ , choose smooth extensions  $u'_R \in D_R$  and  $u'_E \in D_E$  of  $u_E \in D_E$ and  $u_R \in D_R$ , respectively. Then

$$\hat{u} = \bar{u} - \frac{1}{2} [\hat{u}]$$

where  $\bar{u} \in \hat{S}$  and  $[\hat{u}] \in \hat{M}$  are

$$\bar{u} = \frac{1}{2}(u_R' + u_R, u_E' + u_E) \tag{5.23a}$$

$$[\hat{u}] = (u'_R - u_{\hat{R}'}, u'_E - u_E).$$
(5.23b)

Equation (5.22) shows that any  $\hat{u} \in \hat{D}$  can be written

 $\hat{u} = \hat{u}_1 + \hat{u}_2;$   $\hat{u}_1 \in \hat{\mathbf{S}}$  and  $\hat{u}_2 \in \hat{\mathbf{M}}$ 

with

$$\hat{u}_1 = \bar{u}; \qquad \hat{u}_2 = -[\hat{u}]/2.$$
 (5.25)

This establishes the second of equations (5.20), and the proof of Theorem 5.1 is complete.

The fact that the pair  $(\hat{S}, \hat{M})$  constitutes a canonical decomposition of  $\hat{D}$ , implies that given any  $\hat{u} \in \hat{D}$ , the elements  $\bar{u} \in \hat{S}$  and  $[\hat{u}] \in \hat{M}$  are defined up to elements of  $\hat{N}$ ;

more precisely, that  $\bar{u}$  as well as  $[\hat{u}]$ , define unique cosets of the space  $\hat{D}/\hat{N}$ . Elements  $\bar{u}$  and  $[\hat{u}]$  satisfying (5.23) will be called the average and the jump of  $\hat{u}$ , respectively.

By means of Theorem 4.1, it is possible now to define an operator  $\hat{B}: \hat{D} \to \hat{D}^*$  that decomposes  $\hat{A}$  and satisfies (4.11) with  $I_1 = \hat{S}$  and  $I_2 = \hat{M}$ . Such an operator will be denoted by  $\hat{J}$  and will satisfy

$$2\langle \hat{J}\hat{u}, \hat{v} \rangle = 2\langle \hat{A}\hat{u}_2, \hat{v}_1 \rangle = -\langle \hat{A}[\hat{u}], \bar{v} \rangle$$
(5.26)

by virtue of (4.12) and (5.25). The operator  $\hat{J}: \hat{D} \to \hat{D}^*$  defined by (5.26) will be called jump operator. It characterizes  $\hat{S}$  because  $\hat{J}\hat{u} = 0 \Leftrightarrow \hat{u} \in \hat{S}$ .

Equation (5.26) will be used extensively when formulating variational principles for problems with prescribed jumps in discontinuous fields, and it is worthwhile to elaborate it further. Let  $\hat{u} = \hat{u}_1 + \hat{u}_2$ ;  $\hat{v} = \hat{v}_1 + \hat{v}_2$ , where  $\hat{u}_1 = (u_{1R}, u_{1E}) \in \hat{S}$ ,  $\hat{u}_2 = (u_{2R}, u_{2E}) \in \hat{M}$  and similarly for  $\hat{v}$ . Then

$$\langle J\hat{u}, \hat{v} \rangle = \langle A\hat{u}_2, \hat{v}_1 \rangle = \langle A_R u_{2R}, v_{1R} \rangle + \langle A_E u_{2E}, v_{1E} \rangle$$
  
= 2\langle A\_R u\_{2R}, v\_{1R} \rangle = 2\langle \hat{A}\_R \hat{u}\_2, \hat{v}\_1 \rangle = 2\langle \hat{A}\_E \hat{u}\_2, \hat{v}\_1 \rangle = 2\langle \hat

where (5.8), (5.12) and the Definition 5.2 of  $\hat{M}$  have been used. Hence

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = -\langle \hat{A}_{R}[\hat{u}], \bar{v} \rangle = -\langle \hat{A}_{E}[\hat{u}], \bar{v} \rangle$$

by virtue of (5.25). In addition

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle \hat{A}_{R}[\hat{v}], \bar{u} \rangle - \langle \hat{A}_{R}[\hat{u}], \bar{v} \rangle$$

because  $\hat{A} = \hat{J} - \hat{J}^*$ .

The use of formulas (5.28) and (5.29) will be illustrated applying them to the previous example. In view of (5.16), the smooth extensions  $u'_R \in D_R$  and  $u'_E \in D_E$  of  $u_E$  and  $u_R$ , respectively, satisfy

$$u'_{R} = u_{E}; \quad \partial u'_{R} / \partial n_{R} = \partial u_{E} / \partial n_{R}, \text{ on } \partial_{3}R.$$
 (5.30)

In addition

$$\langle A_R u_R, v_R \rangle = \int_{\partial_3 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} d\mathbf{x}.$$

Applying (5.28)

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = -\int_{\partial_{3}R} \left\{ (\bar{v})_{R} \frac{\partial [\hat{u}]_{R}}{\partial n} - [\hat{u}]_{R} \frac{\partial (\bar{v})_{R}}{\partial n} \right\} d\mathbf{x}.$$

Equation (5.23) yields

$$\begin{bmatrix} \hat{u} \end{bmatrix}_{R} = u_{E} - u_{R}; \qquad \frac{\partial \llbracket \hat{u} \rrbracket_{R}}{\partial n} = \frac{\partial u_{E}}{\partial n} - \frac{\partial u_{R}}{\partial n}; \qquad \text{on } \partial_{3}R \qquad (5.33a)$$
$$(\bar{v})_{R} = \frac{1}{2}(v_{E} + v_{R}); \qquad \frac{\partial (\bar{v})_{R}}{\partial n} = \frac{1}{2}\left(\frac{\partial v_{E}}{\partial n} + \frac{\partial v_{R}}{\partial n}\right), \qquad \text{on } \partial_{3}R$$

by virtue of (5.16). Equation (5.32) can be simplified if the component to be used is indicated by the index under the integral sign; thus

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_{3}R} \left\{ \begin{bmatrix} \hat{u} \end{bmatrix} \frac{\partial \bar{v}}{\partial n} - \bar{v} \begin{bmatrix} \partial \hat{u} \\ \partial n \end{bmatrix} \right\} d\mathbf{x}$$
$$= \int_{\partial_{3}E} \left\{ \begin{bmatrix} \hat{u} \end{bmatrix} \frac{\partial \bar{v}}{\partial n} - \bar{v} \begin{bmatrix} \partial \hat{u} \\ \partial n \end{bmatrix} \right\} d\mathbf{x}$$
(5.34)

where  $[\partial \hat{u}/\partial n]_R = \partial u_E/\partial n - \partial u_R/\partial n$ , on  $\partial_3 R$ . The last equality in (5.34) follows from the second equation in (5.28), but can also be seen because there is a double change of signs on each term appearing in the integrals; one due to the change in the sense of the unit normal and the other one due to the change of sign of the jump of  $\hat{u}$ . Equation (5.29) yields

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_{3}R} \left\{ [\hat{u}] \frac{\partial v}{\partial n} + \bar{u} \left[ \frac{\partial v}{\partial n} \right] - \bar{v} \left[ \frac{\partial \hat{u}}{\partial n} \right] - [\hat{v}] \frac{\partial u}{\partial n} \right\} d\mathbf{x}$$

The following definition and results establish a relation between the problem of connecting and problems subjected to restrictions of continuation type.

Definition 5.3. Let  $D, P: D \to D^*$  and a linear sub-space  $I \subset D$  be given. Then the problem with linear restrictions (2.1) will be said to be subjected to a constraint of continuation type, when for some  $\hat{D} = D_R \oplus D_E$ ,  $\hat{P}: \hat{D} \to \hat{D}^*$  and smoothness criterion  $\hat{S} \subset \hat{D}$ :

(a)  $D = D_R$ ,

$$P = P_R : D_R \to D_R^*$$

and

(c) 
$$I = \{ u \in D | \exists \ \hat{u} = (u, u_E) \in \hat{\mathbb{S}}_{\exists} P_E u_E = 0 \}.$$
 (5.36)

THEOREM 5.2. Assume problem (2.1) is subjected to restrictions of continuation type and the associated smoothness condition  $\hat{S}$  is regular. Then, if the associated problem of connecting satisfies existence, the linear sub-space  $I \subset D$  is completely regular for P.

*Proof.* Theorem 3.1 will be applied to show that  $(I, I_P)$  is a canonical decomposition of D. Here, according to Equation (2.12),  $I_P = N + N_P$ . By Theorem 3.1, it is only necessary to prove that  $I \subset D$  is a regular sub-space for P and that the problem with linear restrictions satisfies existence. Given any  $u \in I$  and  $v \in I$ , take  $u_E \in D_E$  satisfying the conditions of (5.36) and similarly  $v_E \in D_E$ . Then

$$\langle Au, v \rangle = -\langle A_E u_E, v_E \rangle = \langle P_E v_E, u_E \rangle - \langle P_E u_E, v_E \rangle = 0$$
(5.37)

where use has been made of (5.12). The condition  $N \subset I$  follows from the fact that  $\hat{N} \subset \hat{S}$ , using (5.9) or equivalently (5.10). This shows that  $I \subset D$  is a regular sub-space for P.

By virtue of Lemma 2.3, it remains to prove that the basic problem

$$Pu = PU; \quad u \in I \tag{5.38}$$

satisfies existence. To prove this, given  $U \in D$ , define  $\hat{U} = (U, 0) \in \hat{D}$  and let  $\hat{u} = (u, u_E)$  be a solution of the problem of connecting

$$\hat{P}\hat{u} = \hat{P}\hat{U}; \quad \hat{u} \in \hat{S}. \tag{5.39}$$

Then, recalling definition (5.36), it is seen that  $u \in D$  satisfies (5.38), and the proof of Theorem 5.2 is complete.

As an example, in Fig. 1, the functions of  $D = H^s(R)$ ,  $(s \ge 2)$ , can be continued smoothly into functions of  $H^s(E)$  that are harmonic on E, vanish on  $\partial_1 E$  and whose normal derivative vanishes on  $\partial_2 E$ , constitutes a completely regular sub-space for  $P: D \to D^*$ , defined by

$$\langle Pu, v \rangle = \int_{R} v \nabla^2 u \, dx + \int_{\partial_1 R} u \, \frac{\partial v}{\partial n} \, dx - \int_{\partial_2 R} v \, \frac{\partial u}{\partial n} \, dx$$

Here, the criterion of smoothness is that u and  $\partial u/\partial n$  are continuous across  $\partial_3 R$ . Such a result can be extended to unbounded regions if suitable radiations conditions are imposed on the functions considered (Herrera & Sabina, 1978).

## 6. Variational Principles

The theory developed in this paper will be used in this section to formulate two types of variational principle for problems with linear restrictions.

The first one applies when there is available a canonical decomposition  $(I, I_c)$ , one of whose elements is the linear sub-space I which specifies the restrictions in problem (2.1). In this case, P-B, where  $B: D \to D^*$  is the operator associated with the canonical decomposition by means of (4.12), is symmetric; by its use one obtains variational principles for which the variations need not be restricted. However, it must be observed that the mere existence of such canonical decomposition is not sufficient to permit the formulation of these variational principles; it is required, in addition, that the actual decomposition of every vector  $u \in D$  in its components  $u_1$ and  $u_2$ , can be carried out without difficulty, because this is necessary in order to construct B by means of (4.12). Problems subjected to restrictions of continuation type, do not fulfil this requirement in spite of the fact that for them  $(I, I_P)$ , frequently constitutes a canonical decomposition; this can be seen by observing that to obtain the components  $U_1, U_2$  of any  $U \in D$  with respect to this canonical decomposition, it is essentially required to solve the problem with linear restrictions (2.1).

When it is difficult to construct the operator B the second type of variational principle can be applied. It is associated with the operator 2P - A, which is always symmetric and can be used if variations are restricted to be in the regular sub-space I; the results are enhanced when the sub-space is completely regular, as is often the case.

Applications are made to the problem of connecting, for which the construction of B (the jump operator) is possible, as shown in Section 5, and to problems with restrictions of continuation type, for which, as already mentioned, such construction is not possible and the operator 2P - A has to be used.

The following lemmas lead to the desired variational principles.

LEMMA 6.1. Let  $I \subset D$  be a completely regular sub-space for P, then given  $U \in D$  and  $V \in D$ , an element  $u \in D$  is solution of the problem with linear constraints (2.1), if and only if

$$Pu = PU \tag{6.1}$$

and

$$\langle A(u-V), v \rangle = 0, \quad \forall \ v \in I.$$
(6.2)

When I is regular, but not completely regular, the above assertion holds for elements  $u \in V+I$ .

**Proof.** The mere regularity of  $I \subset D$  is enough to guarantee that equation (2.1) implies (6.1) and (6.2). When, in addition,  $I \subset D$  is completely regular, conversely, (6.2) implies that  $u - V \in I$ ; hence, equation (2.1) follows from (6.1) and (6.2), in this case. The second part of the lemma is now straightforward.

LEMMA 6.2. Assume  $(I, I_c)$  constitutes a canonical decomposition of D with respect to P, and let  $B: D \to D^*$  be defined by (4.12), taking  $u_2$  and  $v_1$  as components of vectors on  $(I, I_c)$ . Then  $u \in D$  is a solution of the problem with linear constraints (2.1) if and only if

$$Pu = PU$$
 and  $Bu = BV$ . (6.3)

*Proof.* By Theorem 4.1, equation (4.11a),  $u - V \in I$  if and only if B(u - V) = 0. Definition 6.1. An operator  $P: D \to D^*$  is said to be formally symmetric when for every  $u \in D$ 

$$\langle Pu, v \rangle = 0, \quad \forall \ v \in N \Rightarrow Pu = 0.$$
 (6.4)

It is customary to call a differential operator L, formally symmetric, when

$$\int_{R} \{v \mathbf{L} u - u \mathbf{L} v\} d\mathbf{x} = \text{boundary terms.}$$
(6.5)

To such an operator one can associate a  $P: D \to D^*$  which is formally symmetric in the sense of Definition 6.1 by means of

$$\langle Pu, v \rangle = \int_{R} v \mathbf{L} u \, d\mathbf{x}.$$
 (6.6)

As an example, the operator associated by means of (6.6) to the Laplacian, is formally symmetric in the sense of Definition 6.1. Indeed, in this case  $P: D \rightarrow D^*$  is given by equation (2.2) and the null sub-space [equation (2.10b)] is the set of functions which together with their normal derivatives, vanish on the boundary. Property (6.4), in this case, amounts to the so-called, fundamental lemma of calculus of variations.

LEMMA 6.3. Assume  $P: D \to D^*$  is formally symmetric and  $I \subset D$  is regular for P. Then

(a) (6.1) and (6.2) hold simultaneously if and only if

$$\langle (2P-A)u, v \rangle = \langle 2PU-AV, v \rangle, \quad \forall \ v \in I.$$
(6.7)

(b) Equations (6.3) hold, if and only if

$$(P-B)u = PU - BV. ag{6.8}$$

*Proof.* Re-arranging, equation (6.7) becomes

$$\langle 2P(u-U), v \rangle = \langle A(u-V), v \rangle, \quad \forall \ v \in I.$$
(6.9)

Clearly, (6.1) and (6.2) imply (6.9). Conversely, (6.9) implies

$$\langle 2P(u-U), v \rangle = 0, \quad \forall \ v \in N \subset I$$
 (6.10)

which in turn implies (6.1), because P is formally symmetric. Once this has been shown, (6.9) reduces to (6.2). This proves (a).

Equation (6.8) can be obtained by subtracting one of equations (6.3) from the other. Conversely, (6.8) implies

$$\langle P(u-U), v \rangle = \langle B(u-V), v \rangle = 0, \quad \forall \ v \in N \subset D$$
(6.11)

because according to Lemma 4.2,  $B^*$  is a boundary operator (i.e.  $N_{B^*} \supset N$ ). The first of equations (6.3) follows from (6.11), because P is formally symmetric. Once that equation has been proved, (6.8) reduces to the second equation in (6.3).

**THEOREM 6.1.** Assume  $P: D \to D^*$  is formally symmetric and  $(I, I_c)$  constitutes a canonical decomposition of D. Then  $u \in D$  is a solution of the problem with linear restrictions (2.1), if and only if

$$\Omega'(u) = 0 \tag{6.12}$$

where

$$\Omega(u) = \frac{1}{2} \langle (P-B)u, u \rangle - \langle PU - BV, u \rangle.$$
(6.13)

Here  $B: D \to D^*$  is the operator associated with  $(I, I_c)$  by means of (4.12).

*Proof.* Recall that  $P - P^* = A = B - B^*$ ; hence, P - B is symmetric. Applying (2.16) to this symmetric operator, Theorem 6.1 follows from Lemmas 6.2 and 6.3.

**THEOREM 6.2.** Assume P is formally symmetric and  $I \subset D$  is a completely regular subspace for P. Define

$$X(u) = \langle Pu, u \rangle - \langle 2PU - AV, u \rangle.$$
(6.14)

Then  $u \in D$  is a solution of the problem with linear restrictions (2.1), if and only if

$$\langle X'(u), V \rangle = 0, \quad \forall \ v \in I.$$
 (6.15)

When I is regular but not completely regular, an element  $u \in V + I$  is a solution of (2.1), if and only if (6.15) holds.

*Proof.* 2P - A is symmetric with quadratic form  $\langle 2Pu, u \rangle$ , because A is antisymmetric. From (6.14), it follows that

$$X'(u) = (2P - A)u - (2PU - AV).$$
(6.16)

Theorem 6.2, follows from Lemmas 6.1 and 6.3, by virtue of (6.16).

The following variational principles are corollaries of Theorems 6.1 and 6.2.

**THEOREM 6.3.** Take  $\hat{P}: \hat{D} \to \hat{D}^*$  as in Section 5 and let  $\hat{S} \subset \hat{D}$  be a completely regular smoothness relation for  $\hat{D}$ . Define  $\hat{J}: \hat{D} \to \hat{D}^*$  by means of (5.28). Then, when  $\hat{P}$  is formally symmetric  $\hat{u} \in \hat{D}$  is a solution of the problem of connecting (5.11), if and only if

$$\Omega'(\hat{u}) = 0 \tag{6.17}$$

where

$$\Omega(\hat{u}) = \langle \hat{P}(\hat{u} - 2\hat{U}), \hat{u} \rangle - \langle \hat{J}(\hat{u} - 2\hat{V}), \hat{u} \rangle.$$

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*Proof.* According to Theorem 5.1, the pair  $(\hat{\mathbf{S}}, \hat{\mathbf{M}})$  constitutes a canonical decomposition of  $\hat{D}$ , where  $\hat{\mathbf{M}}$  is given by Definition 5.2. Hence, Theorem 6.3 follows Theorem 6.1, because  $\hat{J}: \hat{D} \to \hat{D}^*$  is the operator that decomposes  $\hat{A}$ , associated by Theorem 4.1 with  $(\hat{\mathbf{S}}, \hat{\mathbf{M}})$ .

THEOREM 6.4. Assume problem (2.1) is subjected to restrictions of continuation type (Definition 5.3) and the associated smoothness condition  $\hat{S}$  is regular for  $\hat{P}$ . Let the functional  $X : D \to F$  be given by (6.14). Then, when the problem of connecting satisfies existence and  $P : D \to D^*$  is formally symmetric,  $u \in D$  fulfils (2.1), if and only if, (6.15) holds.

Proof. This result follows from Theorem 6.2, by virtue of Theorem 5.2.

#### 7. Applications

The variational principles for the problem with linear constraints presented in Section 6, supply a systematic frame-work for the formulation of such principles associated with boundary value problems and boundary methods. There are many classical problems of partial differential equations that can be cast in this framework; here, however, it will only be applied to two types of problem : problems formulated in discontinuous fields subjected to prescribed jump conditions; and problems subjected to restrictions of continuation type. The corresponding variational principles will be special cases of Theorem 6.3 and 6.4, respectively.

These two kinds of principles will be derived for potential theory, reduced wave equation, heat and wave equations, and Elasticity (static, periodic motions and dynamical). Variational principles for the linearized theory of free surface flows have also been obtained by this method (Herrera, 1977a). It is of interest to notice that problems involving two phases can also be formulated in this manner; to illustrate this fact variational principles are derived for a problem in which the region R (Fig. 1) is occupied by an inviscid liquid while E is occupied by an elastic solid. For static and quasi-static problems the regions to be considered are illustrated in Fig. 1. The regions illustrated in Fig. 3 apply to time dependent problems, which will be formulated in a finite time interval [0, T]. For simplicity the regions R and E shown in the figures are bounded, but the results can also be applied in unbounded regions if suitable conditions such as radiation conditions are imposed on the elements of the spaces  $D_R$  and  $D_E$ . Thus, for example, diffraction problems formulated in a half-space (Fig. 4) can be treated in this manner.

#### 7.1. Potential Theory and Reduced Wave Equation

The function spaces  $D_R$  and  $D_E$  can be taken as suitable Sobolev spaces (Babuska & Aziz, 1972; Lions & Magenes, 1968); generally,  $D_R = H^s(R)$ ,  $D_E = H^s(E)$ , with  $s \ge 2$ . A slight modification has to be made when complex valued functions are considered. Given  $\rho$  and non-zero constants  $k_R$ ,  $k_E$ , define

$$\mathbf{L}(u) = \nabla^2 u + \rho u, \tag{7.1}$$

$$\langle P_R u_R, v_R \rangle = k_R \left\{ \int_R v \mathbf{L}(u) \, d\mathbf{x} + \int_{\partial_1 R} u \, \frac{\partial v}{\partial n} \, d\mathbf{x} - \int_{\partial_2 R} v \, \frac{\partial u}{\partial n} \, d\mathbf{x} \right\}$$
(7.2)



FIG. 3



FIG. 4

and  $P_E: D_E \to D_E^*$ , replacing R by E in equation (7.2). Then, using (5.10) it can be seen that

$$\widehat{N} = \{(u_R, u_E) \in \widehat{D} | u_R = u_E = \partial u_E / \partial n = \partial u_R / \partial n = 0; \text{ on } \partial_3 R\}$$
(7.3)

and it is easy to verify that  $\hat{P} : \hat{D} \to \hat{D}^*$  is formally symmetric, because it satisfies (6.4). Let the smoothness relation be

$$\mathbf{S} = \{ \hat{u} \in \hat{D} | u_R = u_E, k_R \partial u_R / \partial n = k_E \partial u_E / \partial n; \text{ on } \partial_3 R \}.$$
(7.4)

For  $\hat{v} \in \hat{S}$  and arbitrary  $\hat{u} \in \hat{D}$ 

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \left\{ k[\hat{u}] \frac{\partial v}{\partial n} - v \left[ \hat{k} \frac{\partial \hat{u}}{\partial n} \right] \right\} d\mathbf{x}$$
 (7.5)

where

$$[\hat{u}]_{R} = u_{E} - u_{R}; \qquad [\hat{k} \,\partial\hat{u}/\partial n]_{R} = k_{E} \,\partial u_{E}/\partial n_{R} - k_{R} \,\partial u_{R}/\partial n_{R}.$$
(7.6)

Here, as in what follows, the components (R or E) to be used when carrying out the integration, are indicated by the sub-index under the integral sign. From (7.5) and Lemma 5.1, it can be seen that  $\hat{S}$  is completely regular for  $\hat{P}$ . Applying (5.28), one gets

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \left\{ \overline{k} \frac{\partial v}{\partial n} \left[ \hat{u} \right] - \overline{v} \left[ \hat{k} \frac{\partial \hat{u}}{\partial n} \right] \right\} d\mathbf{x}$$
(7.7)

where

$$(\bar{v})_{R} = (u_{E} + u_{R})/2;$$
  $\left(k\frac{\partial v}{\partial n}\right)_{R} = \frac{1}{2}\left(k_{E}\frac{\partial v_{E}}{\partial n_{R}} + k_{R}\frac{\partial v_{R}}{\partial n_{R}}\right).$  (7.8)

Given  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$ , the problem of connecting (5.11), is equivalent to

$$\mathbf{L}(\hat{u}) = f_{R \cup E} = \mathbf{L}(\hat{U}); \quad \text{on } R \cup E,$$

$$\hat{u} = f_1 = \hat{U}; \quad \text{on } \partial_1(R \cup E),$$

$$\frac{\partial \hat{u}}{\partial n} = f_2 = \frac{\partial \hat{U}}{\partial n}, \quad \text{on } \partial_2(R \cup E),$$
(7.9c)

subjected to prescribed jump conditions

$$u_E - u_R = f_{J1} = V_E - V_R;$$
  

$$k_E \frac{\partial u_E}{\partial n_R} - k_R \frac{\partial u_R}{\partial n_R} = f_{J2} = k_E \frac{\partial v_E}{\partial n_R} - k_R \frac{\partial V_R}{\partial n_R}; \quad \text{on } \mathcal{E}_s \mathcal{P}. \quad (7.10)$$

This problem can be formulated variationally by means of Theorem 6.3. The corresponding functional is

$$\Omega(u) = \int_{R \cup E} u(u - 2f_{R \cup E}) \, dx + \int_{\partial_1(R \cup E)} (u - 2f_1) \frac{\partial u}{\partial n} \, dx - \int_{\partial_2(R \cup E)} u\left(\frac{\partial u}{\partial n} - 2f_2\right) \, dx - \int_{\partial_3 R} \left\{ \left( [\hat{u}] - 2f_{J_1} \right) \overline{k} \frac{\partial u}{\partial n} - \overline{u} \left( \left[ \hat{k} \frac{\partial \hat{u}}{\partial n} \right] - 2f_{J_2} \right) \right\} \, dx. \quad (7.11)$$

The problem with restrictions of continuation type of Definition 5.3, in this case corresponds to

$$\mathbf{L}\boldsymbol{u} = \boldsymbol{f}_{\boldsymbol{R}} = \mathbf{L}\boldsymbol{U}; \quad \text{on } \boldsymbol{R}, \tag{7.12a}$$

$$u = f_1 = U; \quad \text{on } \partial_1 R, \tag{7.12b}$$

$$\frac{\partial u}{\partial n} = f_2 = \frac{\partial U}{\partial n}; \text{ on } \partial_2 R.$$
 (7.12c)

The restriction is that there exists a function  $u_E \in D_E$ , such that

$$u-V = u_E;$$
  $k_R\left(\frac{\partial u}{\partial n} - \frac{\partial V}{\partial n}\right) = k_E \frac{\partial u_E}{\partial n};$  on  $\partial_3 R.$  (7.13)

Here

$$Lu_E = 0$$
, on  $E$ ;  $u_E = 0$ , on  $\partial_1 E$ ;  $\partial u_E / \partial n = 0$ , on  $\partial_2 E$  (7.14)  
This problem occurs in diffraction studies.

Taking  $I \subset D$  as the linear sub-space that satisfies (7.13) with  $V \equiv 0$ , Theorem 6.4 is applicable. Equation (6.14) yields

$$X(u) = \int_{R} u \{ \mathbf{L}u - 2f_{R} \} dx + \int_{\partial_{1}R} (u - 2f_{1}) \frac{\partial u}{\partial n} dx - \int_{\partial_{2}R} u \left( \frac{\partial u}{\partial n} - 2f_{2} \right) d\mathbf{x} + \int_{\partial_{3}R} \left\{ u \frac{\partial V}{\partial n} - V \frac{\partial u}{\partial n} \right\} d\mathbf{x}.$$
 (7.15)

Here the factor  $k_R$  was deleted because it was superfluous.

## 7.2. Heat Equation

A similar application can be made to the heat equation. In this case (Fig. 3),  $R = R_x \times [0, T], E = E_x \times [0, T]$  and

$$\mathbf{L}\boldsymbol{u} = \nabla^2 \boldsymbol{u} - \partial \boldsymbol{u} / \partial t. \tag{7.16}$$

The operator  $P_R: D_R \to D_R^*$  can be defined by

$$\langle P_R u_R, v_R \rangle = \int_{R_x} v * \operatorname{Lu} d\mathbf{x} + \int_{\partial_1 R_x} u * \frac{\partial v}{\partial n} d\mathbf{x} - \int_{\partial_2 R_x} v * \frac{\partial u}{\partial n} d\mathbf{x} - \int_{R_x} u(0) v(T) d\mathbf{x}. \quad (7.17)$$

The following notation

$$u * v = \int_0^T u(T-t)v(t) dt$$
 (7.18)

is adopted.  $P_E: D_E \to D_E^*$  is obtained replacing R by E in (7.17). The smoothness condition can be taken as

$$\hat{\mathbf{S}} = \{ \hat{u} \in \hat{D} | u_R = u_E; \quad \partial u_R / \partial n = \partial u_E / \partial n; \quad \text{on } \partial_3 \mathbf{R} \}$$
(7.19)

where the subsets  $\partial_i R = [0, T] \times \partial_i R_x$  (i = 1, 2, 3), do not cover the boundary  $\partial R$  of R. When  $\hat{v} \in \hat{S}$  and  $\hat{u} \in \hat{D}$  is arbitrary,

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R_x} \left\{ [\hat{u}] * \frac{\partial v}{\partial n} - v * \left[ \frac{\partial \hat{u}}{\partial n} \right] \right\} d\mathbf{x}.$$
(7.20)

Again, use of Lemma 6.1, permits establishing that  $\hat{S}$  is completely regular for  $\hat{D}:\hat{D}\to\hat{D}^*$ . Equation (5.28), yields

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R_x} \left\{ \frac{\partial v}{\partial n} * [\hat{u}] - \bar{v} * \left[ \frac{\partial \hat{u}}{\partial n} \right] \right\} d\mathbf{x}.$$
(7.21)

Given  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$ , the problem of connecting (5.11), is equivalent to equation (7.9) supplemented by

$$\hat{u}(\mathbf{x}, 0) = f_0 = \hat{U}(\mathbf{x}, 0); \text{ on } R_x \cup E_x$$
 (7.22)

subjected to jump conditions

$$u_E - u_R = f_{J1} = V_E - V_R, \quad \frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n} = f_{J2} = \frac{\partial V_E}{\partial n} - \frac{\partial V_R}{\partial n}; \quad \text{on } \partial_3 R.$$

Here,  $\partial_3 R = [0, T] \times \partial_3 R_x$ ; thus,  $f_{J1}$  and  $f_{J2}$  are also functions of time t. The variational formulation of Theorem 6.3, yields the functional

$$\Omega(\hat{u}) = \int_{R_x \cup E_x} u * (\mathbf{L}u - 2f_{R \cup E}) \, d\mathbf{x} + \int_{\partial_1(R_x \cup E_x)} (u - 2f_1) * \frac{\partial u}{\partial n} \, d\mathbf{x} - \int_{\partial_2(R_x \cup E_x)} u * \left(\frac{\partial u}{\partial n} - 2f_2\right) \, d\mathbf{x} - \int_{R_x \cup E_x} \{u(0) - 2f_0\} u(T) \, d\mathbf{x} + \int_{\partial_3 R_x} \left\{ \bar{u} * \left( \left[\frac{\partial \hat{u}}{\partial n}\right] - 2f_{J_2} \right) - \left( [\hat{u}] - 2f_{J_1} \right) * \frac{\partial u}{\partial n} \right\} \, d\mathbf{x}.$$
(7.24)

The problem with restrictions of continuation type of Definition 5.3 in this case is governed by equations (7.12), supplemented by

$$u(\mathbf{x}, 0) = f_0 = U(\mathbf{x}, 0); \text{ on } R_x.$$
 (7.25)

The restriction is obtained by taking  $k_R = k_E = 1$  in equation (7.13) and supplementing (7.14) with

$$u_E(\mathbf{x}, 0) = 0;$$
 on  $E_x$ 

The functional of Theorem 6.4, is

$$X(u) = \int_{R_x} u * \{ \mathbf{L}u - 2f_R \} d\mathbf{x} + \int_{\partial_1 R_x} (u - 2f_1) * \frac{\partial u}{\partial n} d\mathbf{x} - \int_{\partial_2 R} u * \left( \frac{\partial u}{\partial n} - 2f_2 \right) d\mathbf{x} - \int_{R_x} \{ u(0) - 2f_0 \} u(T) d\mathbf{x} + \int_{\partial_3 R_x} \left\{ u * \frac{\partial V}{\partial n} - V * \frac{\partial u}{\partial n} \right\} d\mathbf{x}.$$
(7.27)

## 7.3. The Wave Equation

The results corresponding to the wave equation are listed below.

(a) 
$$R = R_x \times [0, T]; \quad E = E_x \times [0, T].$$
  
(b)  $I_{\mu} = \nabla^2 u - \partial^2 u / \partial t^2$  (7.28)

(c) 
$$P_R: D_R \to D_R^*$$
 is (7.28)

$$\langle P_R u_R, v_R \rangle = \int_{R_x} v \mathbf{L} u \, d\mathbf{x} + \int_{\partial_1 R_x} u * \frac{\partial v}{\partial n} \, d\mathbf{x} - \int_{\partial_2 R_x} v * \frac{\partial u}{\partial n} \, d\mathbf{x} - \int_{R_x} \{u(0)v'(T) + u'(0)v(T)\} \, d\mathbf{x} \quad (7.29)$$

where the primes stand for the partial derivatives with respect to t. To obtain  $P_E: D_E \to D_E^*$ , R has to be replaced by E in (7.29).

- (d) Equations (7.19) to (7.21) also hold in this case.
- (e) Given  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$ , the problem of connecting (5.11) is equivalent to equation (7.9), supplemented by

$$\widehat{u}(\mathbf{x},0) = f_0 = \widehat{U}(\mathbf{x},0);$$

$$\partial \hat{u}(\mathbf{x}, 0)/\partial t = f'_0 = \partial \hat{U}(\mathbf{x}, 0)/\partial t; \text{ on } R_x \cup E_x$$

subjected to (7.23).

Here,  $\partial_3 R = [0, T] \times \partial_3 R_x$ ; thus,  $f_{J1}$  and  $f_{J2}$  are also functions of time t. The variational formulation of Theorem 6.3, yields the functional

$$\Omega(\hat{u}) = \int_{R_x \cup E_x} u * (\mathbf{L}u - 2f_{R \cup E}) \, d\mathbf{x} + \int_{\partial_1(R_x \cup E_x)} (u - 2f_1) * \frac{\partial u}{\partial n} \, d\mathbf{x} - \int_{\partial_2(R_x \cup E_x)} u * \left(\frac{\partial u}{\partial n} - 2f_2\right) \, d\mathbf{x} - \int_{R_x \cup E_x} \{u(0) - 2f_0\} u(T) \, d\mathbf{x} + \int_{\partial_3 R_x} \left\{ \bar{u} * \left( \left[\frac{\partial \hat{u}}{\partial n}\right] - 2f_{J_2} \right) - ([\hat{u}] - 2f_{J_1}) * \frac{\partial u}{\partial n} \right\} \, d\mathbf{x}.$$
(7.24)

The problem with restrictions of continuation type of Definition 5.3 in this case is governed by equations (7.12), supplemented by

$$u(\mathbf{x}, 0) = f_0 = U(\mathbf{x}, 0); \text{ on } R_x.$$
 (7.25)

The restriction is obtained by taking  $k_R = k_E = 1$  in equation (7.13) and supplementing (7.14) with

$$u_E(\mathbf{x}, 0) = 0; \text{ on } E_{\mathbf{x}}.$$
 (7.26)

The functional of Theorem 6.4, is

$$X(u) = \int_{R_x} u * \{ \mathbf{L}u - 2f_R \} d\mathbf{x} + \int_{\partial_1 R_x} (u - 2f_1) * \frac{\partial u}{\partial n} d\mathbf{x} - \int_{\partial_2 R} u * \left( \frac{\partial u}{\partial n} - 2f_2 \right) d\mathbf{x} - \{ u(0) - 2f_0 \} u(T) d\mathbf{x} + \int_{\partial_3 R_x} \left\{ u * \frac{\partial V}{\partial n} - V * \frac{\partial u}{\partial n} \right\} d\mathbf{x}.$$
(7.27)

## 7.3. The Wave Equation

The results corresponding to the wave equation are listed below.

(a) 
$$R = R_x \times [0, T]; \quad E = E_x \times [0, T].$$
  
(b) 
$$Lu = \nabla^2 u - \partial^2 u / \partial t^2. \quad (7.28)$$

(c) 
$$P_R: D_R \to D_R^*$$
 is

$$\langle P_R u_R, v_R \rangle = \int_{R_x} v \mathbf{L} u \, d\mathbf{x} + \int_{\partial_1 R_x} u * \frac{\partial v}{\partial n} \, d\mathbf{x} - \int_{\partial_2 R_x} v * \frac{\partial u}{\partial n} \, d\mathbf{x} - \int_{R_x} \{u(0)v'(T) + u'(0)v(T)\} \, d\mathbf{x} \quad (7.29)$$

where the primes stand for the partial derivatives with respect to t. To obtain  $P_E: D_E \to D_E^*$ , R has to be replaced by E in (7.29).

- (d) Equations (7.19) to (7.21) also hold in this case.
- (e) Given  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$ , the problem of connecting (5.11) is equivalent to equation (7.9), supplemented by

$$\widehat{u}(\mathbf{x},0) = f_0 = \widehat{U}(\mathbf{x},0);$$

$$\partial \hat{u}(\mathbf{x}, 0)/\partial t = f'_0 = \partial \hat{U}(\mathbf{x}, 0)/\partial t; \text{ on } R_x \cup E_x$$
 (7.30)

subjected to (7.23).

(f) The functional of Theorem 6.3, is

$$\Omega(\hat{u}) = \int_{R_x \cup E_x} u * (\mathbf{L}u - 2f_{R \cup E}) d\mathbf{x} + \int_{\partial_1(R_x \cup E_x)} (u - 2f_1) * \frac{\partial u}{\partial n} d\mathbf{x} - \int_{\partial_2(R_x \cup E_x)} u * \left(\frac{\partial u}{\partial n} - 2f_2\right) d\mathbf{x} - \int_{R_x \cup E_x} \{u(0) - 2f_0\} u'(T) d\mathbf{x} - \int_{R_x \cup E_x} \{u'(0) - 2f'_0\} u(T) d\mathbf{x} + \int_{\partial_3 R_x} \left\{ \bar{u} * \left( \left[ \frac{\partial \hat{u}}{\partial n} \right] - 2f_{J_2} \right) - ([\hat{u}] - 2f_{J_1}) * \frac{\partial u}{\partial n} \right\} d\mathbf{x}.$$
 (7.31)

(g) The problem with restrictions of continuation type of Definition 5.3 is given by (7.12), (7.25), supplemented by

$$\partial u(\mathbf{x}, 0)/\partial t = f'_0 = \partial U(\mathbf{x}, 0)/\partial t; \text{ on } R_x$$
 (7.32)

subjected to the restriction that there exists  $u_E \in D_E$ , that satisfies (7.13) with  $k_R = k_E = 1$ , (7.14) and (7.26), together with

$$\partial u_E(\mathbf{x}, 0)/\partial t = 0; \quad \text{on } E_x.$$
 (7.33)

(h) The functional of Theorem 6.4, is

^

$$X(u) = \int_{R_x} u * \{ \mathbf{L}u - 2f_R \} d\mathbf{x} + \int_{\partial_1 R_x} (u - 2f_1) * \frac{\partial u}{\partial n} d\mathbf{x} - \int_{\partial_2 R} u * \left( \frac{\partial u}{\partial n} - 2f_2 \right) d\mathbf{x} - \int_{R_x} \{ u(0) - 2f_0 \} u'(T) d\mathbf{x} - \int_{R_x} \{ u'(0) - 2f'_0 \} u(T) d\mathbf{x} + \int_{\partial_3 R_x} \left\{ u * \frac{\partial V}{\partial n} - V * \frac{\partial u}{\partial n} \right\} d\mathbf{x}$$

## 7.4. Elasticity

In order to formulate the problems of Elasticity, the elastic tensor  $C_{ijpq}$  is assumed to be defined in the regions R and E. It will be assumed to be sufficiently differentiable on R and on E, separately; for example, it is not too restrictive to assume that  $C_{ijpq}$ possesses continuous derivatives of all orders on R and on E, that can be extended continuously to the boundaries of these regions. In addition,  $C_{ijpq}$  is assumed to satisfy the usual symmetry conditions (Gurtin, 1972)

$$C_{ijpq} = C_{pqij} = C_{jipq} \tag{7.35}$$

and to be strongly elliptic; i.e.

$$C_{ijpq}\xi_i\eta_j\xi_p\eta_q > 0, \quad \text{whenever } \xi_i\xi_i \neq 0; \ \eta_i\eta_i \neq 0. \tag{7.36}$$

7.4.1. Static and periodic motions. The elements of the linear spaces  $D_R$  and  $D_E$  can be taken as vector valued functions whose components belong to  $H^s(R)$  and  $H^s(E)$ ,  $s \ge 2$ ,

respectively. When treating periodic motions in unbounded domains, it is frequently convenient to consider complex valued vector fields. Let

$$\tau_{ij}(\mathbf{u}) = C_{ijpq} \frac{\partial u_p}{\partial x_q}; \quad \text{on } R \cup E,$$
(7.37a)

$$\mathbf{L}_{i}(\mathbf{u}) = \frac{\partial \tau_{ij}}{\partial x_{i}} + ku_{i}; \quad \text{on } R \cup E$$
(7.37b)

and

$$T_i(\mathbf{u}) = \tau_{ij}(\mathbf{u})n_j;$$
 on  $\partial R$  and  $\partial E$ . (7.37c)

Here k is a function of position which satisfies continuity conditions similar to those of the elastic tensor. The definition of the tractions T(u) depends on the sense of the unit normal vector, so that two such tractions which have opposite signs, are defined on the common boundary  $\partial R = \partial E$ . As in the case of the normal vector, sometimes they will be represented by  $T_R(u)$  and  $T_E(u)$ ; more often, however, the ambiguity will be resolved by the suffix used under the integral sign. Observe that when considering the problem of connecting the following combinations can occur  $T_R(u_R)$ ,  $T_R(u_E)$ ,  $T_E(u_R)$  and  $T_E(u_E)$ .

The definitions and results for static and periodic motions in Elasticity are listed below:

(a)  $P_R: D_R \to D_R^*$  is

$$\langle P_R u_R, v_R \rangle = \int_R v_i \mathbf{L}_i(\mathbf{u}) \, d\mathbf{x} + \int_{\partial_1 R} u_i T_i(\mathbf{v}) \, d\mathbf{x} - \int_{\partial_2 R} v_i T_i(\mathbf{u}) \, d\mathbf{x}$$

and  $P_E: D_E \to D_E^*$  is obtained replacing R by E in (7.38).

(b) The smoothness condition can be taken as

$$\widehat{\mathbf{S}} = \{ \widehat{u} \in \widehat{D} | u_{Ri} = u_{Ei}; \quad T_i(u_R) = T_i(\mathbf{u}_E); \quad \text{on } \partial_3 R \}.$$

(c) When  $\hat{v} \in \hat{S}$  and  $\hat{u} \in \hat{D}$  is arbitrary

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \int_{\partial, R} \left\{ \left[ \hat{u}_i \right] T_i(\mathbf{v}) - v_i \left[ T_i(\hat{u}) \right] \right\} d\mathbf{x}.$$
(7.40)

Here

$$[\hat{u}_i] = u_{Ei} - u_{Ri}; \qquad [T_i(\hat{u})]_R = T_{Ri}^E(\mathbf{u}_E) - T_{Ri}^R(\mathbf{u}_R), \tag{7.41}$$

where the sub-indices R and E in the tractions refer to the normal vector used, while the super-indices refer to the elastic tensor used; thus, for example

$$T_{R_i}^E(\mathbf{u}_E) = C_{ijpq}^E \frac{\partial u_{Ep}}{\partial x_q} n_{Rj}.$$

- (d)  $\hat{S}$  is completely regular for  $\hat{P}: \hat{D} \to \hat{D}^*$ . This result can be established using Lemma 5.1 and strong ellipticity (equation (7.36)).
- (e) Equation (5.28), yields

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{ [\hat{u}_i] T_i(\mathbf{v}) - \bar{v}_i [T_i(\mathbf{u})] \} d\mathbf{x}$$
(7.42)

(f) Given any  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$ , the problem of connecting (5.11) is equivalent to

$$\mathbf{L}_{i}\hat{\mathbf{u}} = f_{R\cup Ei} = \mathbf{L}_{i}\hat{U}; \quad \text{on } R\cup E \tag{7.43a}$$

$$\hat{u}_i = f_{1i} = \hat{U}_i;$$
 on  $\partial_1(R \cup E)$  (7.43b)

$$T_i(\hat{\mathbf{u}}) = f_{2i} = T_i(\hat{\mathbf{U}}); \quad \text{on } \partial_2(R \cup E)$$
(7.43c)

subjected to the jump conditions

$$[\hat{\mathbf{u}}] = \mathbf{f}_{J1} = [\hat{\mathbf{V}}], \qquad [\mathbf{T}(\hat{\mathbf{u}})] = \mathbf{f}_{J2} = [\mathbf{T}(\mathbf{V})]; \quad \text{on } \partial_3 R. \tag{7.44}$$

(g) For this problem, the variational formulation of Theorem 6.3 yields the functional

$$\Omega(\hat{u}) = \int_{R \cup E} u_i (\mathbf{L}_i \mathbf{u} - 2f_{R \cup Ei}) \, d\mathbf{x} + \int_{\partial_1(R \cup E)} (u_i - 2f_{1i}) T_i \mathbf{u} \, d\mathbf{x} - \int_{\partial_2(R \cup E)} u_i (T_i \mathbf{u} - 2f_{2i}) \, d\mathbf{x} + \int_{\partial_3 R} \{ \overline{u}_i ([T_i(\mathbf{u})] - 2f_{J_{2i}}) - ([u_i] - 2f_{J_{1i}}) \overline{T_i(\mathbf{u})} \} \, d\mathbf{x}.$$
(7.45)

(h) The functional of Theorem 6.4, for the problem with restrictions of continuation type is given by

$$X(\mathbf{u}) = \int_{R} u_i \{ \mathbf{L}_i \mathbf{u} - 2\mathbf{f}_{Ri} \} d\mathbf{x} + \int_{\partial_1 R} (u_i - 2f_{1i}) T_i \mathbf{u} d\mathbf{x} - \int_{\partial_2 (R \cup E)} u_i (T_i \mathbf{u} - 2f_{2i}) d\mathbf{x} + \int_{\partial_3 R} \{ u_i T_i(\mathbf{v}) - V_i T_i(\mathbf{u}) \} d\mathbf{x}.$$
(7.46)

7.4.2 Dynamics. The extension from elastostatics to dynamic elasticity is very similar to that carried out when going to the wave equation from Laplace's. The operators have to be defined as

$$\mathbf{D}_{i}\mathbf{u}=\mathbf{L}_{i}\mathbf{u}-\rho\,\frac{\partial^{2}u_{i}}{\partial t^{2}}$$

where  $L_i$  is given by (7.37b) with  $k \equiv 0$ ; then

$$\langle P_R \mathbf{u}_R, \mathbf{v}_R \rangle = \int_{R_x} v_i * \mathbf{D}_i \mathbf{u} \, d\mathbf{x} + \int_{\partial_1 R_x} u_i * T_i(\mathbf{v}) \, d\mathbf{x} - \int_{\partial_2 R_x} v_i * T_i(\mathbf{u}) \, d\mathbf{x} - \int_{R_x} \rho \{ u_i(0) v_i'(T) + u_i'(0) v_i(T) \} \, d\mathbf{x} \quad (7.48)$$

where, as in (7.29), the primes stand for the partial derivatives with respect to time. The regions are shown in Fig. 3. The smoothness condition is given (7.39), with the new interpretation of  $\partial_3 R$ . It can be shown that  $\hat{S}$  is completely regular for  $\hat{P}: \hat{D} \to \hat{D}^*$ , so that Theorems 6.3 and 6.4 can be applied.

For the jump operator, it is obtained

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_3 R} \{ [u_i] * \overline{T}_i(\mathbf{v}) - \bar{v}_i * [T_i(\mathbf{u})] \} d\mathbf{x}$$
(7.49)

## 7.5. A Two Phase Problem

Let R in Fig. 1 be occupied by a linear elastic solid, while E will be occupied by an inviscid compressible fluid. It will be assumed that the motion in E is potential and the governing equations have been linearized.

For periodic motions of angular frequency  $\omega$ , equations (7.43) apply on R, with  $k = \rho \omega^2$ . In general, when the motion is non-periodic, the equations in E are (Meyer, 1972; Landau & Lifshitz, 1959)

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0; \quad \text{on } E$$
(7.50)

where p is the pressure and  $c^2 = (dp/d\rho)_0$  will be taken as constant. The acceleration  $a_i$  is

$$a_i = -\frac{\partial p}{\rho \, \partial x_i}.\tag{7.51}$$

The smoothness conditions across the connecting boundary  $\partial_3 R = \partial_3 E$  are: continuity of tractions and continuity of the normal components of displacements. For periodic motions  $p = u_E e^{i\omega t}$ , this leads to

$$u_E n_i + T_i(\mathbf{u}) = 0; \quad \text{on } \partial_3 R, \tag{7.52}$$

$$\frac{\partial u_E}{\partial n} - \rho \omega^2 u_i n_i = 0; \quad \text{on } \partial_3 R \tag{7.53}$$

The inhomogeneous form of (7.50), for such periodic motions, is

$$\nabla^2 u_E + \rho \omega^2 u_E = f_E; \quad \text{on } E.$$

Therefore, the problem is governed by (7.43) on R and (7.54), subjected to the smoothness conditions (7.52). In order to consider the more general problem, for which the right-hand side in equation (7.53) may be prescribed non-zero functions, the operator  $P_R: D_R \to D_R^*$  will be defined multiplying the right-hand side of (7.38) by  $\rho\omega^2$ , while  $P_E: D_E \to D_E^*$  is defined when replacing  $\rho$  by  $\rho\omega^2$ , R by E and setting  $k_E = 1$ , in (7.2). Notice that functions of  $D_R$  are vector valued, while those of  $D_E$  have only one component. Then

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \rho \omega^2 \int_{\partial_2 R} \left\{ v_i T_i(\mathbf{u}) - u_i T_i(\mathbf{v}) \right\} \, \mathbf{dx} + \int_{\partial_3 E} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} \, d\mathbf{x}. \tag{7.55}$$

The smoothness relation  $\hat{\mathbf{S}} \subset \hat{D}$  is defined as the set whose elements satisfy (7.52). When  $\hat{v} = (v_R, v_E) \in \hat{\mathbf{S}}$ , while  $\hat{u} = (u_R, u_E) \in \hat{D}$  is arbitrary, equation (7.55) reduces to

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \rho \omega^2 \int_{\partial_3 R} v_i \{ T_i(\mathbf{u}) + u_E n_i \} d\mathbf{x} + \int_{\partial_3 R} T_i(\mathbf{v}) \left\{ \frac{\partial u_E}{\partial n} n_i - \rho \omega^2 u_i \right\} d\mathbf{x}.$$

When strong ellipticity (7.36) is satisfied, it can be shown that  $v_i$  and  $T_i(v)$  can be varied independently. Using this fact and equation (7.56), it is not difficult to see that

$$\langle A\hat{u}, \hat{v} \rangle = 0, \quad \forall \ \hat{v} \in \mathbf{S} \Leftrightarrow \hat{u} = (\mathbf{u}_R, u_E) \text{ satisfies } (7.52);$$
(7.57)

hence,  $\hat{\mathbf{S}}$  is completely regular for  $\hat{P}$ . That  $\hat{P}$  is formally symmetric follows from the fact that

$$\hat{N} = \{ \hat{u} \in \hat{D} | u_{Ri} = u_E = \partial u_{Ri} / \partial n = \partial u_E / \partial n = 0; \text{ on } \partial_3 R \}$$
(7.58)

which involves boundary conditions on  $\partial_3 R$ , only. Thus, the general theory developed previously is applicable to  $\hat{P}$  and the variational principles of Theorem 6.3 and 6.4 are applicable to this problem. It is now a straightforward exercise to obtain corresponding formulae, but the details will not be included here.

#### REFERENCES

- BABUSKA, I. & AZIZ, A. K. 1972 Survey lectures on the mathematical foundations of the finite element method. In *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (A. K. Aziz, Ed.), pp. 3–859. New York: Academic Press.
- BATES, R. H. T. 1975 Analytic Constraints on Electro-magnetic Field Computations, I.R.E. Trans. microw. Theory Tech. 23, 605–623.
- BREBBIA, C. A. (Ed.) 1978 Recent Advances in Boundary Element Methods. London: Plymouth Pentech Press, pp. 424.
- CRUSE, T. A. 1974 An improved boundary-integral equation method for three dimensional elastic stress analysis, *Comput. Struct.* 4, 741–754.
- CRUSE, T. A. & RIZZO, F. J. 1975 Boundary-integral equation method: Computational applications in applied mechanics. Am. Soc. Mech. Eng. 11, 000-000.
- GURTIN, M. E. 1972 The Linear Theory of Elasticity. In *Encyclopedia of Physics*, VIa/2, pp. 1–295. Berlin: Springer-Verlag.
- HEISE, U. 1978 Numerical properties of integral equations in which the given boundary values and the sought solutions are defined on different curves. *Comput. Struct.* 8, 199–205.
- HERRERA, I. 1974 A General Formulation of Variational Principles. Instituto de Ingenieriá, Universidad Nal. A. de México, México, E-10.
- HERRERA, I. 1977a General variational principles applicable to the hybrid element method. Proc. Nat. Acad. Sci. USA 74, 2595–2597.
- HERRERA, I. 1977b Theory of connectivity for formally symmetric operators. Proc. Nat. Acad. Sci. USA 74, 4722-4725.
- HERRERA, I. 1977c On the variational principles of mechanics. In Trends in Applications of Pure Mathematics to Mechanics, Vol. II, pp. 115-128. London: Pitmans.
- HERRERA, I. 1978a Theory of connectivity: A systematic formulation of boundary element methods. Appl. Math. Mod. 3, 151-156.
- HERRERA, I. 1978b Theory of connectivity as a unified approach to boundary methods. Proc. IUTAM Symposium on Variational Principles of Mechanics. Evanston, Ill.: Pitmans. (In press.)
- HERRERA, I. & BIELAK, J. 1976 Dual variational principles for diffusion equations. Q. Appl. Math. 34, 85-102.
- HERRERA, I. & SABINA, F. J. 1978 Connectivity as an alternative to boundary integral equations. Construction of bases. Proc. Nat. Acad. Sci. USA 75, 2059–2063.
- HERRERA, I. & SEWELL, M. J. 1978 Dual extremal principles for Non-negative unsymmetric operators. J. Inst. Maths Applics 21, 95-115.

- KANTOROVICH, L. V. & KRYLOV, V. I. 1964 Approximate Methods of Higher Analysis. (Translated from Russian by C. D. Benster). New York: John Wiley.
- KUPRADZE, V. D. 1965 Potential Methods in the Theory of Elasticity. London: Oldbourne Press.
- KUPRADZE, V. D., GEGELIA, T. G., BASCHELEJSCHWILI, M. O. & BURTSCHULADGE, T. V. Threedimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. (In Russian.) Moskow: Nauka.
- LANDAU, L. D. & LIFSHITZ, E. M. 1959 Fluid Mechanics. London: Pergamon Press.
- LIONS, J. L. & MAGENES, E. 1968 Problemes aux Limits non Homogenes et Applications, Vol. 1. Paris: Dunod.
- MEI, C. C. & CHEN, H. S. 1976 A hybrid element method for steady linearized free surface flows. Int. J. Num. Meth. Engng. 10, 1153–1175.
- MEYER, R. E. 1971 Introduction to Mathematical Fluid Dynamics. London: John Wiley.
- MILLAR, R. F. 1973 The Rayleigh hypothesis and a related least-squares solution to scattering problems for periodic surfaces and other scatterers. *Radio Sci.* 8, 785–796.
- NASHED, M. Z. 1971 Differentiability and related properties of non-linear operators: some aspects of the role of differentials in non-linear functional analysis. In Non-linear Functional Analysis and Applications (L. B. Rall, Ed.), pp. 103–109. New York: Academic Press.
- NEMAT-NASSER, S. 1972a General variational principles in non-linear and linear elasticity with applications. In *Mechanics Today*, Vol. 1 (S. Nemat-Nasser, Ed.), pp. 214–261. London: Pergamon Press.
- NEMAT-NASSER, S. 1972b On variational methods in finite and incremental elastic deformation problems with discontinuous fields. Q. Appl. Math. 30, 143–156.
- OLIVEIRA, E. R. 1968 Plane stress analysis by a general integral method. J. Engng. Mech. Div. ASCE 94, 79-101.
- RIEDER, G. 1962 Iterationsverfahren und Operatorgleichiengen in der Elastizitätstheorie. Abh. Braunschweig. Wiss. Ges. 14, 109–343.
- REIDER, G. 1968 Mechanische Deuting und Klassifizierung einiger Integralverfahren der ebenen Elastizitätstheorie I, II. Bull. Acad. Pol. Sci. Ser. Sci. Techn. 16, 101-114.
- RIZZO, F. J. 1967 An integral equation approach to boundary value problems of classical elastostatics. *Quart. J. Appl. Math.* 25, 83–95.
- SABINA, F. J., HERRERA, I. & ENGLAND, R. 1978 Theory of connectivity. Applications to seismic waves. I. SH-wave motion. In Proc. 2nd Int. Conf. Microzonation, San Francisco, 1978. Available as Comunicaciones Técnicas 9 (175), IIMAS-UNAM, México. To appear.
- SANCHEZ-SESMA, F. J. & ROSENBLUETH, 1979 Ground motion at canyons of arbitrary shape under incident SH waves. Int. J. Earthquake Engng. Struct. Dyn. 7, 3–9.