

# Boundary methods in water resources

I. HERRERA

*Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, (IIMAS), Universidad Nacional Autónoma de México, Apdo. Postal 20-726, México 20, D.F., Mexico*

A brief review of boundary methods is presented. Emphasis is placed on alternatives to boundary integral equations. Different theoretical questions required to give a firm foundation to these procedures are discussed. Extensions to non-linear problems are explained. Examples of application to fluid problems are included.

## INTRODUCTION

Boundary methods are being used to treat many problems. When they are applicable the size of the regions that need to be treated numerically can be reduced.

In general, two situations can be distinguished. One may consider a boundary value problem in a region such as  $R$  (Fig. 1a), in which general analytical solutions are known in the whole region  $R$ , or alternatively, the problem may be formulated in a region such as  $R \setminus E$  (Fig. 1b), and the general analytical solutions may be known in the subregion  $R$ , only. In the first case, the application of boundary methods permits restricting the numerical treatment to the boundary  $\partial R$  and the dimensionality of the problem is reduced. In the second case, the subregion  $R$  has to be treated numerically but the region  $E$  is eliminated.

Most frequently, boundary methods have been formulated by means of integral equations<sup>1-5</sup>. However, there are alternatives which present advantages. In general, a complete family of solutions is required in order to apply boundary methods. There are two possible in which such family can be supplied; one is a denumerable family of regular solutions<sup>6</sup> and another one is by means of a fundamental singular solution. The boundary integral equation method is a special case in which the singularity is placed on the boundary of the region considered and due to this fact the resulting equations are singular.

When a denumerable family of regular solutions is used the resulting equations are non-singular; a fact that offers numerical advantages. What is probably more important in complicated problems, is the observation that generally, it is easier to construct complete families of solutions, than fundamental solutions; indeed, there are methods available for synthesizing fundamental solutions starting from complete families (e.g. in terms of plane waves<sup>7</sup>).

Several recent accounts of boundary integral equations (BIE) are available<sup>1-3</sup>. However, alternative approaches that have been the subject of systematic research by the author<sup>8-16</sup> will be the main concern of this paper. The theoretical basis of these alternatives is not complicated, but the applicability of the methods has been unnecessarily restricted due to lack of clarity. For example, in acoustics and electromagnetic field computations 'Rayleigh hypothesis' introduces severe restrictions<sup>17</sup> which can be avoided altogether if a different point of view is adopted<sup>18</sup>.

Generally, the solution of a boundary value problem

that is well-posed depends continuously on the boundary data<sup>19</sup>; thus, all what is required to construct the arbitrary solution, is to have available a family of solutions in terms of which one can approximate any given boundary values.

The general description of the method is given first, followed by a discussion of the theoretical questions which require study. An abstract framework, developed by the author, which can be used to make the subject more systematic is briefly described next. This can be interpreted in terms of abstract Green's formulae; here, it is only presented in a form which is suitable for application to formally symmetric operators, but has recently been extended to general non-symmetric operators<sup>20</sup>. One of the applications of this framework is the formulation of variational principles; an exposition of this subject has just been published<sup>13</sup> and it is briefly explained later. Problems formulated in discontinuous fields with prescribed jump conditions such as those that were surveyed by Nemat-Nasser<sup>21</sup>, constitute a general application of the theory; another example, are problems subjected to continuation type restrictions, i.e. problems formulated in a region such as  $R$  (Fig. 1b), subjected to the restriction that the solutions can be continued smoothly into solutions of given differential equations in a neighbouring region such as  $E$ .

## DESCRIPTION OF THE METHOD

To fix ideas we first consider a simple example; this is Laplace equation subjected to Dirichlet type boundary conditions:

$$\Delta u = 0 \text{ on } R; \quad u = f_{\partial R} \text{ on } \partial R \quad (1)$$

The region  $R$  and its boundary  $\partial R$ , are illustrated in Fig. 1a. General results on the existence and continuity properties of solutions of elliptic equations<sup>19</sup>, grant that for every real  $s$ , when  $f_{\partial R} \in H^s(\partial R)$  there is a unique  $u \in H^{s+1/2}(R)$  which satisfies equation (1); even more, the solution  $u$  depends continuously on the boundary data  $f_{\partial R}$ .

Assume  $w_n \in H^{s+1/2}(R)$ ,  $n = 1, 2, \dots$ , satisfy  $\Delta w_n = 0$  and their boundary values constitute a basis of the Hilbert space  $H^s(\partial R)$ . When this is the case, it is possible to construct a sequence of approximations:

$$u^N = \sum_{n=1}^N a_n^N w_n; \quad N = \quad (2)$$

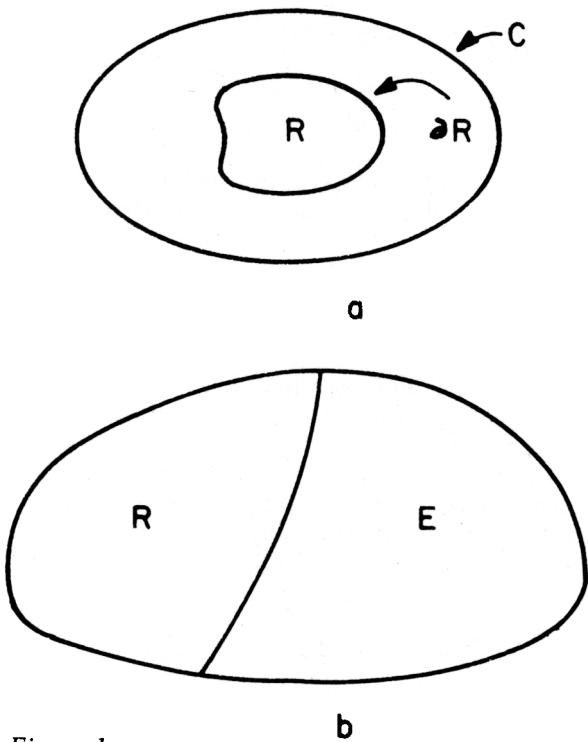


Figure 1.

such that  $u^N \rightarrow f_{\partial R}$  in the boundary with respect to metric of  $H^s(\partial R)$ . The optimal choice of the coefficients  $a_n^N$  would correspond to an element  $u^N$  of the subspace of  $H^s(\partial R)$ , generated by  $\{w_1, \dots, w_N\}$ , which is closest to  $f_{\partial R}$  in the metric of  $H^s(\partial R)$ . Clearly,  $u^N \rightarrow u$  on  $R$ , with respect to the norm of  $H^{s+1/2}(R)$ , because  $u^N f_{\partial R}$  on  $H^s(\partial R)$ . Also  $u^N \rightarrow u$  on  $R$ , with respect to the metric of  $H^s(R)$ , because convergence on  $H^{s+1/2}(R)$  implies convergence on  $H^s(R)$ .<sup>19</sup>

Of special interest is the case  $s=0$ , because  $H^0(\partial R) = \mathcal{L}^2(\partial R)$ , is the space of square integrable functions. In applications, it offers numerical advantages to restrict to this space, because of the simplicity of the inner product; also, it is easier to obtain Hilbert space bases. The above discussion shows that if the sequence defined by equation (w) converges in  $\mathcal{L}^2(\partial R)$  to  $f_{\partial R}$ , then  $u$  converges to the solution in the region  $R$ , in the least square sense.

The procedure described here, is closely related with the series expansion method that has been applied in acoustics and electromagnetic field computations<sup>17</sup>. However, for such kind of applications, severe restrictions were imposed by the introduction of the so-called 'Rayleigh hypothesis'. The above discussion shows that when a more convenient criterion is adopted for the choice of the coefficients  $a_n^N$  in equation (2), Rayleigh hypothesis is not required; this fact had been pointed out by Millar<sup>18</sup>, in connection with such applications.

The computation of the normal derivative on the boundary  $\partial R$ , using  $\mathcal{L}^2(\partial R)$  inner product only, becomes more involved. If the boundary values  $f_{\partial R} \in H^0(\partial R)$ , one can only guarantee that  $\partial u / \partial n \in H^{-1}(\partial R)$  and in general,  $\partial u^N / \partial n$  converges to  $\partial u / \partial n$  in this norm only. However, if  $f_{\partial R} \in H^1(\partial R)$ , then  $\partial u / \partial n \in H^0(\partial R) = \mathcal{L}^2(\partial R)$  and it is possible to evaluate  $\partial u / \partial n$  on  $\partial R$ , without resource to any other inner product, as we show next.

Indeed, assume the boundary values of the functions  $w_n$  at the boundary are in  $H^1(\partial R) \subset H^0(\partial R)$ . Then

$\partial w_n / \partial n \in H^0(\partial R)$  and therefore, are square integrable on the boundary  $\partial R$ . Hence, we can apply Green's formula to obtain:

$$b_n = \int_{\partial R} w_n \frac{\partial u}{\partial n} dx = \int_{\partial R} u \frac{\partial w_n}{\partial n} dx = \int_{\partial R} f_{\partial R} \frac{\partial w_n}{\partial n} dx \quad (3)$$

When  $w_n$  ( $n=1, 2, \dots$ ) are orthonormal on the boundary; i.e.

$$\int_{\partial R} w_n w_m dx = \delta_{nm} \quad (4)$$

$b_n$  are the Fourier coefficients of  $f_{\partial R}$ . More generally, the sequence of functions

$$q_N = \sum_{n=1}^N c_n^N w_n; \quad \text{on } \partial R \quad (5)$$

where  $c_n^N$  satisfy

$$\sum_{n=1}^N c_n^N \int_{\partial R} w_n w_m dx = b_m \quad (6)$$

converges to  $\partial u / \partial n$  in the sense of  $\mathcal{L}^2(\partial R)$ .

The procedure just described is closely related with a method proposed by Picone<sup>23</sup> and studied more extensively by Amerio<sup>24-26</sup> and Fichera<sup>27-28</sup>, in which one requires to obtain first the normal derivative and use of fundamental solution to obtain the desired solution in the region. However, in the approach explained here, a fundamental solution is not needed when use is made of the sequence of approximations (2).

These ideas can be applied to very general boundary value problems associated with differential operators ' $\mathcal{L}$ ' and ' $\mathcal{B}$ ' defined in a generalized sense, on a region  $R$  (Fig. 1a) and on its boundary  $\partial R$ , respectively. Assume  $H(R)$  and  $H(\partial R)$  are Hilbert-spaces of functions defined on  $R$  and  $\partial R$ , respectively. Let  $H'(R)$  and  $H'(\partial R)$  be also Hilbert-spaces such that  $\mathcal{L}: H(R) \rightarrow H'(R)$  and  $\mathcal{B}: H(\partial R) \rightarrow H'(\partial R)$ .

Consider the problem which consists in finding  $u \in H(R)$ , such that:

$$\mathcal{L}u = f_R, \text{ on } R; \quad \mathcal{B}u = f_{\partial R}, \text{ on } \partial R \quad (7)$$

where  $f_R \in H'(R)$  and  $f_{\partial R} \in H'(\partial R)$  are given. In addition, let  $w_n \in H(R)$ , ( $n=1, 2, \dots$ ), satisfy  $\mathcal{L}w_n = 0$  and  $\mathcal{B}w_n \in H'(\partial R)$  be such that  $\mathcal{B}w_n$  is a Hilbert-space basis of  $H'(\partial R)$ . Choose  $u^0 \in H(R)$  such that:

$$\mathcal{L}u^0 = f_R \quad (8)$$

Then, it is possible to define a sequence of functions:

$$u^N = \sum_{n=1}^N a_n^N w_n + u^0; \quad N = 1, 2, \dots \quad (9)$$

such that  $\mathcal{B}u^N \rightarrow f_{\partial R}$  in the metric of  $H'(\partial R)$ . When the solution depends continuously on the boundary data, then  $u^N$  tends to the solution of  $u$  of equation (7) in the metric of  $H(R)$ .

As mentioned previously, it is numerically advantageous to use  $\mathcal{L}^2$  norms, only. Many of the questions associated with continuity and convergence properties in these norms can be answered by use of the results of the general theory of partial differential equations<sup>19</sup>. The use of Green's formulae, can also be useful in this more general situation, to compute boundary derivatives.

It must be recalled that the procedures here discussed can be applied whenever the basic equations are linear; thus, in addition to steady state situations, they can be used to treat time-dependent problems. Some extensions to non-linear problems, will be discussed later.

## SCOPE OF THE THEORY

In the previous section the use of complete systems of functions to construct any other solution in a given region has been explained. The numerical treatment of many problems can be simplified using such methods, because the size of the regions requiring to be treated numerically is reduced in this manner. Two general situations can occur in applications; the first one, illustrated in Fig. 1a, corresponds to a boundary value problem for which the basic set of functions is known in the whole region  $R$  and in which only the boundary  $\partial R$  has to be treated numerically, the second one, illustrated in Fig. 1b, in which the problem is formulated in region  $R \cup E$ , but the basic set of solutions is only known in  $R$ , so that  $E$  has to be treated numerically. In both cases, the size of the regions in which the numerical schemes are applied, are reduced. A variant of the above, is the case, when the basic sets of solutions are known both on  $R$  and on  $E$ ; in this case it is required to connect them across the common boundary between  $R$  and  $E$ . This problem will be referred as the problem of connecting which corresponds in applications to problems formulated in discontinuous fields and with prescribed jumps conditions<sup>13</sup>.

In general the application of the methods described in this paper poses the following theoretical questions: (a) development of complete systems of solutions; (b) convergence of the approximating procedures; and (c) formulation of variational principles.

Both (a) and (c) are discussed later. Questions of convergence can be discussed using the results of available theory for partial differential equations in a manner similar to that presented in the previous section. This is not difficult to do, for example, for general elliptic equations by means of the results presented in the first volume of Lions and Magene's book<sup>19</sup> (for many fluid flow problems see Temam<sup>22</sup>).

## SYSTEMATIC DEVELOPMENT

The systematic discussion of the questions proposed above can be carried out in an abstract setting recently developed by the author<sup>13</sup>.

Let  $D$  be a linear space and  $D^*$  its algebraic dual (the space of linear functionals defined on  $D$ ). In this paper attention will be restricted to functional valued operators  $P: D \rightarrow D^*$  which are linear. Such operators are characterized by the bilinear functional  $\langle Pu, v \rangle$ ; the transposed functional  $\langle Pv, u \rangle$  is associated with the adjoint operator  $P^*: D \rightarrow D^*$ .

The theory is based on the development of abstract Green's formulae. This will be done in a manner which is

suitable for application to formally symmetric operators, although the theory has just been extended to general non-symmetric operators<sup>20</sup>. Let

$$A = P - P^* \quad (10)$$

An operator  $B: D \rightarrow D^*$  decomposes  $A$ , when  $B$  and  $B^*$  can be varied independently (see definition in Herrera<sup>13</sup>), while

$$A = B - B^* \quad (11)$$

When  $B$  decomposes  $A$ ,  $B$  and  $B^*$  are necessarily boundary operators and equation (11) is an abstract Green's formula. In this abstract framework, by a boundary operator  $B: D \rightarrow D^*$ , it is meant one such that  $N_B \supset N_A$ , where  $N_B$  and  $N_A$  stand for the null subspace of  $B$  and  $A$ , respectively.

When  $B$  decomposes  $A$ , the subspace  $I_1 = N_B$ , enjoys the following properties: (i)  $I_1 \subset D$  is a commutative subspace for  $P$ ; (ii)  $N_A \subset I_1$ ; (iii) for every  $u \in D$ , one has

$$\langle Au, v \rangle = 0 \quad v \in I_1 = u \in I_1$$

Subspaces  $I_1 \subset D$  which satisfy (i) and (ii) are called regular; when, in addition, (iii) is fulfilled,  $I_1$  is said to be completely regular.

Subspaces which are completely regular can be characterized by connectivity bases<sup>16</sup>. By this we mean a subset  $B \subset I$  such that  $A|_B \subset D^*$  is linearly independent, while for every  $u \in D$ , one has:

$$\langle Au, w \rangle = 0 \quad w \in B, u \in I$$

When  $B$  decomposes  $A$ , the pair of subspaces  $I_1 = N_B$ ,  $I_2 = N_{B^*}$  are completely regular and satisfy:

$$D = I_1 + I_2 \quad N_A = I_1 \cap I_2 \quad (14)$$

An ordered pair  $(I_1, I_2)$  of completely regular subspaces, is said to be a canonical decomposition of  $D$ . There is one-to-one correspondence between canonical decompositions of  $D$  and operators  $B$  that decompose  $A$  (i.e. Green's formulae). Given a canonical decomposition  $(I_1, I_2)$  the desired operator  $B: D \rightarrow D^*$  that decomposes  $A$  is associated with the bilinear form:

$$\langle Bu, v \rangle = \langle Au_2, v_1 \rangle \quad (15)$$

Here  $u_2$  and  $v_1$  are the components of the corresponding vector on  $I_2$  and  $I_1$ , respectively<sup>13</sup>. This is an interesting fact that will be useful when developing abstract formulae; an application is given later.

There is a close connection between Hilbert space bases and connectivity bases. For the purpose of this paper, it is enough to know that when a connectivity basis is available, corresponding Hilbert space bases for the boundary values can be easily derived. The precise statement of this result is given by Herrera<sup>14</sup>.

## VARIATIONAL PRINCIPLES

When an abstract Green's formula (11) is available and the operator  $P: D \rightarrow D^*$  is formally symmetric, variational principles are easily formulated.

The property of formal symmetry is usually defined for differential operators. A functional valued operator  $P:D \rightarrow D^*$  is said to be formally symmetric<sup>13</sup> when for every  $u \in D$  one has

$$\langle Pu, v \rangle = 0 \quad v \in N_A = Pu = 0 \quad (16)$$

Given  $U \in D$  and  $V \in D$ , consider the problem of finding  $u \in D$  such that

$$Pu = PU; \quad Bu = BV \quad (17)$$

where  $B:D \rightarrow D^*$  satisfies the abstract Green's formula (11) and  $P:D \rightarrow D^*$  is formally symmetric. Define the functional:

$$\Omega(u) = \frac{1}{2} \langle (P - B)u, u \rangle - \langle (PU - BV), u \rangle \quad (18)$$

Then equation (17) holds if and only if  $\Omega'(u) = 0$ ; where  $\Omega'(u)$  is the derivative of the functional  $\Omega$  in the sense of additive Gateaux variation.

There are cases<sup>13</sup> where the subspace  $I_1 = N_B$  can be characterized easily while it is not possible to give the operator  $B:D \rightarrow D^*$  explicitly. An example of this situation arises in finite element formulations when the region  $R$  is treated numerically subjected to the restrictions that the sought solutions in  $R$ , can be continued smoothly into  $E$  (Fig. 1b), as solutions of given differential equations there<sup>13</sup>. When this happens, a more convenient variational principle is associated with the functional:

$$X(u) = \langle Pu, u \rangle - \langle 2PU - AV, u \rangle \quad (19)$$

Then equation (17) holds if and only if:

$$\langle X'(u), v \rangle = 0 \quad v \in I_1 \quad (20)$$

A good sample of applications of these principles has been given in refs. 8-13.

## PROBLEMS WITH PRESCRIBED JUMP CONDITIONS

A very general example of application of the theory is the problem of connecting solutions of given differential equations, subjected to a smoothness criterion, or more generally, when they are required to satisfy prescribed jump conditions across the common boundary.

This problem has been formulated abstractly as the 'problem of connecting'<sup>13</sup>. Let  $D_R$  and  $D_E$  be linear spaces of functions defined in  $R$  and in  $E$  (Fig. 1b), respectively. Take  $\hat{D} = D_R + D_E$  as the product space; elements  $\hat{u} \in \hat{D}$  are pairs  $(u_R, u_E)$ . An operator  $\hat{P}:\hat{D} \rightarrow \hat{D}^*$ , having the additive property:

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \langle \hat{P}u_R, v_R \rangle + \langle \hat{P}u_E, v_E \rangle \quad (21)$$

is considered.

There is a class of elements  $\hat{\mathcal{S}} \in \hat{D}$  called 'smooth' which satisfy a smoothness condition across the common boundary separating  $R$  and  $E$  (Fig. 1b). Given  $\hat{U} \in \hat{D}$  and  $\hat{V} \in \hat{D}$  one considers the problem of finding  $\hat{u} \in \hat{D}$  such that:

$$\hat{P}\hat{u} = \hat{P}\hat{U} \quad \text{and} \quad \hat{u} - \hat{V} \in \hat{\mathcal{S}} \quad (22)$$

This is an abstract version of problems with prescribed

jumps. Applications to potential theory, elasticity, heat flow and many others have been given<sup>13</sup>.

When  $D$  is a complete regular subspace the set

$$\mathcal{M} = \{(u_R, u_E) \in \hat{D} \mid (u_R - u_E) \in \hat{\mathcal{S}}\} \quad (23)$$

is also a completely regular subspace and the pair  $(\hat{\mathcal{S}}, \mathcal{M})$  is a canonical decomposition of  $\hat{D}$ . Associated with this pair of subspaces there is an abstract Green's formula:

$$\hat{A} = \hat{P} - \hat{P}^* = \hat{J} - \hat{J}^* \quad (24)$$

where the operator  $\hat{J}:\hat{D} \rightarrow \hat{D}^*$  characterizes the jump conditions.

Using the results of the previous section and the Green's formula (24) variational principles from problems with prescribed jumps and also for others subjected to continuation type restrictions, have been derived<sup>13</sup>.

## COMPLETE SYSTEMS

There are many ways in which complete systems of functions can be developed. Here, only a few examples are given. The discussion is concerned with connectivity bases; when a connectivity basis is available it permits one to obtain Hilbert space bases for corresponding boundary conditions. An advantage of using connectivity bases is that this concept is independent of the Hilbert space structure and therefore, the results acquire greater generality.

A procedure to obtain connectivity bases has been described by Herrera and Sabina<sup>16</sup>. This is closely related to an idea originated by Picone, Amerio and Fichera, and generalized by Kupradze<sup>29</sup>. We illustrate this method by applying it to Laplace's equation.

Consider region  $R$  (Fig. 1a) and define (the linear space  $D$  can be taken as  $H^{3/2}(R)$ , for example)

$$\langle Pu, v \rangle = \int_R v \Delta u dx$$

Then

$$I_P = N_P + N_A \quad (26)$$

Here  $A = P - P^*$  and  $N_P$  is the null subspace of  $P$ . It can be seen that  $I_P \subset D$  is the subspace of functions whose boundary values  $\{u, \partial u / \partial n\}$  coincide on  $\partial R$ , with those of a harmonic functions on  $R$ . Standard arguments show that  $u \in I_P$  if and only if, the function:

$$W(y) = \int_{\partial R} \left\{ u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right\} dx$$

vanishes identically in the exterior of the region  $R$ . Here,  $G(x, y)$  stands for the fundamental solution of Laplace's equation in the whole space. In view of equation (27),  $W(y)$  is harmonic in the exterior of  $R$ . If  $C$  (Fig. 1a) is a curve such that the exterior Dirichlet problem for Laplace's equation satisfies uniqueness, then it can be shown<sup>16</sup> that a sufficient condition for  $W(x)$  to be identically zero, is that it vanishes identically on  $C$ . If  $\{y_1, y_2, \dots\}$  is a dense

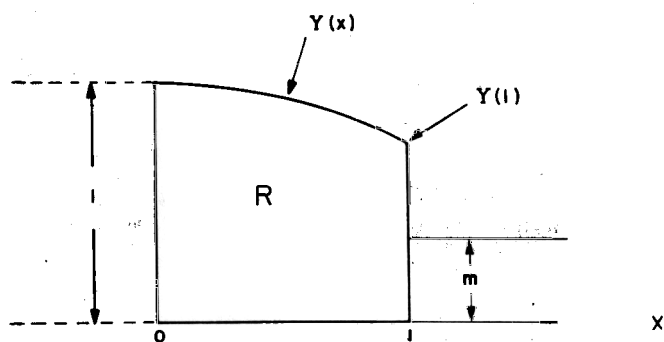


Figure 2.

denumerable subset of points on the curve  $C$ , define

$$w_\alpha(\mathbf{x}) = G(\mathbf{x}, \mathbf{y}_\alpha); \quad \alpha = 1, 2, \dots \quad (28)$$

Then the above discussion shows that:

$$\int \left\{ w_\alpha \frac{\partial u}{\partial n} - u \frac{\partial w_\alpha}{\partial n} \right\} d\mathbf{x} = \langle Au, w_\alpha \rangle = 0$$

$$\forall \alpha = u \in I_p \quad (29)$$

This shows that  $B = \{w_1, w_2, \dots\}$  is a connectivity basis for Laplace's equation (this result is essentially Kupradze's<sup>29</sup>).

However, there are many other ways of construction connectivity bases. One is to choose a complete system of functions  $\{p_1, p_2, \dots\}$  on the curve  $C$ , and require<sup>16</sup>:

$$\int P_\alpha(y) W(y) dy = 0; \quad \alpha = 1, 2, \dots \quad (30)$$

This yields the system of functions:

$$w_\alpha(\mathbf{x}) = \int_C P_\alpha(y) G(\mathbf{x}, \mathbf{y}) dy; \quad \alpha = 1, 2, \dots \quad (31)$$

as connectivity basis. In a similar fashion one can prove that given any point  $\mathbf{y}_0$  of the exterior of  $R$ , the systems of functions consisting of  $G(\mathbf{x}, \mathbf{y}_0)$  together with all its partial derivatives with respect to  $\mathbf{y}$  at  $\mathbf{y}_0$ , is a connectivity basis. These procedures for constructing connectivity bases are applicable to a very general class of equations including time dependent ones<sup>16,29</sup>.

The availability of different connectivity bases for a given problem is important, because in numerical applications it is necessary to obtain reliable systems. For example, Aleksidze<sup>31</sup> has shown that the connectivity basis defined by equation (28) is unreliable. For the case when the region  $R$  is a circle, Herrera and Sabina's<sup>16</sup> procedure yields an orthogonal system, which is reliable; one can expect that the numerical properties of this system deteriorate steadily as the region is deformed away from the circle, a fact that has been confirmed numerically for a related system<sup>6</sup>.

The methods just described possess considerable generality; they can be applied to many problems governed by linear differential equations, such as

potential flow, fluids in porous media, heat flow, wave propagation and elasticity (static and dynamic). Connectivity bases for Stokes problems have just been constructed<sup>15</sup>.

The possibility of constructing connectivity bases which are independent of the regions considered, is relevant for the treatment of non-linear problems. Herrera and Sabina<sup>16</sup> obtained systems satisfying this property both for bounded and unbounded regions. In general, problems in which the differential equations are linear and the non-linearity is introduced through the boundary conditions can be treated by the methods described in this paper. This is the case of problems subjected to floating boundary conditions such as seepage flow, Stefan problem, some problems in plasticity and contact problems.

## EXTENSION TO NON-LINEAR PROBLEMS

As mentioned previously the boundary methods here described can be applied to non-linear problems, when the non-linearity is introduced through the boundary conditions. This is the case of seepage flow (Fig. 2), which is presented here as an example.

This, as well as Stefan's problem, was treated in collaboration with Prof. Nima Geffen, of the University of Tel-Aviv and Dr. Hervé Gourgeon, of the University of Paris XI, Orsay, France.

A normalized version of this problem is subjected to:

(a) Governing equations:

$$\nabla^2 \phi = 0; \quad \text{on } R \quad (32)$$

(b) Boundary conditions:

$$\phi(0, y) = 1; \quad 0 \leq y < 1 \quad (33)$$

$$\phi(1, y) = \begin{cases} m & 0 \leq y < m \\ y & m \leq y \leq Y(x) \end{cases}$$

$$\frac{\partial \phi}{\partial y}(x, 0) = 0; \quad 0 \leq x \leq 1 \quad (34)$$

$$\frac{\partial \phi}{\partial n}(x, Y(x)) = 0 \quad (35a)$$

$$\phi(x, Y(x)) = Y(x)$$

(c) Points of the region  $R$  satisfy:

$$0 \leq x \leq 1; \quad 0 \leq y \leq Y(x) \quad (36)$$

Thus,  $Y(x)$  defines the free surface.

Equation (35a) can be rewritten as:

$$Y'(x) = \frac{\phi_y(x, Y(x))}{\phi_x(x, Y(x))} \quad (37)$$

A convenient procedure to solve the system (35b), (37) is iterative. Define for  $n=0, 1, \dots$

$$\phi_n(x, Y_n(x)) = Y_n(x) \quad (38a)$$

$$Y'_{n+1}(x) = \frac{\phi_{ny}(x, Y_n(x))}{\phi_{nx}(x, Y_n(x))}$$

The scheme can be started by choosing  $Y_0(x)$  arbitrarily (e.g.  $Y_0(x) \equiv 1$ ).

$$\varphi_n(x, y) = \varphi_0(x, y) + u_n(x, y) \quad (39)$$

It is seen that:

$$u_n(x, Y_n(x)) = Y_n(x) - \varphi_0(x, Y_n(x)) \quad (40)$$

In addition

$$u_n(0, y) = u_n(1, y) = 0 \quad (41a)$$

$$\frac{\partial u_n}{\partial y}(x, 0) = 0; \quad 0 \leq x \leq 1 \quad (41b)$$

A connectivity basis convenient for this problem, because it satisfies equation (41) is

$$w_\alpha(x, y) = \sin \alpha \pi x \cosh \alpha \pi y; \quad \alpha = 1, 2, \dots \quad (42)$$

In this manner only the free boundary needs to be treated numerically. The approximation for  $u_n$  is (keeping fixed  $N$ ):

$$u_n(x, y) = \sum_{\alpha=1}^N A_\alpha^n w_\alpha(x, y) \quad (43)$$

Numerically, it is advantageous to determine the  $N$  coefficients  $A_\alpha^n$  by collocation; i.e. by requiring that equation (40) be satisfied at  $N$  points  $x_1 < x_2 < \dots < x_N$  of the interval  $[0, 1]$ .

Stefan problems has been treated in a similar fashion.

Free-boundary problems have received considerable attention in recent years using several approaches. Baiocchi, *et al.*<sup>32</sup>, for example, have based their method on variational inequalities. Liggett, on the other hand, has applied boundary integral equations to this class of problems; in particular the example presented here has been treated by that method<sup>33</sup>.

## REFERENCES

- Brebbia, C. A., (Ed.) *Recent Advances in Boundary Element Methods*, Pentech Press, London, 1978
- Cruse, T. A. An improved boundary-integral equation method for three dimensional elastic stress analysis, *Comput. Struct.*, 1974, **4**, 741
- Cruse, T. A. Mathematical foundations of the boundary-integral equation method in solid mechanics, *Sp. Scient. Rep., AFSC(F44620-74-C-0060)*, 1977
- Cruse, T. A. and Rizzo, F. J. Boundary-integral equation method: Computational applications in applied mechanics, *Am. Soc. Mech. Eng. AMD*, 1975, p. 11
- Rizzo, F. J. An integral equation approach to boundary value problems of classical elastostatics, *Q. J. Appl. Math.*, 1967, **25**, 83
- England, R., Sabina, F. J. and Herrera, I. Scattering of SH waves by surface cavities of arbitrary shape using boundary methods, *Physics of the Earth and Planetary Interiors*, 1980 in press
- Courant, R. and Hilbert, D. *Methods of Mathematical Physics*, Vol. 2, John Wiley, New York, 1962
- Herrera, I. General variational principles applicable to the hybrid element method, *Proc. Nat. Acad. Sci. USA*, 1977, **74**, 2595
- Herrera, I. Theory of connectivity for formally symmetric operators, *Proc. Nat. Acad. Sci. USA*, 1977, **74**, 4722
- Herrera, I. On the variational principles of mechanics, *Trends in Applications of Pure Mathematics to Mechanics*, Vol. II, Pitman, London, 1979, pp. 115-128
- Herrera, I. Theory of connectivity: a systematic formulation of boundary element methods, *Appl. Math. Modelling*, 1979, **3**, 151
- Herrera, I. Theory of connectivity as a unified approach to boundary methods, *Proc. IUTAM Symposium on Variational Principles of Mechanics*, 1978, University of Wisconsin-Madison, MRC Technical Summary Report No. 1938, 1979
- Herrera, I. Variational principles for problems with linear constraints. Prescribed jumps and continuation type restrictions, *J. Inst. Math. Appl.* 1980, in press; University of Wisconsin-Madison, MRC Technical Summary Report No. 1939, 1979
- Herrera, I. Boundary methods. A criterion for completeness, *Comunicaciones Técnicas, IIMAS-UNAM, México*, 1980, in press
- Herrera, I. Connectivity bases for Stokes problems, *Comunicaciones Técnicas, IIMAS-UNAM, Serie NA*, 1980, in press
- Herrera, I. and Sabina, F. J. Connectivity as an alternative to boundary integral equations. Construction of bases, *Proc. Nat. Acad. Sci. USA*, 1978, **75**, 2059
- Bates, R. H. T. Analytic constraints on electro-magnetic field computations, *IEEE Trans. Microwave Theory and Techniques*, 1975, **23**, 605
- Millar, R. F. The Rayleigh hypothesis and a related least-squares solution to scattering problems for periodic surfaces and other scatterers. *Radio Science*, 1973, **8**, 785
- Lions, J. L. and Magenes, E. *Non-homogeneous Boundary Value Problems and Applications*, Springer-Verlag, New York, 1972
- Herrera, I. Abstract Green's formulas and applications, *Comunicaciones Técnicas, IIMAS-UNAM, Serie NA*, 1980, in press
- Nemat-Nasser, S. On variational methods in finite and incremental elastic deformation problems with discontinuous fields, *Q. Appl. Math.* 1972, **30**, 143
- Temam, R. *Navier-Stokes equations: Theory and numerical analysis*, North-Holland, Amsterdam, 1977
- Picone, M. Nuovi metodi risolutivi per i problemi d'integrazione delle equazioni lineari a derivati parziali e nuova applicazione della trasformata multipla di Laplace nel caso delle equazioni a coefficienti costanti, *Atti Acc. Sc. Torino*, 1940, **76**, 413
- Amerio, L. Sull'integrazione delle equazioni lineari a derivate parziali del secondo ordine di tipo ellittico, *Acta Pont. Ac. Sc.* 1945, **9**, 213
- Amerio, L. Sul calcolo delle autosoluzioni dei problemi al contorno per le equazioni lineari del secondo ordine di tipo ellittico, *Rend. Acc. Lincei*, 1946, **1**, 352
- Amerio, L. Su un metodo di integrazione delle equazioni differenziali lineari a derivate parziali, *Rend. Sem. Mat. Fis. Milano*, 1947, **18**, 114
- Fichera, G. Teoremi di completezza sulla frontiera di un dominio per taluni sistema di funzioni, *Ann. Mat. Pura Appl.*, 1948, **27**, 1
- Fichera, G. Applicazione della teoria del potenziale di superficie ad alcuni problemi di analisi funzionale lineare, *Giornale Mat. Battaglini*, 1949, **78**, 71
- Kupradze, V. D. On the approximate solution of problems in mathematical physics, *Russ. Math. Surveys*, 1967, **22**(2), 58; *Uspehi Mat. Nauk.*, 1967, **22**(2), 59
- Christiansen, S. On Kupradze's functional equations for plane harmonic problems, *Functional Theoretic Methods in Differential equations*, (Eds. R. P. Gilbert and R. J. Weinacht) Pitman, London, 1976
- Aleksidze, M. A. Notes on an approximate method of solving boundary value problems, *Soviet Math. Dokl.*, 1967, **8**, 297
- Baiocchi, C., Comincioli, V., Magenes, E. and Pozzi, G. A. Fluid flow through porous media, *Laboratorio di Analisi Numerica del Consiglio Nazionale delle Ricerche, Pavia*, 1974, p. 69
- Liggett, J. A. Location of free surface in porous media, *J. Hydr. Div., ASCE*, 1977, **103**(4), 533