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A FINITE ELEMENT IN TIME AND SPACE MODEL FOR QUASI-THREE-DIMENSIONAL MULTIAQUIFER SIMULATION

by

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ABSTRACT.

A finite element in time and space model is applied to the quasi-three-dimensional approach to multiaquifer simulation based on the integrodifferential formalism developed by Herrera and his coworkers [1-4]. The resulting model is applicable to general heterogeneous multiaquifer systems over arbitrarily shaped regions and is presently being applied to the problem of Mexico City land subsidence.

1 INTRODUCTION

In a series of papers, Herrera and his coworkers [1-4] developed an integrodifferential formalism for the modelling of multiaquifer systems, based on the quasi-three-dimensional theory for such systems where the flow is assumed to be horizontal in the aquifers and vertical in the intervening aquitards. In most practical situations where the permeability contrast between aquifers and adjoining aquitards is two orders of magnitude or greater, the error resulting from this assumption is generally less than 5% [5], which is acceptable for most engineering purposes. In the original model, the method was formulated for homogeneous aquifers in terms of dimensionless time and space variables. This restriction has been removed in the present model and general heterogeneous aquifers can be treated in demensional variables as shown in Section 2. A general computer code based on the formalism of Section 2 was developed and is being applied to the problem of land subsidence in Mexico City. Some details of its implementation are described in Section 3, leading eventually to a consistent finite element in time and space model.

2. INTEGRODIFFERENTIAL MODEL FOR HETEROGENEOUS AQUIFERS.

Considering for the sake of simplicity a system of two aquifers separated by one aquitard (see Figure 1), the governing equations are

$$\frac{\partial}{\partial x}\left(T_1 \frac{\partial}{\partial x} s_1\right) + \frac{\partial}{\partial y}\left(T_1 \frac{\partial}{\partial y} s_1\right) + K' \frac{\partial s'}{\partial z}\Big|_{z=0} = S_1 \frac{\partial}{\partial t} s_1 + Q_1,$$

$$\frac{\partial}{\partial z}\left(K' \frac{\partial s'}{\partial z}\right) = S'_s \frac{\partial}{\partial t} s'$$

and

$$\frac{\partial}{\partial x}\left(T_2 \frac{\partial}{\partial x} s_2\right) + \frac{\partial}{\partial y}\left(T_2 \frac{\partial}{\partial y} s_2\right) - K' \frac{\partial s'}{\partial z}\Big|_{z=b'} = S_2 \frac{\partial}{\partial t} s_2 + Q_2 , \qquad (1c)$$

where Darcy's law has been used to express the leakage fluxes. The notation used is standard in aquifer mechanics.

In Eqs. (1), all the coefficients are allowed to vary horizontally. K' and S'_s would normally also exhibit a vertical variation. To simplify this presentation, we shall assume that they are constant vertically or can be suitably averaged in that direction.

This is a quite frequent situation in practice, especially when the field data are too scattered to provide more than reasonable average values. Anyhow this is an unnecessary restriction of the method that has been subsequently removed [6]. In a recent comparison between existing quasi-three-dimensional models [7], it has been shown that the method here presentedhas important advantages over an alternative one that has just been published [8]. This is due to the application of a boundary element method which permits eliminating the aquitard from the numerical treatment and restrict attention to the main aquifers only. Under the averaging assumption for instance, Eq. (1b) can be integrated out and the set of Eqs. (1) reduces to

$$\frac{\partial}{\partial x}\left(T_1 \frac{\partial}{\partial x} s_1\right) + \frac{\partial}{\partial y}\left(T_1 \frac{\partial}{\partial y} s_1\right) - \frac{K'}{b'} \int_0^t \frac{\partial s_1}{\partial t}(t-\tau)\, f\left(\frac{\alpha'\tau}{b'^2}\right) d\tau + \frac{K'}{b'} \int_0^t \frac{\partial s_2}{\partial t}(t-\tau)\, h\left(\frac{\alpha'\tau}{b'^2}\right) d\tau = S_1 \frac{\partial s_1}{\partial t} + Q_1 \quad , \tag{2a}$$

and

$$\frac{\partial}{\partial x}\left(T_2 \frac{\partial}{\partial x} s_2\right) + \frac{\partial}{\partial y}\left(T_2 \frac{\partial}{\partial y} s_2\right) - \frac{K'}{b'} \int_0^t \frac{\partial s_2}{\partial t}(t-\tau) f\left(\frac{\alpha'\tau}{b'^2}\right) d\tau + \frac{K'}{b'} \int_0^t \frac{\partial s_2}{\partial t}(t-\tau) h\left(\frac{\alpha'\tau}{b'^2}\right) d\tau = S_2 \frac{\partial s_2}{\partial t} + Q_2 \quad . \tag{2b}$$

In (2), $\alpha' = K'/S_s'$ while $f = 1+g$ and h are the memory and influence functions introduced in [1], namely

$$f(t') = 1 + 2 \sum_{n=1}^{\infty} \exp(-n^2\pi^2 t') \tag{3a}$$

and

$$h(t') = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp(-n^2\pi^2 t'). \tag{3b}$$

For computational purposes, g and h can be approximated [4] by finite expansions like

$$g(t') \approx g_{N_1}(t') = A_{N_1}\, \delta(t') + \sum_{n=1}^{N_1} a_n \exp(-n^2\pi^2 t') \quad , \tag{4a}$$

and

$$h(t') \approx h_{N_2}(t') = 1 + \sum_{n=1}^{N_2} b_n \exp(-n^2\pi^2 t') \tag{4b}$$

Any standard finite element discretization would then look for

$$s_k(x,y,t) \approx \sum_{j=1}^{N} P_{kj}(t)\, u_j(x,y), \quad k=1,2, \tag{5}$$

where the $u_j(x,y)$ are given piecewise polynomial basis functions, whose support is a small patch of elements. In the original integrodifferential model [4], homogeneity was assumed and consequently t' was a convenient dimensional time which did not depend on position. Here, however, t' is a function of position and the memory and influence terms become quite involved in the final Galerkin semidiscrete equations, unless, as we found out, reduced integration is used [9, p. 537]. Under any numerical integration scheme

consistent with the expected accuracy implicit in the choice of the finite elements the mass-like matrix elements are evaluated as a summation

$$\int_{\Omega} f(\bar{r},t)\, u_i(\bar{r}) u_j(\bar{r}) d\bar{r} \approx \sum_k w_k\, f(\bar{r}_k,t) u_i(\bar{r}_k) u_j(\bar{r}_k) \quad , \qquad (6)$$

where $\bar{r}_k$ is the sampling point where the integrand has to be found and w_k, the corresponding weighting. If the sampling points are located at the mesh nodes, as all except one basis function are zero there, the corresponding matrix becomes diagonal. As a result, the integral terms are diagonalized and t' is only needed at the mesh points, which is quite convenient. This lumping by reduced integration presents some additional advantages: the resulting Galerkin semidiscrete equations are less stiff and their time integration is therefore easier, the treatment of general boundary conditions is greatly simplified and finally considerable reductions in computer time and memory requirement can be achieved by careful programming, as we indeed verified. The final set of equations obtained from (2) after lumping by reduced integration has been performed exhibits the following general structure

$$A_1 \dot{P}_1 = - B_1 P_1 - C \left(\sum_{n=1}^{N_1} a_n D_{1n} - \sum_{n=0}^{N_2} b_n D_{2n} \right) + S_1 \quad , \qquad (7a)$$

and

$$A_2 \dot{P}_2 = - B_2 P_2 - C \left(\sum_{n=1}^{N_1} a_n D_{2n} - \sum_{n=0}^{N_2} b_n D_{1n} \right) + S_2 ,$$

assuming zero initial conditions for $P_k = [P_{k1},\ldots,P_{kN}]^T$ and taking into account (4). In (7), the matrices A_k and B_k are symmetric and banded, C is a diagonal matrix, the source vectors S_k directly derive from the pumping rates Q_k while the vectors D_{kn} have components $D_{kni}(t)$ given by

$$D_{kni}(t) = \exp(-n^2\pi^2\lambda_i t) \int_0^t \dot{P}_{ki}(\tau) \exp(+ n^2\pi^2\lambda_i \tau)\, d\tau \quad , \qquad (8)$$

where $\lambda_i = \alpha'/b'^2\,(\bar{r}_i)$

Once we have coped with the major difficulty due to heterogeneity, namely the treatment of the memory and influence terms, thanks to the reduced integration technique, the final equations (8) exhibit essentially the same properties as in the previous integro-differential model [4,7]. For the sake of completness, we shall briefly recall them: first of all, Eqs. (7a) and (7b) may be regarded as uncoupled, owing to the shape of the influence function h from one aquifer to the other one. Furthermore, the memory and influence terms can be evaluated at each time step by a simple updating procedure for the vectors D_{kn}, which is independent of any reference time and does not involve past history. Finally, the drawdown in the aquitard need not be calculated unless it is required at some time, in which case it can be evaluated simply in terms of the values in the adjacent aquifers and of auxiliary terms which occur in the calculation [see e. g. Ref 4].

3. FINITE ELEMENT IN SPACE AND TIME IMPLEMENTATION OF THE INTEGRODIFFERENTIAL MULTIAQUIFER MODEL.

Starting from an early version [4] applicable to homogeneous aquifers in terms of dimensionless space and time variables, we gradually developed a much more general computer code, which can treat heterogeneous systems of one or more aquifers separated by arbitrary aquitards. A specific application of this procedure has been the development of a model for the Valley of Mexico City to study land subsidence due to withdrawal of ground water. The preprocessing phase, i.e. the finite element mesh generation and the subsequent matrix element evaluation, has been considerably simplified by use of a procedure

which is applicable to star-shaped regions,which is the case of most two-dimensional regions of practical interest. Under this assumption, it is easy to find a one-to-one (non-conformal) global mapping which maps Ω into a circle of radius a, namely $\Omega^* \equiv [0,a] \times [0,2\pi]$ in (ρ,ϕ) coordinates. If we assume that the boundary of Ω is described by a known function $r=g(\Theta)$ obtained for instance by cubic spline interpolation, then the simple radially expansive map defined by

$$\rho = ar/g\ (\Theta), \tag{9a}$$

and

$$\varphi = \Theta\ , \tag{9b}$$

puts the points of Ω into one-to-one correspondence with the points of Ω^*. Over Ω^*, standard tensor product discretizations are easily implemented. Since all the mass and stiffness like matrix elements are evaluated by numerical integration, this mapping does not introduce any additional complexity, especially when reduced integration is used. Of course, some care has to be exercized in the node numbering because of the periodicity condition at $\varphi=0$ and $\varphi=2\pi$: we found a solution that essentially yields the bandwidth normally expected in a 2D situation, by numbering the nodes alternatively from $\varphi=0$ and $\varphi=2\pi$ for a given ρ.

All these implementation details are particularly well suited to the specific applications that we are presently considering. It should however be pointed out that other finite element discretizations, for instance with triangles or isoparametric quadrilaterals, would equally fit in the proposed multiaquifer model and lend themselves to the reduced integration technique of Section 2, which was the main simplification required in order to overcome the difficulties encountered in the treatment of heterogeneous aquifers.

In the previous versions of our code, time integration of Eqs. (7) was restricted to the Crank-Nicolson procedure which is known to exhibit poor asymptotic stability properties especially in presence of fast transients. This disadvantage has also been eliminated from the present model by the introduction of a consistent finite element approximation in space and time [10]. Looking back at Eqs. (7), it is clear that each of them has the following general structure

$$A\,\dot{P} = -\,BP - CI(P) + S \tag{10}$$

where I(P) stands for the integral terms depending on P. If Eq. (10) is integrated over $[t_i,\ t_{i+1}]$, we get

$$\begin{aligned} A\,[P(t_{i+1}) - P(t_i)] = &- B \int_{t_i}^{t_{i+1}} P\ dt \\ &- C \int_{t_i}^{} I(P)\ dt \\ &+ \int_{t_i}^{t_{i+1}} S\ dt\ , \end{aligned} \tag{11}$$

using the fact that the matrices A, B and C are time-independent. In the particular case, where not only the coefficients which appear in A, B and C, but also the boundary conditions and the pumping rates are constant, the general equations (1) constitute a system

of coupled parabolic partial differential equations, whose solution is a linear combination of exponentials decaying with different time constants. After the fast transients have sufficiently decayed or in any case some time after any modification ocurred to the boundary conditions and (or) the pumping rates, the solution approximately exhibits a single mode exponential behavior and it is therefore reasonable to assume that the components $P_j, j=1,\ldots,N$ of P over $[t_i, t_i+h]$ are of the form

$$P_j(t) \approx \tilde{P}_j(t) = a_j + b_j \exp(\mu t) \quad , \tag{12}$$

where μ is some real negative value, which should ideally approximate the algebraically largest inverse characteristic time, on physical or numerical grounds. Introducing that kind of behavior in the righthand side of Eq. (11), we obtain the approximate scheme

$$A\,[\,P_{i+1} - P_i\,] = -B \int_{t_i}^{t_{i+1}} \tilde{P}\,dt - C \int_{t_i}^{t_{i+1}} I(\tilde{P})\,dt + \int_{t_i}^{t_{i+1}} S\,dt \tag{13}$$

where $\tilde{P} = [\,\tilde{P}_1, \ldots, \tilde{P}_N\,]^T$ is completely defined by the interpolation conditions

$$\tilde{P}(t_i) = P_i \tag{14a}$$

and

$$\tilde{P}(t_{i+1}) = P_{i+1} \tag{14b}$$

so that $\tilde{P} = \tilde{P}(P_i, P_{i+1})$, namely

$$\tilde{P} = P_i \frac{\exp z - \exp \mu(t-t_i)}{\exp z - 1} + P_{i+1} \frac{\exp \mu(t-t_i) - 1}{\exp z - 1} \tag{15}$$

with $z \equiv \mu\,(t_{i+1}-t_i) = \mu h$. Eq (13) thus defines a one-step scheme for P_{i+1} knowing P_i. In particular, we have

$$\int_{t_i}^{t_{i+1}} \tilde{P}\,dt = h\,P_i\,[1 - \Theta(\mu,h)] + h\,P_{i+1}\,\Theta(\mu,h) \tag{16}$$

with

$$\Theta(\mu,h) = (\exp z - 1 - z)/\,z(\exp z - 1) \tag{17}$$

relating this class of schemes to the classical Θ- schemes [11], with the difference that here Θ is normally not fixed but actually depends on the time step Θ and on the inverse

characteristic time μ of the mode we want to reproduce. It is easy to show that any μ satisfying $-\infty \leq \mu \leq 0$ provides an A-stable integration scheme. Some particular cases are well-known:for instance as μ tends to zero, Θ tends to 1/2 so that (16) becomes

$$\int_{t_i}^{t_{i+1}} \tilde{P}\, dt = h(P_i + P_{i+1})/2 \tag{18}$$

corresponding to the Crank-Nicolson scheme. Also as μ tends to $-\infty$, Θ tends to 1 and (16) becomes

$$\int_{t_i}^{t_{i+h}} \tilde{P}\, dt = h\, P_{i+1} \tag{19}$$

corresponding to the Backward Euler scheme. The interpolation properties of these schemes and some other ones are sketched on Figure 2. If P is actually of the form $a + b \exp \mu t$, $P \equiv \tilde{P}$ and no approximation is introduced in the calculation of the second member of (13) if of course the integrals can be performed analytically, which is in general the case for this class of problems. In fact, one of the major virtues of this approach is that it provides us with consistent rules to derive the updating rules for the vectors D_{kn}, based on Eq. (8), mentioned in Section 2. When large times are reached, the previous considerations also allow us to increase h considerably without any loss of accuracy.

As a final comment, let us point out that the proposed implementation is of the finite element type in space and time, since the values of P at any point $(\bar{r},t)$ can be retrieved from the nodal values. The present version of our code exhibits all the details of implementation mentioned in this paper and is presently applied to the problem of Mexico City land subsidence. A detailed report of this problem will be published elsewhere. However, we can advance that the results thus far obtained have been quite satisfactory.

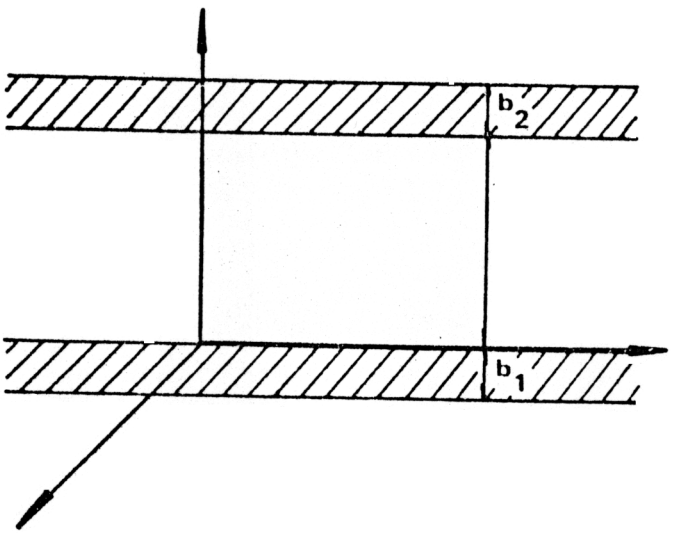

Figure aquifer

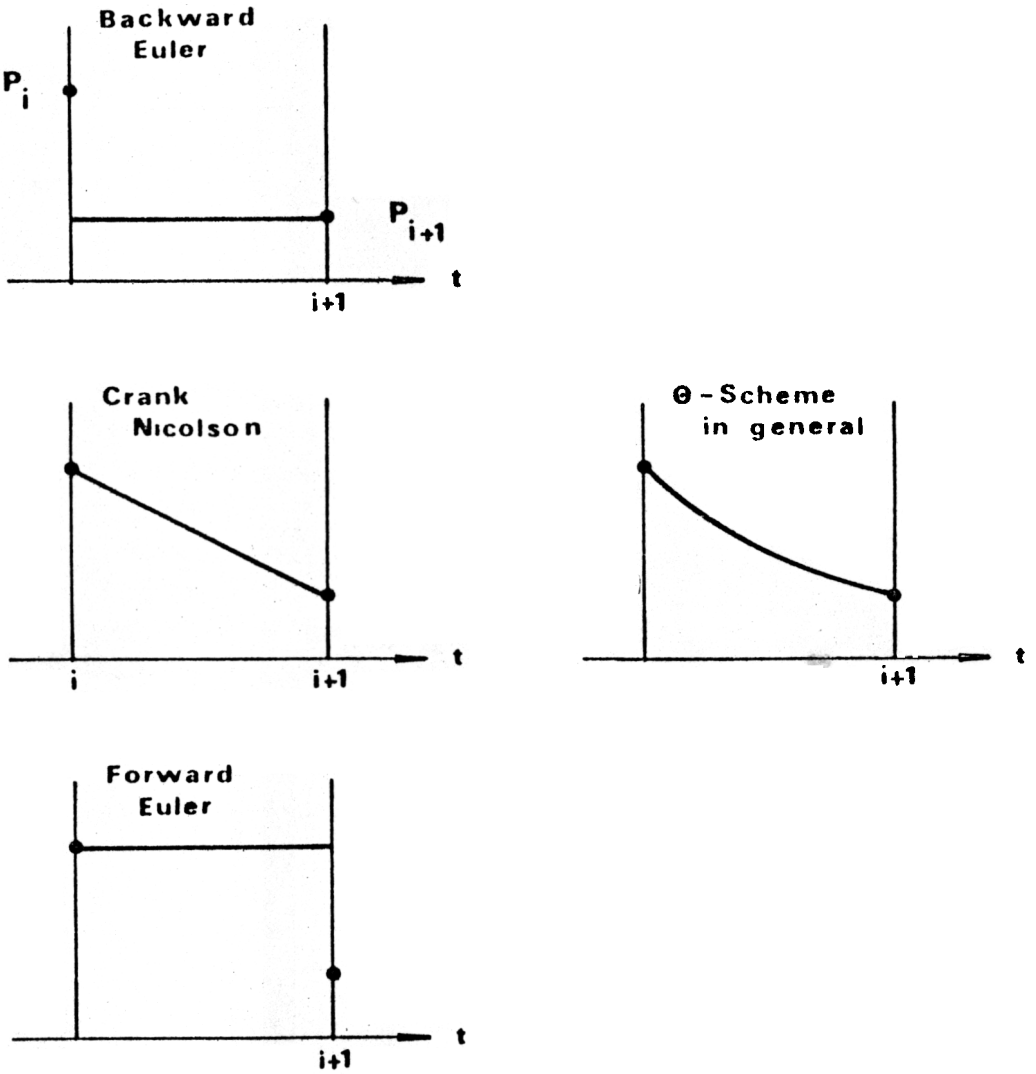

Figure 2. Some time integration schemes and their interpolation properties.

REFERENCES.

1. HERRERA, I. and RODARTE, L., "Integrodifferential Equations for Systems of Leaky Aquifers and Applications. 1. The Nature of Approximate Theories", Water Resour. Res., Vol. 9, pp. 995-1005, 1973.

2. HERRERA, I., "Integrodifferential Equations for Systems of Leaky Aquifers and Applications. 2. Error Analysis of Approximate Theories", Water Resour. Res., Vol.10, pp. 811-820, 1974.

3. HERRERA, I., YATES, R., and PEREZ DE CELIS, C., "A Family of Approximate Numerical Procedures for Leaky Aquifer Systems", Proc. Second International Symposium of Finite Element Methods in Flow Problems, pp. 335-364, S. Margherita Ligure, Italy, 1976.

4. HERRERA, I. and YATES, R., "Integrodifferential Equations for Systems of Leaky Aquifers and Applications. 3. A Numerical Method of Unlimited Applicability", Water Resour. Res., Vol. 13, pp. 725-732, 1977.

5. NEUMAN, C. P. and WITHERSPOON, P. A., "Theory of Flow in a Confined Two-Aquifer System", Water Resour. Res., Vol. 5, pp. 817-829, 1969.

6. YATES, R., HERRERA, I., and HENNART, J. P., "Extension of the Integrodifferential Approach to Heterogeneous Multiaquifer Systems", Technical Report, to appear en IIMAS-UNAM, Comunicaciones Técnicas, Serie NA.

7. HERRERA, I., HENNART, J. P., and YATES, R., "A Critical Discussion of Numerical Models for Multiaquifer Systems, to appear in Proc. Colloque International sur Certains Aspects du Calcul des Caractéristiques Hydrologiques pour la Planification des Ressources en Eau, Leningrad, USSR, 1979.

8. CHORLEY, D. W. and FRIND, E. O., "An Iterative Quasi-Three-Dimensional Finite Element Model for Heterogeneous Multiaquifer Systems", Water Resour. Res., Vol. 14, pp. 943-952, 1978.

9. ZIENKIEWICZ, O. C., "The Finite Element Method, 3rd. Edition", McGraw-Hill, London 1977.

10. HENNART, J. P., "Multiple Collocation Finite Elements in Time for Parabolic Evolution Problems", to appear in MAFELAP 1978. The Mathematics of Finite Elements and Applications, WHITEMAN, J. R., Ed., Academic Press, London, 1979.

11. DOUGLAS, J., Jr. and DUPONT, T., "Galerkin Methods for Parabolic Equations", SIAM J. Numer. Anal., Vol. 7, pp. 575-626, 1970.