Proc. Natl. Acad. Sci. USA
Vol. 77, No. 8, pp. 4395–4398, August 1980
Applied Mathematical and Physical Sciences

Boundary methods: A criterion for completeness

(numerical methods/finite element method/partial differential equations/functional equations/scattering problems)

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Communicated by Emilio Rosenblueth, March 31, 1980

ABSTRACT The application of methods which constitute an alternative to boundary integral equations to specific problems depends on development of complete systems of solutions, convergence of approximating procedures, and formulation of variational principles. This paper establishes a criterion for completeness. In this manner, greater flexibility of the theory is achieved; for example, systems of functions which are complete for different types of boundary conditions are developed.

When boundary methods are applicable, the size of the regions considered can be reduced. In general, two situations can be distinguished. One may consider a boundary value problem in a region such as R (Fig. 1 *Upper*), in which general analytical solutions are known in the whole region R. Alternatively, the problem may be formulated in a region such as $R \cup E$ (Fig. 1 *Lower*), and the general analytical solutions may be known only in the subregion R. In the first case, the application of boundary methods permits restricting the numerical treatment to the boundary ∂R and the dimensionality of the problem is reduced. In the second case, the subregion R has to be treated numerically, but the region E is eliminated.

Most frequently, boundary methods have been formulated by means of integral equations (1–5). However, there are alternatives which present advantages. In general, a complete family of solutions is required in order to apply boundary methods. There are two possible ways in which such family can be supplied: one is a denumerable family of regular solutions (6) and the other is by means of a fundamental singular solution. The boundary integral equations method is a special case in which the singularity is placed on the boundary of the region considered, and due to this fact the resulting equations are singular. When a denumerable family of regular solutions is used the resulting equations are nonsingular, a fact that offers numerical advantages. What is probably more important in complicated problems is the observation that, generally, it is easier to construct complete families of solutions than fundamental solutions; indeed, there are methods available for synthesizing fundamental solutions starting from complete families [e.g., in terms of plane waves (7)].

Several recent accounts of boundary integral equations are available (1–4). Alternative approaches have been the subject of systematic research by me (8–15). The theoretical foundations of these alternatives are not complicated, but their applicability has been unnecessarily restricted due to lack of clarity. For example, in acoustics and electromagnetic field computations, the "Rayleigh hypothesis" introduces severe restrictions (16) which can be avoided altogether if a different point of view is adopted (17).

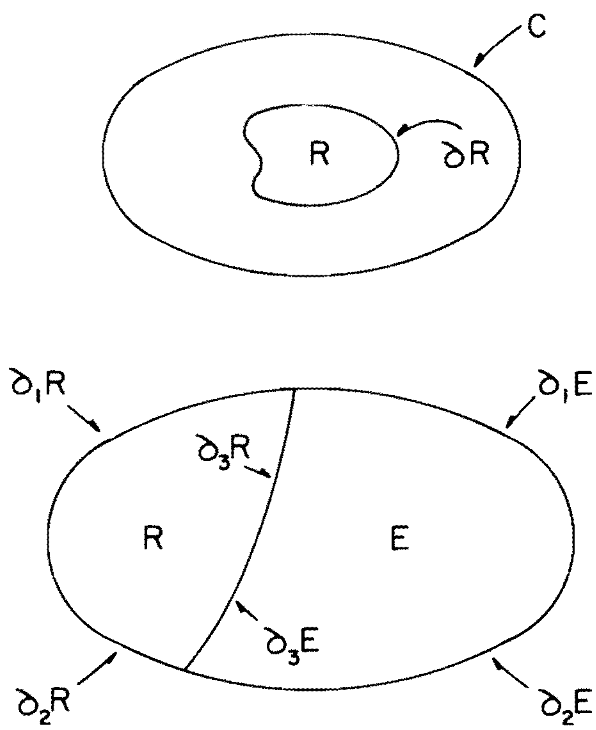

FIG. 1. Regions for boundary methods.

In general, the application of such methods requires the following studies:† (*i*) development of complete systems of solutions; (*ii*) convergence of the approximating procedures; and (*iii*) formulation of variational principles.

Generally, the solution of a boundary value problem that is well posed depends continuously on the boundary data with respect to a suitable norm (18); thus, in order to construct an arbitrary solution, it is only required to have available a family of solutions that is complete with respect to this norm.

Questions of convergence can be discussed by using results of the theory of partial differential equations in a manner similar to that presented elsewhere†; for example, for elliptic equations, the results presented in ref. 18 can be used. General variational principles applicable to boundary methods have been developed recently (13).

Generally, these methods are applicable to linear problems; however, they can also be applied to an important class of nonlinear problems. These are problems for which the governing equations are linear, and the nonlinearity is introduced

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† Herrera, I. (1980) "Boundary Methods in flow problems," *Proceedings Third International Conference on Finite Elements in Flow Problems*, Banff, Canada, 10–13 June, Vol. 1, pp. 30–42.

through the boundary conditions only. This is the case of free boundary problems, such as seepage flow, Stefan problems, and contact problems.

This paper establishes a criterion for completeness of systems of solutions. In previous papers (15), the notion of connectivity basis was introduced. The main result to be presented here establishes a connection between this concept and Hilbert-space bases, which permits one to obtain the latter when connectivity bases are available. An advantage of using connectivity bases springs from the fact that this concept is independent of the Hilbert-space structure and it is more flexible in applications. This allows developing systems of functions that are complete independently of the boundary conditions considered. For Laplace's equation, for example, a system of functions that is complete for the Dirichlet problem is necessarily complete for the Neuman problem, also. In elasticity, this leads to systems that are complete for displacements, tractions, and mixed problems, simultaneously. By using the criterion for completeness given here, it is not difficult to derive Hilbert-space bases, that were obtained by Kupradze (19) for a large class of problems, from connectivity bases that can be obtained by a procedure developed by Herrera and Sabina (15). As a further illustration, later in this paper, Hilbert-space bases are obtained for a general class of problems that was introduced previously under the title "Problem of Connecting" (13).

In previous articles (13), linear spaces D were considered and the null subspace N of an antisymmetric operator A was introduced. The quotient space $\mathcal{D} = D/N$ corresponds to the space of boundary values in applications; e.g., in potential theory, members of this space are defined by the values of the functions and their normal derivatives on the boundary. Here, only these spaces need to be considered. The notation is consistent with that used in other papers by me.

THEORETICAL DEVELOPMENTS

We shall be concerned with functional valued operators $P:\mathcal{D} \to \mathcal{D}^*$ which are linear. Here, $\mathcal{D}$ is any linear space in the field of real or, alternatively, complex numbers and $\mathcal{D}^*$ is the algebraic dual of $\mathcal{D}$. The notation $\langle Pu, v\rangle$ is adopted for the value of the linear functional $Pu\in\mathcal{D}^*$ evaluated at $v\in\mathcal{D}$. The following notations and results were given previously (13).

Definition 2.1: The operators $P:\mathcal{D} \to \mathcal{D}^*$ and $Q:\mathcal{D} \to \mathcal{D}^*$ can be varied independently if and only if, given $U\in\mathcal{D}$ and $V\in\mathcal{D}$, there exists $u\in\mathcal{D}$ such that

$$Pu = PU \text{ and } Qu = QV. \qquad [2.1]$$

In what follows, $A:\mathcal{D} \to \mathcal{D}^*$ will be an antisymmetric operator which will be assumed to be one-to-one; i.e.

$$N = N_A = \{u\in\mathcal{D} \mid Au = 0\} = \{0\}. \qquad [2.2]$$

Definition 2.2: An operator $B:\mathcal{D} \to \mathcal{D}^*$ said to decompose A when B and B^* can be varied independently and

$$A = B - B^*. \qquad [2.3]$$

The concept of canonical decomposition of $\mathcal{D}$ with respect to A was introduced in ref. 13, and a connection between such concept and operators that decompose A was also discussed. For the purpose of this paper it is only necessary to recall that, when $B:\mathcal{D} \to \mathcal{D}^*$ decomposes A, the null subspaces N_B, N_{B^*} constitute a canonical decomposition of $\mathcal{D}$. In such case, one has

$$N_B + N_{B^*} = \mathcal{D};\, N_B \cap N_{B^*} = N_A. \qquad [2.4]$$

In what follows, it will be assumed that an operator $B:\mathcal{D} \to \mathcal{D}^*$ that decomposes A is given and we write $I_1 = N_B, I_2 = N_{B^*}$.

Definition 2.3: Let a linear subspace $I_P\subset\mathcal{D}$, and elements $U\in\mathcal{D}$, $V\in\mathcal{D}$ be given. Then, an element $u\in\mathcal{D}$ is a solution of the reduced problem with linear restrictions if

$$u \vdash U\in I_P, \text{ and } u - V\in N_B = I_1. \qquad [2.5]$$

Definition 2.4: A linear subspace $I_P\subset\mathcal{D}$ is said to be completely regular when

$$\langle Au,w\rangle = 0 \;\forall\; w\in I_P \Leftrightarrow u\in I_P. \qquad [2.6]$$

Definition 2.5: A subset $\mathcal{B}\subset I_P$ is said to be c-complete (or complete in connectivity) when for every $u\in\mathcal{D}$ one has

$$\langle Au,w\rangle = 0 \;\forall\; w\in \mathcal{B} \Rightarrow u\in I_P \qquad [2.7]$$

The set $\mathcal{B}$ is called a connectivity basis if, in addition, $A\mathcal{B}\subset\mathcal{D}^*$ is a linearly independent subset of $\mathcal{D}^*$.

As before, let $B:\mathcal{D} \to \mathcal{D}^*$ decompose A; write

$$I_1 = N_B;\, I_2 = N_{B^*}. \qquad [2.8]$$

In addition, take

$$N_{\mathcal{L}} = I_P \cap I_1. \qquad [2.9]$$

Notice that $u\in\mathcal{D}$ is a solution of the reduced problem with linear restrictions with vanishing data, if and only if $u\in N_{\mathcal{L}}$. Assume:

(*a*) I_1 and I_2 are Hilbert spaces. The inner product will be denoted by (,).

(*b*) There is an (algebraic and topological) isomorphism $G:I_1 \to I_2$ between these Hilbert spaces.

(*c*) For every $u = u_1 + u_2$, $v = v_1 + v_2$ (where $u_1\in I_1$, $u_2\in I_2$ and similarly for v), one has

$$\langle Au,v\rangle = (v_2,Gu_1) - (u_2,Gv_1). \qquad [2.10]$$

In this case I_2 is a Hilbert-space and one may write $(GN_{\mathcal{L}})^\perp$ for the orthogonal complement of $GN_{\mathcal{L}}$ in I_2.

THEOREM 2.1. *Assume* $N_{\mathcal{L}}$ *is closed in* I_1. *Let* I_P *be completely regular and hypotheses,* a, b, *and* c *hold. Given any subset* $\mathcal{B}\subset I_P$ *define*

$$\mathcal{B}_2 = \{w_2\in I_2 \mid \exists\; w_1\in I_1 \ni w = w_1 + w_2\in\mathcal{B}\}. \qquad [2.11]$$

Then $\mathcal{B}_2\subset(GN_{\mathcal{L}})^\perp$ *spans the Hilbert space* $(GN_{\mathcal{L}})^\perp$ *if and only if* $\mathcal{B}\cup N_{\mathcal{L}}\subset I_P$ *is c-complete. In addition,* $\mathcal{B}\subset\mathcal{D}$ *is linearly independent mod* $(N_{\mathcal{L}})$ *if and only if so is* $\mathcal{B}_2$.

Proof: To start with, let us recall that, when I_P is completely regular and $w = w_1 + w_2\in I_P$, then $w_2\in(GN_{\mathcal{L}})^\perp$ necessarily. Indeed, let $v\in N_{\mathcal{L}} = I_1\cap I_P$; then

$$\langle Aw,v\rangle = -(w_2,Gv) = 0.$$

This proves the desired result, because v is an arbitrary element of $N_{\mathcal{L}}$.

First, taking as an assumption that $\mathcal{B}\cup N_{\mathcal{L}}\subset I_P$ is c-complete, it will be shown that for every $u_2\in(GN_{\mathcal{L}})^\perp\subset I_2$, one has

$$(u_2,w_2) = 0 \;\forall\; w_2\in\mathcal{B}_2 \Rightarrow u_2 = 0. \qquad [2.12]$$

Assume that premise 2.12 is satisfied. Write $u_2 = Gv_1$ with $v_1\in N_{\mathcal{L}}^\perp\subset I_1$; clearly, this is possible. Take $v = v_1$, then

$$\langle Av,w\rangle = (w_2,Gv_1) = (w_2,u_2) = 0, \;\forall\; w\in\mathcal{B}\cup N_{\mathcal{L}}. \qquad [2.13]$$

Here, the fact that $w\in N_{\mathcal{L}} \Rightarrow w_2 = 0$ has been used. This implies $v = v_1\in I_P$ because $\mathcal{B}$ is c-complete for I_P. Hence, $v_1\in I_1\cap I_P = N_{\mathcal{L}}$. Therefore, $v_1\in N_{\mathcal{L}}\cap N_{\mathcal{L}}^\perp$; i.e., $v_1 = 0$ and $u_2 = Gv_1 = 0$. This shows that $\mathcal{B}_2$ spans $(GN_{\mathcal{L}})^\perp\subset I_2$ whenever $\mathcal{B}$ is c-complete. The proof of the converse statement is more complicated and I prefer to show the linear independence properties first. Under the hypothesis that $\mathcal{B}$ is linearly independent, assume that for some N there are nonzero scalars $a_1 a_2, \ldots, a_N$ such that $\sum_{\alpha=1}^{N} a_\alpha w_{2\alpha} = 0$. Then, $\sum_{\alpha=1}^{N} a_\alpha w_\alpha \in I_1 \cap I_P = N_{\mathcal{L}}$, a contradiction. Thus, $\mathcal{B}_2$ is necessarily linearly independent. To prove the converse, it is only necessary to observe that

$$\sum_{n=1}^{N} a_\alpha w_\alpha = N_{\mathcal{L}} \Rightarrow \sum_{n=1}^{N} a_\alpha w_{2\alpha} = 0. \qquad [2.14]$$

We proceed now to prove that when $\mathcal{B}_2$ spans $(GN_{\mathcal{L}})^\perp$, then $\mathcal{B} \cup N_{\mathcal{L}}$ is c-complete in I_P. *Let* $\{e_{12}, e_{22}, \ldots\} \subset (GN_{\mathcal{L}})^\perp$ be an orthonormal Hilbert-space basis of $(GN_{\mathcal{L}})^\perp$ obtained by orthonormalization of $\mathcal{B}_2$. Take $e_\alpha \in I_P$ ($\alpha = 1, 2, \ldots$) as the linear combination of elements of $\mathcal{B}$ which has the same coefficients as $e_{\alpha 2}$. Using this notation, the desired result will follow from the following *Lemma*.

LEMMA 2.1. *Given* $\mathcal{B} \subset \mathrm{I_P}$, *assume* $\mathcal{B}_2$ *spans* $(\mathrm{GN}_{\mathcal{L}})^\perp$ *and take* $\{\mathrm{e_1, e_2}, \ldots\}$ *as explained before. Then, when* u, v$\in \mathcal{D}$ *are such that* $\langle \mathrm{Au,w} \rangle = 0\ \forall\ \mathrm{w} \in \mathcal{B} \cup \mathrm{N}_{\mathcal{L}}$ *and similarly for* v, *one has*

$$(i)\ \mathrm{u_2} \in (\mathrm{GN}_{\mathcal{L}})^\perp, \qquad [2.15]$$

$$(ii)\ (\mathrm{u_2, Ge_{\alpha 1}}) = (\mathrm{e_{\alpha 2}, Gu_1}), \qquad [2.16]$$

$$(iii)\ (\mathrm{v_2, Gu_1}) = \sum_{\alpha=1}^{\infty} (\mathrm{u_2, Ge_{\alpha 1}})(\mathrm{v_2, e_{\alpha 2}});$$

$$\mathrm{u_2} = \sum_{\alpha=1}^{\infty} (\mathrm{u_2, e_{\alpha 2}})\mathrm{e_{\alpha 2}} \qquad [2.17]$$

$$(iv)\ \langle \mathrm{Au,v} \rangle = 0\ \forall\ \mathrm{v} \in \mathrm{I_P}. \qquad [2.18]$$

The case when $\mathrm{u = e_\beta}$ *in* **2.16** *has special interest; it is*

$$(\mathrm{e_{\beta 2}, Ge_{\alpha 1}}) = (\mathrm{e_{\alpha 2}, Ge_{\beta 1}}). \qquad [2.19]$$

Proof: Proposition i follows from the remarks made at the beginning of the proof of *Theorem 2.1*. Eq. **2.16** is straightforward when use is made of the fact that every e_α is a linear combination of elements of $\mathcal{B}$ and Eq. **2.10** is applied.

The second of Eqs. **2.17** is clear by virtue of Eq. **2.15**; the first one follows from the fact that $\{e_{\alpha 2}\}$ is an orthonormal basis of $(GN_{\mathcal{L}})^\perp$ and Eq. **2.16**.

Assume $v \in I_P$. By using Eqs. **2.15** and **2.16**, a direct computation yields

$$(v_2, Gu_1) \sum_{\alpha=1}^{\infty} (u_2, Ge_{\alpha 1})(v_2, e_{\alpha 2})$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (u_2, e_{\beta 2})(v_2, e_{\alpha 2})(e_{\alpha 2}, Ge_{\beta 1}). \qquad [2.20]$$

This shows $(v_2, Gu_1) = (u_2, Gv_1)$, because $(Ge_{\alpha 1}, e_{\beta 2})$ is symmetric in α and β by virtue of Eq. **2.19**. This establishes iv and the proof of the *Lemma* is complete. Recall that the complete regularity of I_P together with Eq. **2.18** implies $u \in I_P$. Hence, the proof of *Theorem 2.1* is complete.

The flexibility supplied to the theory by the results contained in *Theorem 2.1* can be better appreciated by considering a few examples. For Laplace's equation, Dirichlet's problem possesses a unique solution; thus, the boundary values associated with any connectivity basis of Laplace's equation constitute a Hilbert-space basis of $H^\circ(\partial R)$. On the other hand, the subspace $N_{\mathcal{L}}$ of solutions of Neuman's problem with vanishing data are the constant functions. Thus, the normal derivatives of any function such that $\int_{\partial R} \partial u/\partial n dx = 0$ can be approximated in $H^\circ(\partial R)$ by means of a connectivity basis. This is the well-known restriction imposed on the data for the boundary values of the normal derivatives, in order for Neuman's problem to possess a solution. *Theorem 2.1* shows that one can use the same system of functions for both Dirichlet's and Neuman's problems and, indeed, for many other combinations of boundary conditions. In elasticity, displacement, traction, and mixed problems can be treated in this manner. More general examples are given next.

HILBERT-SPACE BASES FOR THE PROBLEM OF CONNECTING

In a previous paper (13) the problem of connecting was formulated and general variational principles were derived. In this section a procedure for constructing Hilbert-space bases for such problems is given.

This problem is an abstract version of the problem of obtaining solutions to partial differential equations defined in neighboring regions such as R and E in Fig. 1 *Lower* and which satisfy prescribed jump conditions across the connecting boundary $\partial_3 R = \partial_3 E$. The jump conditions are relative to a smoothness criterion.

Let $A_R: \mathcal{D}_R \to \mathcal{D}_R^*$ and $A_E: \mathcal{D}_E \to \mathcal{D}_E$ be functional valued operators defined on linear spaces $\mathcal{D}_R$ and $\mathcal{D}_E$. Consider $\mathcal{D} = \mathcal{D}_R \oplus \mathcal{D}_E$ and a linear subspace $I_R \subset \mathcal{D}_R$ and let $\mathcal{B}_R \subset I_R$ be a connectivity basis of I_R. Corresponding relations are satisfied by $\mathcal{D}_E$, I_E, and $\mathcal{B}_E$. Define $\hat{\mathcal{D}} = \mathcal{D}_R \oplus \mathcal{D}_E$ and $\hat{I}_P = I_R \oplus I_E$. Take $\hat{A}: \hat{\mathcal{D}} \to \hat{\mathcal{D}}^*$ as

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle \qquad [3.1]$$

for every $\hat{u} = (u_R, u_E) \in \hat{\mathcal{D}}$ and $\hat{v} = (v_R, v_E) \in \hat{\mathcal{D}}$.

Smoothness criteria were defined previously (13). For our purpose it is enough to recall that a smoothness condition is a linear subspace $\hat{\mathcal{S}} \subset \hat{\mathcal{D}}$; a smoothness condition is said to be completely regular when as a linear subspace it is completely regular.

It has been shown (13) that, when a smoothness condition is completely regular, there is an operator $\hat{J}: \hat{\mathcal{D}} \to \hat{\mathcal{D}}^*$ called the jump operator, such that the null subspaces of $\hat{J}$ and its adjoint $\hat{J}^*$ satisfy

$$\hat{\mathcal{S}} = \hat{N}_J;\ \hat{M} = \hat{N}_{J^*} \qquad [3.2]$$

where $\hat{M} \subset \hat{\mathcal{D}}$ is the set of zero mean elements, defined in (13).

Corresponding to assumptions a, b, and c in the preceding section, it will be assumed

(a) $\hat{\mathcal{S}}$ and $\hat{M}$ are Hilbert spaces with inner product (,).

(b) There is an isomorphism $\hat{G}: \hat{\mathcal{S}} \to \hat{M}$ such that

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = (\hat{v}_2, \hat{G}\hat{u}_1) - (\hat{u}_2, \hat{G}\hat{v}_1) \qquad [3.3]$$

for every $\hat{u} = \hat{u}_1 + \hat{u}_2$, $\hat{u}_1 \in \hat{\mathcal{S}}$, $\hat{u}_2 \in \hat{M}$ and correspondingly for every $\hat{v} = \hat{v}_1 + \hat{v}_2$.

In what follows the following notation and assumptions are adopted. $\mathrm{\hat{I}_P = I_R \oplus I_E} \subset \hat{\mathcal{D}}$ where $I_R \subset \mathcal{D}_R$ and $I_E \subset \mathcal{D}_E$ are linear subspaces; $\mathcal{B}_R \subset I_R$ and $\mathcal{B}_E \subset I_E$ are connectivity bases of I_R and I_E, respectively. The reduced problem of connecting (13) is defined as the special case of *Definition 2.3*, in which I_P is taken as $\hat{I}_P$ and I_1 is $\hat{\mathcal{S}}$.

THEOREM 3.1. *Let* $\hat{\mathcal{S}}$ *be a completely regular smoothness condition such that assumptions* a *and* b *are satisfied and the reduced problem of connecting satisfies uniqueness. Write* $\hat{\mathcal{B}} = \mathcal{B}_\mathrm{R} \oplus \mathcal{B}_\mathrm{E} \subset \mathrm{\hat{I}_P}$, *and let*

$$\hat{\mathcal{B}}_M = \{\hat{w}_m \in \hat{M}\ |\ \exists\ \hat{w}_s \in \hat{\mathcal{S}} \ni \hat{w} = \hat{w}_m + \hat{w}_s \in \hat{\mathcal{B}}\}. \qquad [3.4]$$

Then $\hat{\mathcal{B}}_\mathrm{M}$ is a Hilbert-space basis of $\hat{\mathrm{M}}$.

Proof: In view of *Theorem 2.1*, it is enough to prove that $\hat{\mathcal{B}} \subset \hat{I}_P$ is a connectivity basis. The following *Lemma* establishes this result.

LEMMA 3.1. The set $\hat{\mathscr{B}} = \mathscr{B}_R \oplus \mathscr{B}_E$ is a connectivity basis of $\hat{I}_P$.

Proof: It is required to show that

$$<\hat{A}\hat{u},\hat{w}> = 0 \ \forall \ \hat{w} \in \hat{\mathscr{B}} \Rightarrow \hat{u} \in \hat{I}_P. \qquad [3.5]$$

This is immediate because the premise in [3.5] implies that

$$<A_R u_R, w_R> = 0 \ \forall \ w_R \in \mathscr{B}_R \text{ and}$$
$$<A_E u_E, w_E> = 0 \ \forall \ w_E \in \mathscr{B}_E. \qquad [3.6]$$

Hence, $u_R \in I_R$ while $u_E \in I_E$.

SPECIFIC APPLICATIONS

As specific applications of *Theorem 2.1,* consider Laplace's and reduced wave equations. In a previous paper (18), connectivity bases for these equations in two and three dimensions were given. *Theorem 2.1* shows how to derive Hilbert-spaces bases for the boundary values associated with such problems.

A slight modification of the arguments presented in ref. 15 permits deriving the bases that were obtained by Kupradze (19). When the region is two-dimensional and bounded there is an anomalous situation when the exterior problem for Laplace's equation does not have a unique solution (20). Due to this fact, when considering that equation, the constant function has to be included among the members of the resulting system in order to grant that Kupradze's procedure yield a Hilbert-space basis. It can be seen, however, that such restriction does not apply to the connectivity bases derived by Herrera and Sabina (15) because they were obtained by using a circle of arbitrary radius.

In the case of Laplace's and reduced wave equations, it is convenient to define (Fig. 1 *Lower*)

$$<P_R u_R, v_R> = \int_R v_R \nabla^2 u_R d\mathbf{x} - \int_{\partial_1 R} u_R \frac{\partial v_R}{\partial n} d\mathbf{x} + \int_{\partial_2 R} v_R \frac{\partial u_R}{\partial n} d\mathbf{x} \qquad [4.1]$$

while $P_E: D_E \rightarrow D_E^*$ is taken correspondingly. When the linear subspace $\hat{\mathscr{S}} \subset \hat{\mathscr{D}}$ of smooth functions is taken as functions (u_R, u_E) such that they are continuous together with their normal derivatives across the common boundary $\partial_3 R = \partial_3 E$, which vanish on $\partial_1(R \cup E)$ and with vanishing normal derivative on $\partial_2(R \cup E)$, the jump operator is given by

$$<\hat{J}\hat{u},\hat{v}> = \int_{\partial_3 R} \left\{ [\hat{u}] \frac{\overline{\partial v}}{\partial n} - \bar{v} \left[\frac{\partial \hat{u}}{\partial n} \right] \right\} d\mathbf{x} - \int_{\partial_1(R \cup E)} u \frac{\partial v}{\partial n} d\mathbf{x} + \int_{\partial_2(R \cup E)} v \frac{\partial u}{\partial n} d\mathbf{x}. \qquad [4.2]$$

Similar results hold in elasticity, heat equation, wave equation, and many other problems and can be derived by using operators introduced in previous articles (8–15).

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