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BOUNDARY METHODS IN FLOW PROBLEMS

by
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ABSTRACT

A brief review of boundary methods is presented. Emphasis is placed in alternatives to boundary integral equations. Different theoretical questions required to give a firm foundation to these procedures are discussed. Extensions to non-linar problems are explained. Some examples of application to fluid problems are included.

1. INTRODUCTION.

Boundary methods are being used to treat many problems. When they are applicable the size of the regions that need to be treated numerically can be reduced.

In general, two situations can be distinguished. One may consider a boundary value problem in a region such as R (Fig. 1a), in which general analytical solutions are known in the whole region R, or alternatively, the problem may be formulated in a region such as R∪E (Fig. 1b), and the general analytical solutions maybe known only

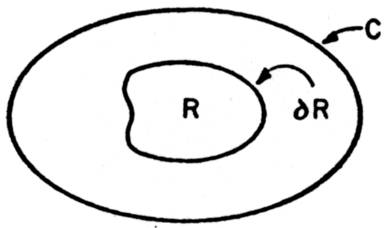

Fig. -a

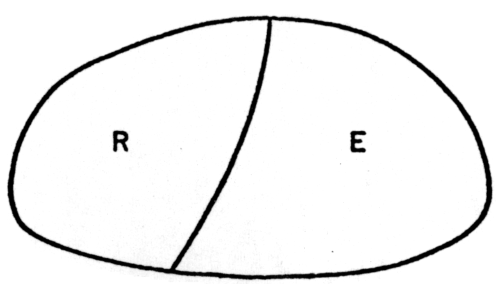

Fig. 1-b

FIG :

in the subregion R. In the first case, the application of boundary methods permits restricting the numerical treatment to the boundary ∂R only, and the dimensionality of the problem is reduced. In the second case, the subregion R has to be treated numerically but the region E is eliminated.

Most, frequently, boundary methods have been formulated by means of integral equations [1-5]. However, there are alternatives which present advantages. In general, a complete family of solutions is required in order to apply boundary methods. There are two possible ways in which such family can be supplied; one is a denumerable family of regular solutions [6] and another one, is by means of a fundamental singular solution. The boundary integral equation method is a special case in which the singularity is placed on the boundary of the region considered and due to this fact the resulting equations are singular.

When a denumerable family of regular solutios is used the resulting equations are non-singular; a fact that offers numerical advantages. What is probably more important in complicated problems, is the observation that generally, it is easier to construct complete families of solutions, than fundamental solutions; indeed, there are methods available for synthesizing fundamental solutions starting from complete families (e.g. in terms of plane waves [7]).

Several recent accounts of boundary integral equations (B.I.E.) are available [1-3]. However, alternative approaches that have been the subject of systematic research by the author [8-16] will be the main concern of this paper. The theoretical basis of these alternatives is not complicated, but the applicability of the methods has been unnecessarily restricted due to lack of clarity. For example, in acoustics and electromagnetic field computations "Rayleigh hypothesis" introduces severe restrictions [17] which can be avoided altogether if a different point of view is adopted [18].

Generally, the solution of a boundary value problem that is well-posed depends continuously on the boundary data [19]; thus, all what is required to consttuct an arbitrary solution, is to have available a family of solutions in terms of which one can approximate any given boundary values.

The general description of the method is given in Section 2. A discussion of the theoretical questions which require study is presented in Section 3. An abstract frame-work, developed by the author, which can be used to make the subject more systematic is briefly described in Section 4. This can be interpreted in terms of abstract Green's formulas; here, it is only presented in a form which is suitable for application to formally symmetric operators, but has recently been extended to general non-symmetric operators [20]. One of the applications of this frame-work is the formulation of variational principles; an exposition of this subject has just been published [13] and it is briefly explained in Section 5. Problems formulated in discontinuous fields with prescribed jump conditions such as those that were surveyed by Nemat-Nasser [21], constitute a general application of the theory; another example, are problems subjected to continuation type restrictions, i.e. problems formulated in a region such as R (Fig. 1b), subjected to the restriction that the solutions can be continued smoothly into solutions of given differential equations in a neighboring region such as E. An abstract version of these problems is given in Section 6. The construction of complete systems is discussed in Section 7, while Section 8 is devoted to explain some extensions to non-linear problems.

SCRIPTION OF THE METHOD

$$\Delta u = 0 \;, \quad \text{on} \quad R \quad ; \quad u = f_{\partial R} \text{ on } \partial R \qquad (2.1)$$

The region R and its boundary ∂R, are illustrated in Fig. 1a. General results on the existence and continuity properties of solutions of elliptic equations [19], grant that for every real s, when $f_{\partial R} \in H^s(\partial R)$ there is a unique $u \in H^{s+1/2}(R)$ which satisfies (2.1); even more, the solution u depends continuously on the boundary data $f_{\partial R}$.

Assume $w_n \in H^{s+1/2}(R)$, n=1,2,..., satisfy $\Delta w_n = 0$ and their boundary values constitute a basis of the Hilbert space $H^s(\partial R)$. When this is the case, it is possible to construct a sequence of approximations

$$u^N = \sum_{n=1}^{N} a_n^N w_n \;; \quad N = 1,2,... \qquad (2.2)$$

such that $u^N \to f_{\partial R}$ in the boundary with respect to metric of $H^s(\partial R)$. The optimal choice of the coefficients a_n^N would correspond to an element u^N of the subspace of $H^s(\partial R)$, generated by $\{w_1, ..., w_N\}$, which is closest to $f_{\partial R}$ in the metric of $H^s(\partial R)$. Clearly, $u^N \to u$ on R, with respect to the norm of $H^{s+1/2}(R)$, because $u^N \to f_{\partial R}$ on $H^s(\partial R)$. Also $u^N \to u$ on R, with respect to the metric of $H^s(R)$, because convergence on $H^{s+1/2}(R)$ implies convergence on $H^s(R)$ [19].

Of special interest is the case s=0, because $H^\circ(\partial R) = \mathcal{L}^2(\partial R)$, is the space of square integrable functions. In applications, it offers numerical advantages to restrict to this space, because of the simplicity of the inner product; also, it is easier to obtain Hilbert space bases. The above discussion shows that if the sequence defined by (2.2) converges in $\mathcal{L}^2(\partial R)$ to $f_{\partial R}$, then u converges to the solution in the region R, in the least square sense.

The procedure described here, is closely related with the series expansion method that has been applied in acoustics and electromagnetic field computations [17]. However, for such kind of applications, severe restriections were imposed by the introduction of the so called "Rayleigh hypothesis". The above discussion shows that when a more convenient criterion is adopted for the choice of the coefficients a_n^N in (2.2), Rayleigh hypothesis is not required; this fact had been pointed out by Millar [18], in connection with such applications.

The computation of the normal derivative on the boundary ∂R, using $\mathcal{L}^2(\partial R)$ inner product only, becomes more involved. If the boundary values $f_{\partial R} \in H^\circ(\partial R)$, one can only guarantee that $\partial u/\partial n \in H^{-1}(\partial R)$ and in general, $\partial u^N/\partial n$ converges to $\partial u/\partial n$ in this norm only. However, if $f_{\partial R} \in H^1(\partial R)$, then $\partial u/\partial n \in H^\circ(\partial R) = \mathcal{L}^2(\partial R)$ and it is possible to evaluate $\partial u/\partial n$ on ∂R, without resource to any other inner product, as we show next.

Indeed, assume the boundary values of the functions w_n at the boundary are in $H^1(\partial R) \subset H^\circ(\partial R)$. Then $\partial w_n/\partial n \in H^\circ(\partial R)$ and therefore, are square integrable on the boundary ∂R. Hence, we can apply Green's formula to obtain

$$b_n = \int_{\partial R} w_n \frac{\partial u}{\partial n}\, d\underset{\sim}{x} = \int_{\partial R} u \frac{\partial w_n}{\partial n}\, d\underset{\sim}{x} = \int_{\partial R} f_{\partial R} \frac{\partial w_n}{\partial n}\, d\underset{\sim}{x} \qquad (2.3)$$

When w_n (n=1,2,...) are orthonormal on the boundary; i.e.

$$\int_{\partial R} w_n w_m \, d\underset{\sim}{x} = \delta_{nm} \qquad (2.4)$$

b_n are the Fourier coefficients of $f_{\partial R}$. More generally, the sequence of functions

$$q_N = \sum_{n=1}^{N} c_n^N w_n \quad \text{on } \partial R \tag{2.5}$$

where c_n^N satisfy

$$\sum_{n=1}^{N} c_n^N \int_{\partial R} w_n w_m \, d\underset{\sim}{x} = b_m \tag{2.6}$$

converges to $\partial u/\partial n$ in the sense of $\mathcal{L}^2(\partial R)$.

The procedure just described is closely related with a method proposed by Picone [23] and studied more extensively by Amerio [24-26] and Fichera [27-28], in which one requires to obtain first the normal derivative and use a fundamental solution to obtain the desired solution in the region. However, in the approach explained here, a fundamental solution is not needed when use is made of the sequence of approximations (2.2).

These ideas can be applied to very general boundary value problems associated with differential operators "$\mathcal{L}$" and "$\mathcal{B}$" defined in a generalized sense, on a region R (Fig. 1a) and on its boundary ∂R, respectively. Assume H(R) and $H(\partial R)$ are Hilbert-spaces of functions defined on R and ∂R, respectively. Let H'(R) and $H'(\partial R)$ be also Hilbert-spaces such that $\mathcal{L}:H(R)\to H'(R)$ and $\mathcal{B}:H(\partial R)\to H'(\partial R)$.

Consider the problem which consists in finding $u\in H(R)$, such that

$$\mathcal{L}u = f_R, \text{ on } R \; ; \; \mathcal{B}u = f_{\partial R}, \text{ on } \partial R \tag{2.7}$$

where $f_R\in H'(R)$ and $f_{\partial R}\in H'(\partial R)$ are given. In addition, let $w_n\in H(R)$, $(n=1,2,\ldots)$, satisfy $\mathcal{L}w_n=0$ and $\mathcal{B}w_n\in H'(\partial R)$ be such that $\mathcal{B}w_n$ is a Hilbert-space basis of $H'(\partial R)$. Choose $u^*\in H(R)$ such that

$$\mathcal{L}u^* = f_R \tag{2.8}$$

Then, it is possible to define a sequence of functions

$$u^N = \sum_{n=1}^{N} a_n^N w_n + u^* \; ; \quad N = 1,2,\ldots \tag{2.9}$$

such that $\mathcal{B}u^N\to f_{\partial R}$ in the metric of $H'(\partial R)$. When the solution depends continuously on the boundary data, then u^N tends to the solution of u of (2.7) in the metric of H(R).

As mentioned previously, it is numerically advantageous to use $\mathcal{L}^2$ norms, only. Many of the questions associated with continuity and convergence properties in these norms can be answered by use of the results of the general theory of partial differential equations [19]. The use of Green's formulas, can also be useful in this more general situation, to compute boundary derivatives.

It must be recalled that the procedures here discussed can be applied whenever the basic equations are linear; thus, in addition to steady state situations, they can be used to treat time dependent problems. Some extensions to non-linear problems, will be discussed in Section 7.

3. SCOPE OF THE THEORY.

In the last section it has been explained the use of complete systems of functions to construct any other solution in a given region.

The numerical treatment of many problems can be simplified using such methods, because the size of the regions requiring to be treated numerically is reduced in this manner. Two general situations can occur in applications; the first one, illustrated in Fig. 1a, corresponds to a boundary value problem for which the basic set of functions is known in the whole region R and in which only the boundary ∂R has to be treated numerically; the second one, illustrated in Fig. 1b, in which the problem is formulated in region $R \cup E$, but the basic set of solutions is only known in R, so that E has to be treated numerically. In both cases, the size of the regions in which the numerical schemes are applied, are reduced. A variant of the above, is the case, when the basic sets of solutions are known both on R and on E; in this case it is required to connect them across the common boundary between R and E. This problem will be referred as the problem of connecting which corresponds in applications to problems formulated in discontinuous fields and with prescribed jumps conditions [13].

In general the application of the methods described in this paper poses the following theoretical questions:

(a) Development of complete systems of solutions;

(b) Convergence of the approximating procedures; and

(c) Formulation of variational principles.

Section 6 is devoted to discuss (a) while the formulation of variational principles is presented in Section 5. Questions of convergence can be discussed using the results of available theory for partial differential equations in a manner similar to that presented in Section 2. This is not difficult to do, for example, for general elliptic equations by means of the results presented in the first volume of Lions and Magene's [19] book (for many fluid flow problems see Teman [22]).

4. SYSTEMATIC DEVELOPMENT.

The systematic discussion of the questions proposed in Section 3, can be carried out in an abstract setting recently developed by the author [13].

Let D be a linear space and D* its algebraic dual (the space of linear functionals defined on D). In this paper attention will be restricted to functional valued operators $P:D \to D^*$ which are linear. Such operators are characterized by the bilinear functional $\langle Pu,v \rangle$; the transposed functional $\langle Pv,u \rangle$ is associated with the adjoint operator $P^*:D \to D^*$.

The theory is based on the development of abstract Green's formulas. This will be done in a manner which is suitable for application to formally symmetric operators, although the theory has just been extended to general non-symmetric operators [20]. Let

$$A = P - P^* \qquad (4.1)$$

An operator $B:D \to D^*$ decomposes A, when B and B* can be varied independently (see definition in Herrera [13]), while

$$A = B - B^* \qquad (4.2)$$

When B decomposes A, B and B* are necessarily boundary operators and (4.2) is an abstract Green's formula. In this abstract frame-work, by a boundary operator $B:D \to D^*$, it is meant one such that $N_B \supset N_A$, where N_B and N_A stand for the null subspaces of B and A, respectively.

When B, decomposes A, the subspace $I_1 = N_B$, enjoys the following properties:

i) $I_1 \subset D$ is a commutative subspace for P;
ii) $N_A \subset I_1$;
iii) For every $u \in D$, one has

$$\langle Au,v\rangle = 0 \quad \forall\, v \in I_1 \Rightarrow u \in I_1 \tag{4.3}$$

Subspaces $I_1 \subset D$ which satisfy i) and ii) are called regular; when, in addition, iii) is fulfilled, I_1 is said to be completely regular.

Subspaces which are completely regular can be characterized by connectivity bases [16]. By this we mean a subset $\mathcal{B} \subset I$ such that $A\mathcal{B} \subset D^*$ is linearly independent, while for every $u \in D$, one has

$$\langle Au,w\rangle = 0 \quad \forall\, w \in \mathcal{B} \Rightarrow u \in I. \tag{4.4}$$

When B decomposes A, the pair of subspaces $I_1 = N_B$, $I_2 = N_{B^*}$ are completely regular and satisfy

$$D = I_1 \dotplus I_2 \; ; \quad N_A = I_1 \cap I_2 \tag{4.5}$$

An ordered pair $(I_1 ; I_2)$ of completely regular subspaces, is said to be a canonical decomposition of D. There is a one-to-one correspondence between canonical decompositions of D and operators B that decompose A (i.e. Green's formulas). Given a canonical decomposition (I_1, I_2) the desired operator $B:D \to D^*$ that decomposes A is associated with the bilinear form

$$\langle Bu,v\rangle = \langle Au_2, v_1\rangle \tag{4.6}$$

Here u_2 and v_1 are the components of the corresponding vector on I_2 and I_1, respectively [13]. This is an interesting fact that will be useful when developing abstract formulas; an application is given in Section 6.

There is a close connection between Hilbert space bases and connectivity bases. For the purpose of this paper, it is enough to know that when a connectivity basis is available, corresponding Hilbert space bases for the boundary values can be easily derived. The precise statement of this result is given in [14].

5. VARIATIONAL PRINCIPLES.

When an abstract Green's formula (4.2) is available and the operator $P:D \to D^*$ is formally symmetric, variational principles are easily formulated.

The property of formal symmetry is usually defined for differential operators. A functional valued operator $P:D \to D^*$ is said to be formally symmetric [13] when for every $u \in D$ one has

$$\langle Pu,v\rangle = 0 \quad \forall\, v \in N_A \Rightarrow Pu = 0 \tag{5.1}$$

Given $U \in D$ and $V \in D$, consider the problem of finding $u \in D$ such that

$$Pu = PU \quad ; \quad Bu = BV \tag{5.2}$$

where $B:D \to D^*$ satisfies the abstract Green's formula (4.2) and $P:D \to D^*$ is formally symmetric. Define the functional

$$\Omega(u) = \frac{1}{2} \langle (P-B)u,u\rangle - \langle (PU-BV),u\rangle \tag{5.3}$$

Then (5.2) holds if and only if $\Omega'(u) = 0$; where $\Omega'(u)$ is the derivative of the functional Ω in the sense of additive Gateaux variation.

There are cases [13] where the subspace $I_1 = N_B$ can be characterized easily while it is not possible to give the operator $B:D\to D^*$ explicitly. An example of this situation arises in finite element formulations when the region R is treated numerically subjected to the restrictions that the sought solutions in R, can be continued smoothly into E (Fig. 1b), as solutions of given differential equations there [13]. When this happens, a more convenient variational principle is associated with the functional

$$X(u) = \langle Pu,u\rangle - \langle 2PU-AV,u\rangle \tag{5.4}$$

Then equation (5.2) holds if and only if

$$\langle X'(u),v\rangle = 0 \quad \forall\ v\in I_1 \tag{5.5}$$

A good sample of applications of these principles has been given in [8-13].

6. PROBLEMS WITH PRESCRIBED JUMP CONDITIONS.

A very general example of application of the theory is the problem of connecting solutions of given differential equations, subjected to a smoothness criterion, or more generally, when they are required to satisfy prescribed jump conditions across the common boundary.

This problem has been formulated abstractly as the "problem of connecting" [13]. Let D_R and D_E be linear spaces of functions defined in R and in E (Fig. 1b), respectively. Take $\hat{D}=D_R\oplus D_E$ as the product space; elements $\hat{u}\in\hat{D}$ are pairs (u_R,u_E). An operator $\hat{P}:\hat{D}\to\hat{D}^*$, having the additive property

$$\langle\hat{P}\hat{u},\hat{v}\rangle = \langle\hat{P}u_R,v_R\rangle + \langle\hat{P}u_E,v_E\rangle \tag{6.1}$$

is considered.

There is a class of elements $\hat{S}\subset\hat{D}$ called "smooth" which satisfy a smoothness condition across the common boundary separating R and E (Fig. 1b). Given $\hat{U}\in\hat{D}$ and $\hat{V}\in\hat{D}$ one considers the problem of finding $\hat{u}\in\hat{D}$ such that

$$\hat{P}\hat{u} = \hat{P}\hat{U} \quad \text{and} \quad \hat{u} - \hat{V} \in \hat{S} \tag{6.2}$$

This is an abstract version of problems with prescribed jumps. Applications to potential theory, elasticity, heat flow and many others have been given [13].

When $\hat{S}\subset\hat{D}$ is a complete regular subspace the set

$$\hat{M} = \{(u_R,u_E)\in\hat{D} \mid (u_R,-u_E)\in\hat{S}\} \tag{6.3}$$

is also a completely regular usbspace and the pair $(\hat{S},\hat{M})$ is a canonical decomposition of $\hat{D}$. Associated with this pair of subspaces there is an abstract Green's formula

$$\hat{A} = \hat{P} - \hat{P}^* = \hat{J} - \hat{J}^* \tag{6.4}$$

where the operator $\hat{J}:\hat{D}\to\hat{D}^*$ characterizes the jump conditions.

Using the results of Section 5 and the Green's formula (6.4) variational principles for problems with prescribed jumps and also for others subjected to continuation type restrictions, have been derived [13].

7. COMPLETE SYSTEMS.

There are many ways in which complete systems of functions can be developed. Here, only a few examples are given. The discussion is concerned with connectivity bases; when a connectivity basis is available a result mentioned in Section 4, permits to obtain Hilbert space bases for corresponding boundary conditions. An advantage of using connectivity bases is that this concept is independent of the Hilbert space structure and therefore, the results acquire greater generarily.

A procedure to obtain connectivity bases has been described by Herrera and Sabina [16]. This is closely related to an idea originated by Picone, Amerio and Fichera, and generalized by Kupradze [29]. We illustrate this method by applying it to Laplace's equation.

Consider region R (Fig. 1a) and define (the linear space D can be taken as $H^{3/2}(R)$, for example)

$$\langle Pu,v\rangle = \int_R v\Delta u d\underset{\sim}{x} \qquad (7.1)$$

Then

$$I_P = N_P + N_A \qquad (7.2)$$

Here $A=P-P^*$ and N_P is the null subspace of P. It can be seen that $I_P \subset D$ is the subspace of functions whose boundary values $\{u, \partial u/\partial n\}$ coincide on ∂R, with those of a harmonic function on R. Standard arguments show that $u \in I_P$, if and only if, the function

$$W(\underset{\sim}{y}) = \int_{\partial R} \{u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}\} d\underset{\sim}{x} \qquad (7.3)$$

vanishes identically in the exterior of the region R. Here, G(x,y) stands for the fundamental solution of Laplace's equation in the whole space. In view of (7.3), W(y) is harmonic in the exterior of R. If C (Fig. 1a) is a curve such that the exterior Dirichlet problem for Laplace's equation satisfies uniqueness*, then it can be shown [16] that a sufficient condition for W(y) to be identically zero, is that it vanishes identically on C. If $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots\}$ is a dense denumerable subset of points on the curve C, define

$$w_\alpha(\underset{\sim}{x}) = G(\underset{\sim}{x}, \underset{\sim}{y}_\alpha) \quad ; \quad \alpha = 1,2,\ldots \qquad (7.4)$$

Then the above discussion shows that

$$\int_{\partial R} \{w_\alpha \frac{\partial u}{\partial n} - u \frac{\partial w_\alpha}{\partial n}\} d\underset{\sim}{x} = \langle Au, w_\alpha\rangle = 0 \quad \forall\, \alpha = 1,2,\ldots \Rightarrow u \in I_P \qquad (7.5)$$

This shows that $\mathcal{B} = \{w_1, w_2, \ldots\}$ is a connectivity basis for Laplace's equation (this result is essentially Kupradze's [29]),

However, there are many other ways of constructing connectivity bases. One is to choose a complete system of functions $\{p_1, p_2, \ldots\}$ on the curve C, and require [16]

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* In two dimensional problems there are anomalous curves for which uniqueness is not granted [30]. However, the results presented in [16] are independent of the curve chosen and therefore, their validity is not restricted by this fact.

$$\int_C p_\alpha(\underset{\sim}{y})W(\underset{\sim}{y})d\underset{\sim}{y} = 0\ ;\ \alpha=1,2,... \qquad (7,6)$$

This yields the system of functions

$$w_\alpha(\underset{\sim}{x}) = \int_C p_\alpha(\underset{\sim}{y})G(\underset{\sim}{x},\underset{\sim}{y})d\underset{\sim}{y}\ ;\ \alpha=1,2,... \qquad (7.7)$$

as connectivity basis. In a similar fashion one can prove that given any point y_0 of the exterior of R, the system of functions consisting of $G(x,y_0)$ together with all its partial derivatives with respecto to $\underset{\sim}{y}$ at y_0, is a connectivity basis. These procedures for constructing connectivity bases are applicable to a very general class of equations including time dependent ones [16 & 29].

The availability of different connectivity bases for a given problem is important, because in numerical applications it is necessary to obtain reliable systems. For example, Aleksidze [31] has shown that the connectivity basis defined by (7.4) is unreliable*. For the case when the region R is a circle, Herrera and Sabina's [16] procedure yields an orthogonal system, which is reliable; one can expect that the numerical properties of this system deteriorate steadily as the region is deformed away from the circle, a fact that has been confirmed numerically for a related system [6].

The methods just described possess considerable generality; they can be applied to many problems governed by linear differential equations, such as potential flow, fluids in porous media, heat flow, wave propagation and Elasticity (static and dynamic). Connectivity bases for Stokes problems have just been constructed [15].

The possibility of constructing connectivity bases which are independent of the regions considered, is relevant for the treatment of non-linear problems. Herrera and Sabina [16] obtained systems satisfying this property both for bounded and unbounded regions. In general, problems in which the differential equations are linear and the non-linearity is introduced through the boundary conditions can be treated by the methods described in this paper. This is the case of problems subjected to floating boundary conditions such as seepage flow, Stefan problem, some problems in Plasticity and contact problems.

8. EXTENSION TO NON-LINEAR PROBLEMS.

As mentioned previously the boundary methods here described can be applied to non-linear problems, when the non linearity is introduced through the boundary conditions. This is the case of seepage flow (Fig. 2), which is presented here as an example.

A normalized version of this problem is subjected to†

(a) Governing equations

$$\nabla^2\varphi = 0 \quad ; \quad \text{on } R \qquad (8.1)$$

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* Unreliable is used here in the sense that the Gram determinant of the system may become arbitrarily small [31].

† This, as well as, Stefan's problem was treated in collaboration with Prof. Nima Geffen, of the University of Tel-Aviv and Dr. Hervá Gourgeon, of the University of Paris XI, Orsay, France.

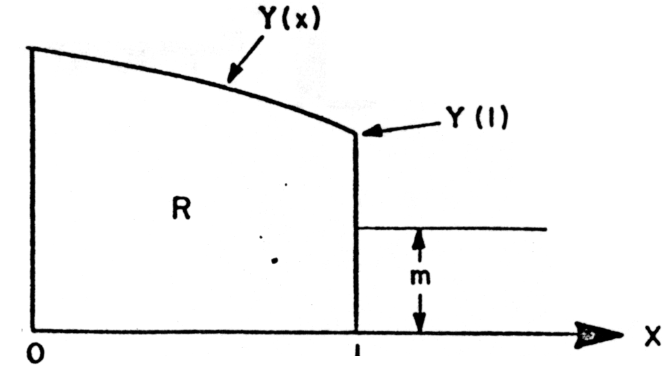

FIGURE 2

(b) Boundary conditions

$$\varphi(0,y) = 1 \qquad 0 \leq y < 1 \tag{8.2a}$$

$$\varphi(1,y) = \begin{cases} m & 0 \leq y \leq m < 1 \\ y & m \leq y \leq Y(1) \end{cases} \tag{8.2b}$$

$$\frac{\partial\varphi}{\partial y}(x,0) = 0 \quad ; \quad 0 \leq x \leq 1 \tag{8.3}$$

$$\frac{\partial\varphi}{\partial n}(x,Y(x)) = 0 \tag{8.4a}$$

$$\varphi(x,Y(x)) = Y(x) \tag{8.4b}$$

(c) Points of the region R satisfy

$$0 \leq x \leq 1 \quad ; \quad 0 \leq y \leq Y(x)$$

Thus, $Y(x)$ defines the free surface.

Equation (8.4a) can be rewritten as

$$Y'(x) = \frac{\varphi_y(x,Y(x))}{-\varphi_x(x,Y(x))} \tag{8.6}$$

A convenient procedure to solve the system (8.4b), (8.6) is iterative. Define for $n=0,1,...$

$$\varphi_n(x,Y_n(x)) = Y_n(x) \tag{8.7a}$$

$$Y'_{n+1}(x) = \frac{\varphi_{ny}(x,Y_n(x))}{\varphi_{nx}(x,Y_n(x))} \tag{8.7b}$$

The scheme can be started by choosing $Y_o(x)$ arbitrarily (e.g. $Y_o(x)\equiv 1$). If one writes

$$\varphi_n(x,y) = \varphi_o(x,y) + u_n(x,y) \quad (8.8)$$

it is seen that

$$u_n(x,Y_n(x)) = Y_n(x) - \varphi_o(x,Y_n(x)) \quad (8.9)$$

In addition

$$u_n(0,y) = u_n(1,y) = 0 \quad (8.10a)$$

$$\frac{\partial u_n}{\partial y}(x,0) = 0 \quad ; \quad 0 < x < 1 \quad (8.10b)$$

A connectivity basis convenient for this problem, because it satisfies (8.10) is

$$w_\alpha(x,y) = \sin \alpha\pi x \; \text{ch}\alpha\pi y \; ; \; \alpha=1,2,\ldots \quad (8.11)$$

In this manner only the free boundary needs to be treated numerically. The approximation for u_n is (keeping fixed N)

$$u_n(x,y) = \sum_{\alpha=1}^{N} A_\alpha^{N} w_\alpha(x,y) \quad (8.12)$$

Numerically, it is advantageous to determine the N coefficients A_α^n, by collocation; i.e. by requiring that (8.9) be satisfied at N points $x_1 < x_2 < \ldots < x_N$ of the interval [0,1].

Stefan problem has been treated in a similar fashion.

Free-boundary problems have received considerable attention in recent year using several approaches. Baiocchi, et al [32], for example, have based their method on variational inequalities. Liggett, on the other hand, has applied boundary integral equations to this class of problems; in particular the example presented here has been treated by that method [33].

REFERENCES

1. Brebbia, A.C., ed. "Recent Advances in Boundary Element Methods", Pentech Press, London, 1978.

2. Cruse, T.A., "An Improved Boundary-Integral Equation Method for Three Dimensional Elastic Stress Analysis", Computers & Structures, Vol. 4, pp. 741-754, 1974.

3. Cruse, T.A., "Mathematical Foundations of the Boundary-Integral Equation Method in Solid Mechanics", Special Scientific Report AFSC(F44620-74-C-0060), 1977.

4. Cruse, T.A. and Rizzo, F.J., "Boundary-Integral Equation Method: Computational Applications in Applied Mechanics", Am. Soc. Mech. Eng. AMD, Vol. 11, 1975.

5. Rizzo, F.J., "An integral Equation Approach to Boundary Value Problems of Classical Elastostatics", Quart. J. Appl. Math., Vol. 25, pp. 83-95, 1967.

6. England, R., Sabina, F.J. and Herrera, I., "Scattering of SH waves

by Surface Cavities of Arbitrary Shape Using Boundary Methods" Physics of the Earth and Planetary Interiors, (In press), 1980

7. Courant, R. and Hilbert D., "Methods of Mathematical Physics", Vol. 2, John Wiley and Sons, New York, 1962.

8. Herrera, I., "General Variational Principles Applicable to the Hybrid Element Method", Proc. Natl. Acad. Sci. USA, Vol. 74, pp. 2595-2597, 1977.

9. Herrera, I., "Theory of Connectivity for Formally Symmetric Operators", Proc. Natl. Acad. Sci. USA, Vol. 74, pp. 4722-4725, 1977.

10. Herrera, I., "On the Variational Principles of Mechanics", Trends in Applications of Pure Mathematics to Mechanicas, Vol. 11, Pitman, London, pp. 115-128, 1979.

11. Herrera, I., "Theory of Connectivity: A Systematic Formulation of Boundary Element Methods", Applied Mathematical Modelling, Vol. 3 No. 2, pp. 151-156, 1979.

12. Herrera, I., "Theory of Connectivity as a Unified Approach to Boundary Methods", Proc. IUTAM Symposium on Variational Principles of Mechanics, (In press). Available as University of Wisconsin-Madison, MRC Technical Summary Report # 1938, 1979.

13. Herrera, I., "Variational Principles for Problems with Linear Constraints. Prescribed Jumps and Continuation Type Restrictions", J. Inst. Maths. Applics. (In press) 1980. Available as University of Wisconsin-Madison, MRC Technical Summary Report # 1939, 1979.

14. Herrera, I., "Boundary Methods. A Criterion for Completeness", Comunicaciones Técnicas, IIMAS-UNAM, México, (In Press), 1980.

15. Herrera, I., "Connectivity Bases for Stokes Problems", Comunicaciones Técnicas, IIMAS-UNAM, México, (In Press), 1980.

16. Herrera, I. and Sabina, F.J., "Connectivity as an Alternative to Boundary Integral Equations. Construction of Bases", Proc. Natl. Acad. Sci. USA, Vol. 75, pp. 2059-2063, 1978.

17. Bates, R.H.T., "Analytic Constraints on Electro-magnetic Field Computations", IEEE Trans. on Microwave Theory and Techniques, Vol 23, pp. 605-623, 1975.

18. Millar, R.F., "The Rayleigh Hypothesis and a Related Least-Squares Solution to Scattering Problems for Periodic Surfaces and other Scatterers", Radio Science, Vol. 8, pp. 785-796, 1973.

19. Lions, J.L. and Magenes, E., "Non-homogeneous Boundary Value Problems and Applications, Springer-Verlag, New York, 1972.

20. Herrera, I., "Abstract Green's Formulas and Application ', Comunicaciones Técnicas, IIMAS-UNAM, México (In Press), 1980.

21. Nemat-Nasser, S., "On Variational Methods in Finite and Incremental Elastic Deformation Problems with Discontinuous Fields", Q. Appl.

Math., Vol. 30, pp. 143-156, 1972.

22. Temam, R., "Navier-Stokes Equations: Theory and Numerical Analysis", Amsterdam, North-Holland, 1977.

23. Picone, M., "Nuovi Metodi Risolutivi per i Problemi d'Integrazione delle Equazioni Lineari a Derivati Parziali e Nuova Applicazione della Transformate Multipla di Laplace nel caso delle Equazioni a Coefficienti Constanti", Atti Acc. Sc. Torino, Vol. 76, pp. 413-426 1940.

24. Amerio, L., "Sull'integrazione delle Equazioni Lineari a Derivate Parziali del Secondo Ordine di Tipo Ellitico", Acta Pont. Ac. Sc. Vol. 9, pp. 213-228, 1945.

Amerio, L., "Sul Calcolo delle Autosoluzioni dei Problemi al Contorno per le Equazioni Lineari del Secondo Ordine di Tipo Ellitico", Rend. Acc. Lincei, Vol. 1, pp. 352-359 e 505-509, 1946

26. Amerio, L., "Su un Metodo di Integrazione delle Equazioni Differentiali Lineari a Derivate Parziali", Rend. Sem. Mat. Fis. Milano, Vol. 18, pp. 114-123, 1947.

Fichera, G., "Teoremi di Completezza sulla Frontiera di un Dominio per Taluni Sistema di Funzioni", Ann. Mat. Pura e Appl., Vol. 27, pp. 1-28, 1948.

28. Fichera, G., "Applicazione della Teoria del Potenziale di Superficie ad Alcuni Problemi di Analisi Funzionale Lineare", Giormale Mat. Battaglini, Vol. 78, pp. 71-80, 1949.

Kupradze, V.D., "On the Approximate Solution of Problems in Mathematical Physics", Russian Math. Survеys, Vol. 22, No. 2, pp. 58-108, 1967 (Uspehi Mat. Nauk. Vol. 22, No. 2, pp. 59-107, 1967).

30. Christiansen, S., "On Kupradze's Functional Equations for Plane Harmonic Problems", in Functional Theoretic Methods in Differential Equations, R.P. Gilbert and R.J. Weinacht, Editors, Pitman, 1976.

Aleksidze, M.A., "Notes on an Approximate Method of Solving Boundary Value Problems", Soviet Math. Dokl., Vol. 8, pp. 297-299, 1967.

32. Baiocchi, C., Comincioli, V., Magenes, E., and Pozzi, G.A., "Fluid Flow through Porous Media", Laboratorio di Analisi Numerica del Consiglio Nazionalle delle Richerche, No. 69, Pavia, 1974.

Liggett, J.A., "Location of Free Surface in Porous Media", J. of the Hydraulics Div. ASCE, Vol. 103, No. 4, pp. 353-365, 1977.