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Extension of the Integrodifferential Approach to Inhomogeneous Multiaquifer Systems

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In the original integrodifferential model developed by Herrera and his co-workers for multiaquifer systems and based on the quasi-three-dimensional theory for such systems, homogeneity was assumed at least in the vertical direction. This restriction is eliminated here, and the integrodifferential approach is extended to general heterogeneous multiaquifer systems. Finite element approximations in time and space are also introduced which include as particular cases most of the well-known time integration schemes and lead to consistent updating schemes for the influence and memory terms.

1. INTRODUCTION

The assumptions of horizontal flow in the aquifers and of vertical flow in the aquitards characterize the mathematical description of leaky aquifer systems. Under these assumptions, leaky aquifers are governed by a system of integrodifferential equations. For the case of vertically homogeneous aquitards such equations were derived by *Herrera and Rodarte* [1973], and a corresponding numerical method was developed by *Herrera and Yates* [1977] and applied to the homogeneous aquifer case. Its application to the short time range has just been analyzed by *Chen and Herrera* [1981].

The mathematical basis of this procedure is, however, not restricted to vertically homogeneous aquitards and can be applied even if the aquitard is completely heterogeneous. What is characteristic of the method is the use of eigenfunction expansions in order to eliminate the treatment of the aquitard. When this is arbitrarily heterogeneous, the eigenfunction expansion has to be derived numerically. However, efficient computational schemes are available to do this [*Wilkinson and Reinsch,* 1971].

This paper is devoted to developing such an approach for an arbitrarily heterogeneous leaky aquifer system in which the properties of the system are arbitrary functions of position, in both the horizontal and the vertical directions. Section 2 is devoted to the treatment of the aquitards and their elimination, leading to a system of integrodifferential equations. In section 3 the details of the numerical implementation of the resulting system are given, with a particular emphasis on the efficient approximation of the horizontal heterogeneity in the aquifers as well as on the use of a class of finite element in time integration schemes, including as particular cases most integration schemes previously used and leading to self-consistent updating formulas for the memory and influence terms. Preliminary comparisons [*Herrera et al.,* 1980] with other available numerical schemes [*Chorley and Frind,* 1978] show that the method presented here offers significant advantages in both computer time and memory requirements.

2. REDUCTION OF A SYSTEM OF INTEGRODIFFERENTIAL EQUATIONS

The basic heterogeneous aquitard equation is

$$\frac{\partial}{\partial z}\left(K' \frac{\partial s'}{\partial z}\right) = S_s' \frac{\partial s'}{\partial t} \qquad (1)$$

subject to boundary conditions

$$s'(0, t) = s_1(t) \qquad (2a)$$

$$s'(b', t) = s_2(t) \qquad (2b)$$

and initial conditions taken, for instance, to be

$$s'(z, 0) = s_0'(z) \qquad (3)$$

To treat this problem, it is convenient to introduce the eigenfunctions $\phi_n(z)$ and the corresponding eigenvalues $\lambda_n^2 (n = 1, 2, \cdots)$ of the auxiliary problem

$$\frac{\partial}{\partial z}\left(K' \frac{\partial \phi_n}{\partial z}\right) + \lambda_n^2 S_s' \phi_n = 0 \qquad z \in (0, b') \qquad (4)$$

with boundary conditions

$$\phi_n(0) = \phi_n(b') = 0 \qquad (5)$$

These eigenfunctions will be assumed to be normalized, that is,

$$\int_0^{b'} S_s' \phi_n \phi_m \, dz = \delta_{nm} \qquad (6)$$

so that

$$\int_0^{b'} K' \frac{\partial \phi_n}{\partial z} \frac{\partial \phi_m}{\partial z} \, dz = \lambda_n^2 \delta_{nm} \qquad (7)$$

Other auxiliary functions to be used in the sequel are $v_k(z)$, $k = 1, 2$, defined by

$$\frac{\partial}{\partial z}\left(K' \frac{\partial v_k}{\partial z}\right) = 0 \qquad z \in (0, b') \qquad (8)$$

and

$$v_1(0) = v_2(b') = 1 \qquad v_1(b') = v_2(0) = 0 \qquad (9)$$

It readily follows that

$$v_1(z) = \frac{1}{t_a} \int_z^{b'} \frac{dz'}{K'} \qquad (10a)$$

$$v_2(z) = \frac{1}{t_a} \int_0^{z} \frac{dz'}{K'}$$

where

$$t_a = \int_0^{b'} \frac{dz'}{K'}$$

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Let us define

$$w(z, t) = s_1(t)v_1(z) + s_2(t)v_2(z) - s'(z, t) \tag{11}$$

implying by (1) and (2) that

$$S_s' \frac{\partial w}{\partial t} - \frac{\partial}{\partial z}\left(K' \frac{\partial w}{\partial z}\right) = S_s' \sum_{k=1}^{2} \frac{\partial s_k}{\partial t} v_k \tag{12}$$

subject to

$$w(0, t) = w(b', t) = 0 \tag{13}$$

With

$$a_n(t) = \int_0^{b'} S_s' w \phi_n \, dz \tag{14}$$

and after multiplying (12) by ϕ_n and integrating the resulting equation from 0 to b' one obtains

$$\frac{\partial a_n}{\partial t} + \lambda_n^2 a_n = \sum_{k=1}^{2} b_{kn} \frac{\partial s_k}{\partial t} \tag{15}$$

Here the boundary conditions (13) have been taken into account, and the constants b_{kn} have been defined as

$$b_{kn} = \int_0^{b'} S_s' v_k \phi_n \, dz \tag{16}$$

From (15) it follows that functions $d_{kn}(t)$ of t can be introduced such that

$$\frac{\partial d_{kn}}{\partial t} + \lambda_n^2 d_{kn} = \frac{\partial s_k}{\partial t} \tag{17}$$

with

$$a_n = \sum_{k=1}^{2} b_{kn} d_{kn} \tag{18}$$

At the given initial time t_0 it follows from (10) and (14) that

$$a_n(t_0) = \sum_{k=1}^{2} b_{kn} s_k(t_0) - c_n \tag{19}$$

where

$$c_n = \int_0^{b'} S_s' s'(z, t_0) \phi_n \, dz \tag{20}$$

The initial conditions for d_{kn} can be selected in many ways, a convenient choice being

$$d_{kn}(t_0) = s_k(t_0) - \theta_k c_n / b_{kn} \tag{21}$$

with

$$\sum_{k=1}^{2} \theta_k = 1$$

An explicit expression for $d_{kn}(t)$ is

$$d_{kn}(t) = \exp\left[-\lambda_n^2 (t - t_0)\right] d_{kn}(t_0) + \int_{t_0}^{t} \exp\left[-\lambda_n^2 (t - \tau)\right] \frac{\partial s_k}{\partial t}(\tau) \, d\tau \tag{22}$$

Since the set $\{\phi_n\}$ of eigenfunctions is complete, definition (14) implies

$$w(z, t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(z) \tag{23}$$

and therefore by virtue of (11) and (18),

$$s'(z, t) = \sum_{k=1}^{2} \left[s_k(t) v_k(z) - \sum_{n=1}^{\infty} b_{kn} d_{kn}(t) \phi_n(z) \right] \tag{24}$$

Let us assume that at $t_0 = 0$ the initial drawdowns in the aquifers and the aquitard are given, namely, $s_k(0)$, $k = 1, 2$, and $s_0'(z)$. Alternatively, the past history of the drawdown may be known from say $t = -\infty$ to $t = 0$. This case is also quite important and will be discussed in Appendix A. Let $w_k(z, t)$, $k = 1, 2$ be solutions of (1) subject to boundary conditions

$$w_1(0, t) = w_2(b', t) \equiv 1 \tag{25a}$$

$$w_1(b', t) = w_2(0, t) \equiv 0 \tag{25b}$$

and initial conditions

$$w_1(z, 0) = w_2(z, 0) = s_0'(z) \tag{26}$$

From (20), (22), and (24) it follows that

$$w_k(z, t) = v_k(z) - \sum_{n=1}^{\infty} b_{kn} \exp\left(-\lambda_n^2 t\right) \phi_n(z) + \sum_{n=1}^{\infty} c_n \exp\left(-\lambda_n^2 t\right) \phi_n(z) \tag{27}$$

and an alternative expression for $s'(z, t)$ in terms of the w_k is

$$s'(z, t) = \sum_{k=1}^{2} s_k(0) w_k(z, t) + \int_0^t w_k(z, t - \tau) \frac{\partial s_k}{\partial t}(\tau) \, d\tau + \sum_{n=1}^{\infty} c_n \exp\left(-\lambda_n^2 t\right) \phi_n(z) \tag{28}$$

This expression is the basis of the modification to the integrodifferential approach [*Herrera and Rodarte*, 1973; *Herrera and Yates*, 1977] presented here; it generalizes equation (3) of [*Herrera and Yates*, 1977]. In particular, (28) takes into account any initial conditions in the aquifers-aquitard system; also, the functions w_k are given in terms of the eigenfunctions ϕ_n. Actually, (28) can be derivated with respect to the z variable to get expressions for the leakage fluxes from the aquitard to the adjacent aquifers at $z = 0$ and $z = b'$. For instance,

$$\left.\frac{\partial s'}{\partial z}\right|_{z=0} = -\frac{1}{t_a K'(0)} \left[\int_0^t f_1\left(\frac{t - \tau}{\bar{S}' t_a}\right) \frac{\partial s_1}{\partial t}(\tau) \, d\tau + s_1(0) f_1(t/\bar{S}' t_a) - \int_0^t h_1\left(\frac{t - \tau}{\bar{S}' t_a}\right) \frac{\partial s_2}{\partial t}(\tau) \, d\tau - s_2(0) h_1(t/\bar{S}' t_a) - t_a \sum_{n=1}^{\infty} c_n b_{1n} \lambda_n^2 \exp\left(-\lambda_n^2 t\right) \right] \tag{29}$$

where we used the definitions

$$f_1(t/\bar{S}' t_a) = -t_a K'(0) \left(\partial w_1 / \partial z\right)|_{z=0} \tag{30a}$$

$$h_1(t/\bar{S}' t_a) = +t_a K'(0) \left(\partial w_2 / \partial z\right)|_{z=0} \tag{30b}$$

$\bar{S}'$ being the mean storage coefficient in the aquitard, and the fact that

$$(\partial \phi_n / \partial z)(0) = \lambda_n^2 b_{1n} / K'(0) \tag{31}$$

as shown in Appendix B. From definitions (30) and equations (27) and (31) it follows that

$$f_1(t/\bar{S}'t_a) = 1 + t_a \sum_{n=1}^{\infty} \lambda_n^2 b_{1n}(b_{1n} - c_n) \exp(-\lambda_n^2 t)$$

$$= 1 + \sum_{n=1}^{\infty} A_{1n} \exp(-\lambda_n^2 t) \tag{32a}$$

and

$$h_1(t/\bar{S}'t_a) = 1 - t_a \sum_{n=1}^{\infty} \lambda_n^2 b_{1n}(b_{2n} - c_n) \exp(-\lambda_n^2 t)$$

$$= 1 + \sum_{n=1}^{\infty} B_{1n} \exp(-\lambda_n^2 t) \tag{32b}$$

Similarly,

$$\left.\frac{\partial s'}{\partial z}\right|_{z=b'} = + \frac{1}{t_a K'(b')} \left[\int_0^t f_2\left(\frac{t-\tau}{\bar{S}'t_a}\right) \frac{\partial s_2(\tau)}{\partial t} d\tau + s_2(0) f_2(t/\bar{S}'t_a) \right.$$

$$\int_0^t h_2\left(\frac{t-\tau}{\bar{S}'t_a}\right) \frac{\partial s_1(\tau)}{\partial t} d\tau - s_1(0) h_2(t/\bar{S}'t_a)$$

$$\left. - t_a \sum_{n=1}^{\infty} c_n b_{2n} \lambda_n^2 \exp(-\lambda_n^2 t) \right] \tag{33}$$

with the definitions

$$f_2(t/\bar{S}'t_a) = +t_a K'(b')(\partial w_2/\partial z)|_{z=b'} \tag{34a}$$

$$h_2(t/\bar{S}'t_a) = -t_a K'(b')(\partial w_1/\partial z)|_{z=b'} \tag{34b}$$

so that

$$f_2(t/\bar{S}'t_a) = 1 + t_a \sum_{n=1}^{\infty} \lambda_n^2 b_{2n}(b_{2n} - c_n) \exp(-\lambda_n^2 t)$$

$$= 1 + \sum_{n=1}^{\infty} A_{2n} \exp(-\lambda_n^2 t) \tag{35a}$$

and

$$h_2(t/\bar{S}'t_a) = 1 - t_a \sum_{n=1}^{\infty} \lambda_n^2 b_{2n}(b_{1n} - c_n) \exp(-\lambda_n^2 t)$$

$$= 1 + \sum_{n=1}^{\infty} B_{2n} \exp(-\lambda_n^2 t) \tag{35b}$$

It is easy to check that in the homogeneous aquitard case, with $s_0'(z) = 0$, $f_1 = f_2$ and $h_1 = h_2$ reduce to the memory and influence functions introduced by *Herrera and Rodarte* [1973] and that in particular the following equivalences hold (where t' is dimensionless time):

$$t/\bar{S}'t_a \to t' \tag{36a}$$

$$\lambda_n^2 t \to -n^2\pi^2 t' \tag{36b}$$

$$A_{1n} \to 2 \tag{36c}$$

$$B_{1n} \to 2(-1)^n \tag{36d}$$

For computational purposes, $g_k(\equiv f_k - 1)$ and h_k, $k = 1, 2$, can be approximated [*Herrera and Yates*, 1977] by truncated expansions like

$$g_k(t/\bar{S}'t_a) \simeq g_k^{N_k}(t/\bar{S}'t_a)$$

$$= A_k^{N_k}\delta(t/\bar{S}'t_a)$$

$$+ \sum_{n=1}^{N_k} a_{kn} \exp(-\lambda_n^2 t) \qquad k = 1, 2 \tag{37a}$$

and

$$h_k(t/\bar{S}'t_a) \simeq h_k^{M_k}(t/\bar{S}'t_a)$$

$$= 1 + \sum_{n=1}^{M_k} b_{kn} \exp(-\lambda_n^2 t) \qquad k = 1, 2 \tag{37b}$$

In a similar way the last term of the right-hand side of (29) and (33) will be limited to N' terms.

To conclude this section, a few comments are offered about a practical implementation of what has been discussed above. At each node in the plane of the aquifers, a string of linear elements can be established in the vertical direction. If K denotes the number of nodes in that direction, one should have

$$K \geq \max(N_k, M_k, N') \qquad k = 1, 2 \tag{38}$$

Then at each node in the plane, one looks for v_k and ϕ_n of the form

$$v_1 = \sum_{i=1}^{K} v_{1i} u_i(z) + u_0(z) \tag{39a}$$

$$v_2 = \sum_{i=1}^{K} v_{2i} u_i(z) + u_{N+1}(z) \tag{39b}$$

$$\phi_n = \sum_{i=1}^{K} \phi_{ni} u_i(z) \tag{39c}$$

where the u_i are given piecewise polynomial basis functions, whose support is a small patch of elements. Typically, u_i would be associated with node i such that $z = z_i$, with $z_0 = 0$ and $z_{K+1} = b'$, and, moreover, $u_i(z_j) = \delta_{ij}$, so that the expressions (39) automatically take into account the boundary conditions on the v_k (equation (9)) and on the ϕ_n (equation (5)). Standard Galerkin equations for the unknown vectors $\mathbf{v}_k = [v_{k1}, \cdots, v_{kK}]^T$ and $\phi_n = [\phi_{n1}, \cdots, \phi_{nK}]^T$ read

$$-K\mathbf{v}_1 = \mathbf{f}_1 \tag{40a}$$

$$-K\mathbf{v}_2 = \mathbf{f}_2 \tag{40b}$$

$$(-K + \lambda_n^2 M)\phi_n = 0 \tag{40c}$$

where $K = (k_{ij})$, $M = (m_{ij})$ are $K \times K$ matrices with elements

$$k_{ij} = \int_0^{b'} K' \frac{\partial u_i}{\partial z} \frac{\partial u_j}{\partial z} dz \tag{41a}$$

$$m_{ij} = \int_0^{b'} S_s' u_i u_j dz \tag{41b}$$

while $\mathbf{f}_1$ and $\mathbf{f}_2$ are vectors with K components f_{1i} and f_{2i} given by

$$f_{1i} = \int_0^{b'} K' \frac{\partial u_i}{\partial z} \frac{\partial u_0}{\partial z} dz \tag{42a}$$

$$f_{2i} = \int_0^{b'} K' \frac{\partial u_i}{\partial z} \frac{\partial u_{N+1}}{\partial z} dz \tag{42b}$$

The source problems (40*a*) and (40*b*) and the eigenproblem (40*c*) are quite standard (see, for instance, *Wilkinson and Reinsch* [1971]) and especially easy to solve, since they are one-dimensional ones. Moreover, they have to be solved once for all, since the coefficients are time-independent.

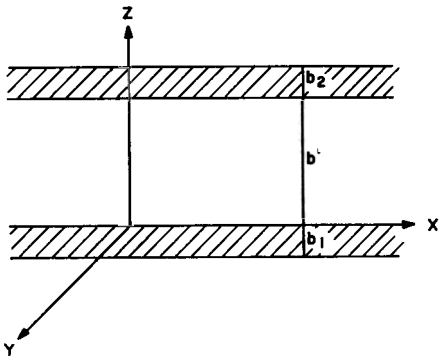

Fig. 1. The aquifer system.

3. NUMERICAL TREATMENT OF THE AQUIFER EQUATIONS

Coupled to the aquitard equation (1), the aquifer equations read

$$\frac{\partial}{\partial x}\left(T_1\frac{\partial s_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_1\frac{\partial s_1}{\partial y}\right)+K'\left.\frac{\partial s'}{\partial z}\right|_{z=0}=S_1\frac{\partial s_1}{\partial t}+Q_1 \qquad (43a)$$

and

$$\frac{\partial}{\partial x}\left(T_2\frac{\partial s_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_2\frac{\partial s_2}{\partial y}\right)-K'\left.\frac{\partial s'}{\partial z}\right|_{z=b'}=S_2\frac{\partial s_2}{\partial t}+Q_2 \qquad (43b)$$

in $\Omega x(0, T]$. They are, moreover, subject to boundary conditions

$$s_k(x, y, t)=0 \qquad k=1, 2 \qquad (44)$$

on $\Gamma x(0, T]$, where $\Gamma=\bar{\Omega}-\Omega$, and initial conditions taken, for instance, to be

$$s_k(x, y, 0)=s_{k0}(x, y) \qquad k=1, 2 \qquad (45)$$

in Ω. Here for the sake of simplicity the particular case of a two-aquifer system separated by one aquitard is considered (see Figure 1), with homogeneous Dirichlet boundary conditions. Extensions to multiaquifer systems with nonhomogeneous Dirichlet, Neumann, or mixed boundary conditions are straightforward.

Any standard finite element discretization of equations (43) would then lead to

$$s_k(x, y, t)\simeq\sum_{j=1}^{N}p_{kj}(t)u_j(x, y) \qquad k=1, 2 \qquad (46)$$

where the $u_j(x, y)$ are given piecewise polynomial basis functions, whose support is a small patch of elements. Using (46) for s_k in equations (43) and the standard Galerkin semidiscretization, after some algebraic manipulations we get

$$-(K_1+M)P_1-G_1+H_1=M_1\dot{P}_1+F_1 \qquad (47a)$$

$$-(K_2+M)P_2-G_2+H_2=M_2\dot{P}_2+F_2 \qquad (47b)$$

where $K_k=(k_{k,ij})$, $M=(m_{ij})$, and $M_k=(m_{k,ij})$ are $N\times N$ matrices with elements

$$k_{k,ij}=\int_\Omega T_k\nabla u_i^T\nabla u_j\,d\bar{r} \qquad (48a)$$

$$m_{ij}=\int_\Omega\frac{1}{t_a}u_iu_j\,d\bar{r} \qquad (48b)$$

$$m_{k,ij}=\int_\Omega S_ku_iu_j\,d\bar{r} \qquad (48c)$$

while $G_k=(g_{ki})$, $H_k=(h_{ki})$, and $F_k=(f_{ki})$ are N vectors with components

$$g_{ki}=\sum_{j=1}^{N}\int_\Omega u_i\frac{1}{t_a}\int_0^t g_k^{N_k}\left(\frac{t-\tau}{\bar{S}'t_a}\right)\dot{P}_{kj}(\tau)u_j\,d\bar{r}\,d\tau \qquad (49a)$$

$$h_{ki}=\sum_{j=1}^{N}\int_\Omega u_i\frac{1}{t_a}\int_0^t h_k^{M_k}\left(\frac{t-\tau}{\bar{S}'t_a}\right)\dot{P}_{kj}(\tau)u_j\,d\bar{r}\,d\tau \qquad (49b)$$

$$f_{ki}=\int_\Omega u_iQ_k+\int_\Omega u_i\frac{1}{t_a}s_k(0)g_k^{N_k}\left(\frac{t}{\bar{S}'t_a}\right)d\bar{r}-\int_\Omega u_i\frac{1}{t_a}s_{k^*}(0)h_k^{M_k}\left(\frac{t}{\bar{S}'t_a}\right)d\bar{r} \qquad (49c)$$

where $k^*=3-k$, $k=1, 2$. Clearly,

$$P_k=[p_{k1}, \cdots, p_{kN}]^T \qquad k=1, 2$$

In the original integrodifferential model [*Herrera and Yates*, 1977], homogeneity was assumed, and consequently, a convenient dimensionless time t' could be introduced which did not depend on position. Here $t/\bar{S}'t_a$ is a function of position, and the memory and influence terms become quite involved in the final Galerkin semidiscrete equations (47) unless, as we have found, reduced integration is used [*Zienkiewicz*, 1977, p. 537]. Under any numerical quadrature scheme consistent with the expected accuracy associated with the choice of the u_j in (46), the masslike matrix elements are approximated as follows:

$$\int_\Omega f(\bar{r}, t)u_i(\bar{r})u_j(\bar{r})\,d\bar{r}\simeq\sum_K w_kf(\bar{r}_k, t)u_i(\bar{r}_k)u_j(\bar{r}_k) \qquad (50)$$

where $\bar{r}_k$ is the sampling point where the integrand has to be evaluated and w_k is the corresponding weight. If the sampling points are taken to be the mesh nodes, as all except one basis function are zero there, the corresponding matrix becomes diagonal, namely,

$$\sum_K w_kf(\bar{r}_k, t)u_i(\bar{r}_k)u_j(\bar{r}_k)=w_if(\bar{r}_i, t)\delta_{ij} \qquad (51)$$

since $u_i(\bar{r}_j)=\delta_{ij}$. As a result, all the masslike matrices are diagonalized. This is true in particular for the memory and influence terms, and $t/\bar{S}'t_a$ is only needed at the mesh points, which is quite convenient. By careful programing, considerable reductions in computer time and memory requirement can be achieved, as we indeed verified. Also, when lumping by reduced integration is used, the resulting Galerkin semidiscrete equations are less stiff, and their time integration is therefore easier.

To be more specific, assume that lumping is being used, and define the following $N\times N$ diagonal matrices:

$$W=\text{diag}\,(w_i) \qquad (52a)$$

$$T=\text{diag}\,[t_a(\bar{r}_i)] \qquad (52b)$$

$$S_k=\text{diag}\,[S_k(\bar{r}_i)] \qquad k=1, 2 \qquad (52c)$$

$$S=\text{diag}\,[\bar{S}'(\bar{r}_i)] \qquad (52d)$$

$$A_k^*=\text{diag}\,[A_k^{N_k}(\bar{r}_i)] \qquad k=1, 2 \qquad (52e)$$

$$A_{kn}=\text{diag}\,[a_{kn}(\bar{r}_i)] \qquad k=1, 2 \qquad n=1, \cdots, N_k \qquad (52f)$$

$$B_{kn}=\text{diag}\,[b_{kn}(\bar{r}_i)] \qquad k=1, 2 \qquad n=0, \cdots, M_k \qquad (52g)$$

With these notations the general aquifer equations (47) become, after lumping,

$$W(S_1 + SA_1^*)\dot{P}_1 = -(K_1 + WT^{-1})P_1 - WT^{-1}\sum_{n=1}^{N_1} A_{1n}D_{1n} + WT^{-1}\sum_{n=0}^{M_1} B_{1n}D_{2n} - F_1 \quad (53a)$$

and

$$W(S_2 + SA_2^*)\dot{P}_2 = -(K_2 + WT^{-1})P_2 - WT^{-1}\sum_{n=1}^{N_2} A_{2n}D_{2n} + WT^{-1}\sum_{n=0}^{M_2} B_{2n}D_{1n} - F_2 \quad (53b)$$

where

$$D_{kn} = [d_{kn1}, \cdots, d_{knN}]^T \qquad k = 1, 2$$

with

$$d_{kni}(t) = \exp[-\lambda_n^2(\bar{r}_i)t]\int_0^t \exp[+\lambda_n^2(\bar{r}_i)\tau]p_{ki}(\tau)\,d\tau \quad (54)$$

Moreover, it was assumed that $b_{k0}(\bar{r}_i) = 1$ and $\lambda_0^2(\bar{r}_i) = 0$ to extend the last summations in equations (53) to $n = 0$.

Consequently, the final set of equations (53) exhibits the following general structure:

$$A_1\dot{P}_1 = -B_1P_1 - C\left(\sum_{n=1}^{N_1} A_{1n}D_{1n} - \sum_{n=0}^{M_1} B_{1n}D_{2n}\right) - F_1 \quad (55a)$$

$$A_2\dot{P}_2 = -B_2P_2 - C\left(\sum_{n=1}^{N_2} A_{2n}D_{2n} - \sum_{n=0}^{M_2} B_{2n}D_{2n}\right) - F_2 \quad (55b)$$

where the matrices A_k and C are diagonal, the matrices B_k are symmetric and banded, while the source vectors F_k directly derive from the extraction rates and the initial conditions (45) for the aquifers.

Once we have coped with the major difficulty due to heterogeneity in the aquifer treatment, namely, the approximation of the memory and influence terms, thanks to the reduced integration technique, the final equations (55) have essentially the same properties as in the previous integrodifferential model [*Herrera and Yates,* 1977; *Herrera et al.,* 1980]. For the sake of completeness these properties will be briefly recalled here: first, owing to the shape of the influence functions h_k, $k = 1, 2$, from one aquifer to the other, (55*a*) and (55*b*) may be regarded as uncoupled. Furthermore, the memory and influence terms can be evaluated at each time step by a simple updating procedure for the vectors D_{kn}, $k = 1, 2$, which does not depend upon a particular reference time and does not involve past history (see discussion below). Finally, the drawdown s' in the aquitard need not be calculated unless it is required at some time, in which case it can be evaluated simply in terms of the s_k and of auxiliary terms already present in the calculation.

In previous versions of this model [*Herrera and Yates,* 1977; *Yates and Herrera,* 1977] the Crank-Nicolson (CN) finite difference procedure was used to carry out the time integration of equations (55). This procedure is known to exhibit poor asymptotic stability properties; thus it has been improved by imbedding it in a wider class of time integration schemes which, besides the CN, includes forward Euler (FE) and backward Euler (BE) procedures. All these are θ schemes [*Douglas and Dupont,* 1970], although the point of view adopted here is in terms of finite elements in time, with an underlying exponential fitting capability. This allows greater generality and provides at the same time a consistent piecewise continuous behavior of the solution in space and time; thus the values of P at any point $(\bar{r}, t)$ can be retrieved from the nodal values. Recalling equations (55), it is clear that each of them has the following general structure:

$$A\dot{P} = -BP - CI(P) - F \quad (56)$$

where $I(P)$ stands for the integral terms depending on P. Integrating (56) from t_j to $t_{j+1} = t_j + h$, we get

$$A[P(t_{j+1}) - P(t_j)] = -B\int_{t_j}^{t_{j+1}} P\,dt - C\int_{t_j}^{t_{j+1}} I(P)\,dt - \int_{t_j}^{t_{j+1}} F\,dt \quad (57)$$

where we used the fact that matrices A, B, and C are time-independent. In most cases of practical interest, linear triangular or quadrilateral elements in space would be used to discretize the aquifers. Even if higher-order elements such as quadratic ones were used, the time integration schemes are not likely to exhibit orders higher than 2 as with the CN scheme. Consistent with this order of approximation for the time integration is the so-called constant parameter assumption, whereby all the possible time-dependent effects on the coefficients, boundary conditions, and source terms level are averaged over $[t_j, t_{j+1}]$. After the transients have sufficiently decayed, the solution approximately exhibits a single-mode exponential behavior, and it is therefore reasonable to assume that the components p_i, $i = 1, \cdots, N$ of P over $[t_j, t_{j+1}]$ are of the form

$$p_i(t) \simeq \tilde{p}_i(t) = a_{ji} + b_{ji}\exp(\mu^j t) \quad (58)$$

where μ^j is some real negative value, eventually different from one time step to the following one, which should ideally approximate the algebraically largest inverse characteristic time. Assumption (58) is certainly approximately correct locally, that is, when $\mu^j = \mu_i^j$, $i = 1, \cdots, N$. It should be quite accurate also enough time after any modifications occurred to the boundary conditions and (or) the pumping rates (since we assume the coefficients in the equations to be constant), provided of course that the resulting numerical scheme correctly damps out the fast transients. When this is the case, the time step can be considerably increased without any loss of accuracy. The above rationale for introducing exponentials in (58) should not be misleading and leave the reader with the impression that the functional form for the components p_i has been restricted too much. Actually, as we shall see below, the proposed schemes constitute a broader class of time integration schemes than are normally used and in particular include well-known schemes based on an implicit polynomial behavior of the p_i. Combining (58) and (57), we obtain the approximate scheme

$$A(P^{j+1} - P^j) = -B\int_t^{t_{j+1}} \tilde{P}\,dt - C\int_{t_j}^{t_{j+1}} I(\tilde{P})\,dt - \int_{t_j}^{t_{j+1}} F\,dt \quad (59)$$

where $\tilde{P} = [\tilde{p}_1(t), \cdots, \tilde{p}_N(t)]^T$ is completely defined by the interpolation conditions

$$\tilde{P}(t_j) = P^j \quad (60a)$$

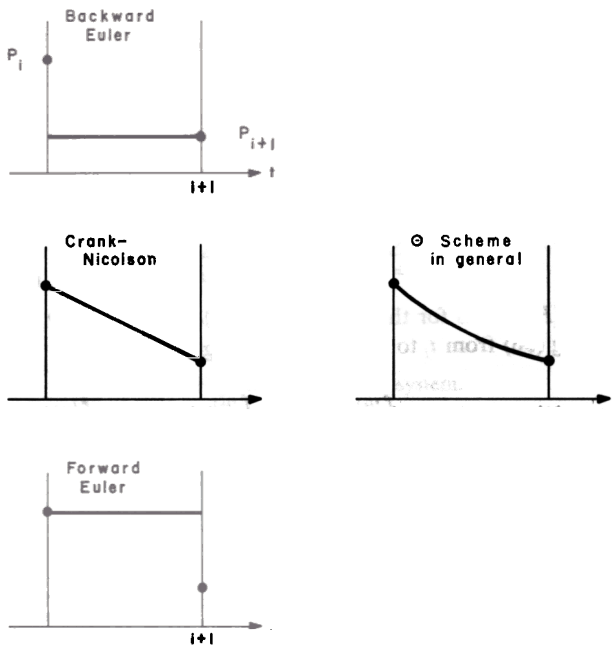

Fig. 2. Some time integration schemes and their interpolation properties.

$$\tilde{P}(t_{j+1}) = P^{j+1} \tag{60b}$$

so that $\tilde{P} = \tilde{P}(P^j, P^{j+1})$, namely,

$$\tilde{P} = P^j \frac{\exp z - \exp \mu(t - t_j)}{\exp z - 1} + P^{j+1} \frac{\exp \mu(t - t_j) - 1}{\exp z - 1} \tag{61}$$

with $z \equiv \mu h$. Equation (59) thus provides a one-step scheme for P^{j+1} knowing P^j, from integration over the previous time interval or from the initial conditions (45). In particular, we have

$$\int_{t_j}^{t_{j+1}} \tilde{P}\, dt = hP^j[1 - \theta(\mu, h)] + hP^{j+1}\theta(\mu, h) \tag{62}$$

with

$$\theta(\mu, h) = (\exp z - 1 - z)/z(\exp z - 1) \tag{63}$$

which relates this class of scheme to the classical θ schemes. Some differences, however, should be pointed out here: in the classical θ schemes, θ is normally fixed, with the consequence that unless $\theta = 0.5$, the resulting schemes are of first order only. Here θ actually depends on μ and h: with a given μ, as h tends to zero, $z = \mu h$ will also tend to zero, and θ to 0.5. It is easy to check that as a result the proposed schemes are always of second order, even when $\mu \neq 0$ [*Hennart and Gourgeon*, 1980]. Moreover, one can show that any μ satisfying $-\infty \leq \mu \leq 0$ provides an A-stable integration scheme. Among the particular cases of interest, let us point out the following ones:

1. As $\mu \to 0$, $\theta \to \frac{1}{2}$,

$$\tilde{p}_i(t) \to p_i^j \left(1 - \frac{t - t_j}{h}\right) + p_i^{j+1} \frac{t - t_j}{h} \tag{64a}$$

and

$$\int_{t_j}^{t_{j+1}} \tilde{P}\, dt \to h(P^j + P^{j+1})/2 \tag{64b}$$

corresponding to the CN scheme.

2. As $\mu \to -\infty$, $\theta \to 1$,

$$\tilde{p}_i(t) \to p_i^{j+1} \tag{65a}$$

and

$$\int_{t_j} \tilde{P}\, dt \to hP^{j+1}$$

corresponding to the BE scheme.

3. As $\mu \to +\infty$, $\theta \to 0$,

$$\tilde{p}_i(t) \to p_i^j$$

and

$$\int_{t_j}^{t_{j+1}} \tilde{P}\, dt \to hP^j \tag{66b}$$

corresponding to the FE scheme.

The interpolation properties of these schemes are sketched in Figure 2. If P is actually of the form $A + B \exp \mu t$, $P \equiv \tilde{P}$ and no approximation is introduced in the calculation of the second member of (59), provided of course that the integrals can be performed analytically or even numerically in a way consistent with the underlying assumption on P. (For more details, the readers are referred to *Hennart* [1979].)

In fact, one of the major virtues of the above approach is that it provides us with consistent rules to update the vectors D_{kn} defined in (54). Using (58), it is easy to verify that in the general case we have

$$d_{kni}(t_{j+1}) = \exp[-\lambda_n^2(\bar{r}_i)h] d_{kni}(t_j) + (p_i^{j+1} - p_i^j) \frac{\mu'}{\lambda_n^2(\bar{r}_i) + \mu'} \frac{1}{\exp(\mu' h) - 1} [\exp(\mu' h) - \exp[-\lambda_n^2(\bar{r}_i)h]] \tag{67}$$

from which it is easy to derive particular updating formulas consistent with the CN scheme,

$$d_{kni}(t_{j+1}) = \exp[-\lambda_n^2(\bar{r}_i)h] d_{kni}(t_j) + (p_i^{j+} - p_i^j) \frac{1 - \exp[-\lambda_n^2(\bar{r}_i)h]}{\lambda_n^2(\bar{r}_i)h} \tag{68}$$

and with the BE scheme,

$$d_{kni}(t_{j+1}) = \exp[-\lambda_n^2(\bar{r}_i)h][d_{kni}(t_j) + p_i^{j+1} - p_i^j] \tag{69}$$

As the contributions of the D_{kn} to the integral terms on the right-hand side of (57) are of the form

$$\int_{t_j}^{t_{j+1}} D_{kn}(t)\, dt \tag{70}$$

a general expression for the ith component is obtained by using (58) in (54). It reads

$$\int_{t_j}^{t_{j+1}} d_{kni}(t)\, dt = h d_{kni}(t_j) \frac{1 - \exp[-\lambda_n^2(\bar{r}_i)h]}{h\lambda_n^2(\bar{r}_i)} + h(p_i^{j+1} - p_i^j) \left[\frac{1}{h(\lambda_n^2(\bar{r}_i) + \mu')} + \frac{\mu'}{\lambda_n^2(\bar{r}_i) + \mu'} \cdot \frac{1}{\lambda_n^2(\bar{r}_i)h} \cdot \frac{\exp[-\lambda_n^2(\bar{r}_i)h] - 1}{\exp(-\mu' j) -}\right] \tag{71}$$

from which again particular expressions may be derived for the CN and BE schemes.

Here it should be pointed out that the physical decoupling mentioned above and due to the shape of the influence functions implies that the second terms in (71) is dropped in the corresponding contributions to the right-hand side of (57).

APPENDIX A

In many practical situations the past history of the drawdowns may be known from, say, $t = t_{-1}$ (often taken to be $-\infty$) to $t = t_0$. In that case, (22) may be rewritten as

$$d_{kn}(t_0) = \exp\left[-\lambda_n^2(t_0 - t_{-1})\right] d_{kn}(t_{-1}) + \int_{t_{-1}}^{t_0} \exp\left[-\lambda_n^2(t_0 - \tau)\right] \frac{\partial s_k(\tau)}{\partial t}\, d\tau \qquad \text{(A1)}$$

At $t = t_{-1}$ the drawdowns are normally assumed to be zero, so that a final expression for $d_{kn}(t_0)$ is

$$d_{kn}(t_0) = \int_{t_{-1}}^{t_0} \exp\left[-\lambda_n^2(t_0 - \tau)\right] \frac{\partial s_k(\tau)}{\partial t}\, d\tau \qquad \text{(A2)}$$

where $\partial s_k(\tau)/\partial t$ is known from the past history of the drawdown in the aquifers. The drawdown in the aquitard at $t = t_0$ is then reconstructed from (24).

APPENDIX B

Useful expressions for $\phi_n'(0)$ and $\phi_n'(b')$ can be obtained without loss of accuracy as follows. Multiply (4) by v_k and (8) by ϕ_n to obtain equations (4′) and (8′), respectively. Then integrate (4′) − (8′) from 0 to b' to get, after integration by parts,

$$\left[v_k K' \phi_n'\right]_0^{b'} + \lambda_n^2 \int_0^{b'} S_s' v_k \phi_n\, dz = 0 \qquad \text{(B1)}$$

where

$$\int_0^{b'} S_s' v_k \phi_n\, dz = b_{kn} \qquad \text{(B2)}$$

By choosing $k = 1, 2$, one gets

$$K'(0)\phi_n'(0) = \lambda_n^2 b_{1n}$$

and

$$K'(b')\phi_n'(b') = -\lambda_n^2 b_{2n} \qquad \text{(B4)}$$

REFERENCES

Chen, B., and I. Herrera, Numerical treatment of leaky aquifers in the short time range, paper presented at John Ferris Symposium on Groundwater Hydraulics, AGU Spring Meeting, Baltimore, Md., 1981. (Available as *Comun. Téc., Ser. NA,* Inst. de Invest. en Mat. Apl. y en Sist., Univ. Nac. Autón. de México, Mexico City, 1981.)

Chorley, D. W., and E. O. Frind, An iterative quasi-three-dimensional finite element model for heterogeneous multiaquifer systems, *Water Resour. Res., 14,* 943–952, 1978.

Douglas, J., Jr., and T. Dupont, Galerkin methods for parabolic equations, *SIAM J. Numer. Anal., 7,* 575–626, 1970.

Hennart, J. P., Multiple collocation finite elements in time for parabolic evolution problems, in *The Mathematics of Finite Elements and Applications III,* edited by J. R. Whiteman, pp. 271–278, Academic, New York, 1979.

Hennart, J. P., and H. Gourgeon, One-step exponentially fitted piecewise continuous methods for initial value problems, *Comun. Téc., Ser. NA,* vol. 263, Inst. de Invest. en Mat. Apl. y en Sist., Univ. Nac. Autón. de México, Mexico City, 1980.

Herrera, I., and L. Rodarte, Integrodifferential equations for systems of leaky aquifers and applications, 1, The nature of approximate theories, *Water Resour. Res., 9*(4), 995–1005, 1973.

Herrera, I., and R. Yates, Integrodifferential equations for systems of leaky aquifers, 3, A numerical method of unlimited applicability, *Water Resour. Res., 13*(4), 725–732, 1977.

Herrera, I., J. P. Hennart, and R. Yates, A critical discussion of numerical models for multiaquifer systems, *Adv. Water Resour., 3,* 159–163, 1980.

Wilkinson, J. H., and C. Reinsch, *Linear Algebra,* Springer-Verlag, New York, 1971.

Yates, R., and I. Herrera, Extension of the Crank-Nicholson procedure to a class of integrodifferential equations, in *Proceedings of the Symposium on Applications to Computer Methods in Engineering,* vol. 1, pp. 377–384, University of Southern California, Los Angeles, 1977.

Zienkiewicz, O. C., *The Finite Element Method,* 3rd. ed., McGraw-Hill, New York, 1977.

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