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Chapter 19

Boundary Methods for Fluids

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19.1 INTRODUCTION

In recent years it is usually understood that a boundary method is a numerical procedure in which a subregion or the entire region is left out of the numerical treatment by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problem, leading to considerable economy in the numerical work. They constitute a very convenient manner of treating adequately unbounded regions by numerical means. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced (1, 2).

There are two main approaches for the formulation of boundary methods; one is based on the use of boundary integral equations and the other on the use of complete systems of solutions. In numerical applications, the first of these methods has received most attention (3). This is in spite of the fact that the use of complete systems of solutions presents important numerical advantages; for example, it avoids the introduction of singular integral equations and it does not require the construction of a fundamental solution. The latter is especially relevant in connection with complicated problems, for which it may be extremely laborious to build up a fundamental solution. This is illustrated by the fact that there are methods for synthetizing fundamental solutions starting from plane waves, which can be shown to be a complete system (4).

One may advance some possible explanations for this situation. Although the principle of superposition is a standard procedure for building up solutions of linear equations, many of its applications have been based on the method of separation of variables. This has led to the frequent but false belief that complete systems of solutions must be constructed specifically for a given region. Of course, this is not the case; as will be seen later, most frequently the completeness of a system of solutions is independent of the detailed shape of the region considered. In Tables 19.1 and 19.2 of section 19.2 we exhibit systems which are complete for any bounded region and others possessing the same property in the exterior of any bounded region.

Also, in some fields of application procedures which constitute particular cases of the approximation by complete systems of solutions have presented severe restrictions and inconveniences. A survey of such difficulties for the case of acoustics and electromagnetic field computations was carried out by Bates (5). For these kinds of studies the so-called 'Rayleigh hypothesis' restricts drastically the applicability of the method. However, work by Millar (6) implies that these difficulties are due mainly to lack of clarity, since he avoided Rayleigh hypothesis altogether by adopting a different point of view. Work by other authors has similar implications (7).

Motivated by this situation the author started a systematic research of the subject (8–15). The aim of the study has been twofold; first, to clarify the theoretical foundations required for using complete systems of solutions in a reliable way, and second, to expand the versatility of such methods, making them applicable to any problem which is governed by partial differential equations that are linear.

The aims of this research have been satisfactorily achieved, to a large extent, and this chapter is based on the results. Preliminary reports have already appeared (16–17), and a more complete study will soon be published (18). The task has been facilitated by progress that has been made in the understanding of partial differential equations (19). The methodology presented here also owes much to the work by Amerio, Fichera, Picone, Kupradze, and Trefftz (20–24). The systematic development of the procedures, in a way which is applicable to any linear problem, was made possible, however, by an abstract theory that has been developed by the author in a sequence of papers (8–13, 25–27). A summary of this theory is given in reference 27, in which the emphasis is placed in applications to variational principles. A more complete summary has appeared (28).

Taking into account that the application of complete systems of solutions in arbitrary regions is a relatively unknown method in which many readers probably lack experience, we prefer to start with an example, to be explained in sections 19.2 and 19.3, leaving for section 19.4 the explanation of the scope of the theory, when it will probably be more meaningful to the reader.

19.2 FOUNDATIONS OF THE METHOD

We first consider a simple example. Take the Laplace equation in a bounded region R, illustrated in Figure 19.1, and subjected to conditions of Dirichlet type on its boundary ∂R. For definiteness, assume $u \in H^{s+1/2}(R)$, where the standard notation for Sobolev spaces is used (19).

Let us denote by $N^{s+1/2}(R) \subset H^{s+1/2}(R)$ the subspace of harmonic functions in R, which belong to $H^{s+1/2}(R)$; i.e. a function u belongs to $N^{s+1/2}(R)$

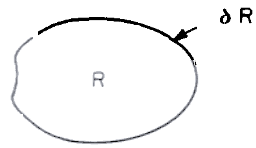

Figure 19.1 The region considered

if and only if $u \in H^{s+1/2}(R)$ and

$$\Delta u = 0, \qquad \text{on } R \tag{19.2.1}$$

If $\mathscr{B} = \{w_1, w_2, \ldots\} \subset N^{s+1/2}(R)$ is a system of harmonic functions which spans $N^{s+1/2}(R)$, then there is a sequence of approximations

$$u^N = \sum_{n=1}^{N} a_n^N w_m; \qquad N = 1, 2, \ldots \tag{19.2.2}$$

such that

$$u^N \rightarrow u \qquad \text{in } N^{s+1/2}(R) \tag{19.2.3}$$

Note that a_n^N depends on the number of terms N, of the approximation. This is essential in order for Equations (19.2.2) and (19.2.3) to hold. When a_n^N is independent of N, Equation (19.2.2) becomes a series, and the approximation by a series can only be granted when the system of functions $\mathscr{B}$ is orthogonal. This fact explains some of the difficulties that were encountered in applications to electromagnetic field studies (5).

In order for Equation (19.2.2) to be useful it will be required to have a procedure for deriving the coefficients a_n^N from boundary data only. This is indeed possible. General results on the existence and continuity properties of solutions of elliptic equations (19) show that when $u \in H^{s+1/2}(R)$ then $u \in H^s(\partial R)$ while $(\partial u/\partial n) \in H^{s-1}(\partial R)$. If the coefficients a_n^N are chosen so that

$$u^N \rightarrow u \qquad \text{on } H^s(\partial R) \tag{19.2.4}$$

then Equation (19.2.3) necessarily holds. Similarly

$$\frac{\partial u^N}{\partial n} \rightarrow \frac{\partial u}{\partial n} \qquad \text{on } H^{s-1}(\partial R) \tag{19.2.5}$$

also imply Equation (19.2.3), except for a constant function on R. Application of Equation (19.2.4) allows solving a Dirichlet problem, while Equation (19.2.5) permits solving a Neuman problem.

If $\mathscr{B} = \{w_1, w_2, \ldots\}$ spans $N^{s+1/2}(R)$, then the continuity properties of elliptic equations imply that

$$\{w_1, w_2, \ \ .\} \text{ spans } N_1^s(\partial R) = H^s(\partial R) \tag{19.2.6a}$$

while

$$\left\{\frac{\partial w_1}{\partial n}, \frac{\partial w_2}{\partial n} \qquad \right\} \text{spans } N_2^{s-1}(\partial R) \subset H^{s-1}(\partial R) \tag{19.2.6b}$$

Here $N_1^S(\partial R)$ represents the range of boundary values that harmonic functions on R take on ∂R, while $N_2^{s-1}(\partial R)$ is the corresponding range of normal derivatives. As is well known, the latter is the subspace of functions with vanishing integral on the boundary ∂R.

These observations show that the coefficients a_n^N can be chosen so that Equation (19.2.4) holds; alternatively, they can be chosen so that Equation (19.2.5) holds. Clearly, Equation (19.2.4) holds if u^N is taken as the projection of the boundary values $u \in H^s(\partial R)$ on the subspace spanned by $\{w_1, \ldots, w_N\}$. On the other hand, Equation (19.2.5) holds if $\partial n^N/\partial n$ is the projection of the boundary values $(\partial u/\partial n) \in H^{s-1}(\partial R)$ on the subspace spanned by $\{(\partial w_1/\partial n), \ldots, (\partial w_N/\partial n)\}$. Therefore in both cases the coefficients can be computed by the standard procedure for projecting on a subspace.

Note that in the first case the projections are taken in the sense of the inner product associated with $H^s(\partial R)$, and in the second, it is associated with $H^{s-1}(\partial R)$. Numerically, it is simpler to use only $\mathscr{L}^2(\partial R) = H^0(\partial R)$ inner products.

This can be done if it is assumed that

$$\{w_1, w_2, \ldots\} \text{ spans } N_1^0(\partial R) = H^0(\partial R) \tag{19.2.7a}$$

$$\left\{\frac{\partial w_1}{\partial n}, \frac{\partial w_2}{\partial n}, \ldots\right\} \text{spans } N_2^0(\partial R) = \{1\}^\perp \subset H^0(\partial R) \tag{19.2.7b}$$

In Equation (19.2.7b) the orthogonal complement of the constant function 1 has been taken with respect to Hilbert space $H^0(\partial R)$.

Conditions (19.2.7) will be satisfied if and only if

$$\mathscr{B} = \{w_1, w_2, \ldots\} \subset N^{3/2}(R) \text{ spans } N^{3/2}(R) \tag{19.2.8}$$

In fact conditions (19.2.7) are granted whenever $\mathscr{B}$ spans $N^{s+1/2}(R)$ with $s \geq 1$, but choice (19.2.8) is optimal in the sense that it corresponds to the least s that can be taken, granting Equations (19.2.7).

There is an alternative method of imposing condition (19.2.8). Let $D_R \subset H^{1/2}(R)$ be the linear subspace† with the property that for every $u \in D_R$ the boundary values satisfy $u \in H^0(\partial R)$, while $(\partial u/\partial n) \in H^0(\partial R)$. Define for every $u \in D_R$ and $v \in D_R$ the bilinear functional

$$\langle A_R u, v\rangle = \int_{\partial R} \left\{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right\} \mathrm{d}x \tag{19.2.9}$$

† The linear subspace D_R, so defined, is not closed.

Let $I_R \subset D_R$ be the linear subspace of D_R with the property that $v \in I_R$ when there is a harmonic function $w \in D_R$ such that

$$v = w; \qquad \text{on } \partial R \tag{19.2.10a}$$

and

$$\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} \qquad \text{on } \partial R \tag{19.2.10b}$$

We should note another fact. Let us define $N(R) \subset D_R$ as the linear subspace of harmonic functions that belong to D_R. From the fact that $(\partial u/\partial n) \in H^0(\partial R)$ it is easy to see that $N(R) = N^{3/2}(R)$, which is a closed space in the metric of $H^{3/2}(R)$.

When a system of functions $\mathcal{B} \subset N(R) \subset D_R$ is given, results that have been reported recently (13, 28) can be used to show that conditions (19.2.7) hold if and only if for every $u \in D_R$ one has

$$\langle Au, w_\alpha \rangle = 0 \qquad \forall\, \alpha = 1, 2, \ldots \Rightarrow u \in I_R \tag{19.2.11}$$

When Equation (19.2.11) is satisfied, the system $\{w_1, w_2, \ldots\} \subset N(R)$ is said to be c-complete.† Thus the system $\mathcal{B}$ is c-complete if and only if Equations (19.2.7a) and (19.2.7b) hold simultaneously. But since Equations (19.2.7) and (19.2.8) are equivalent, we can summarize our results as follows.

Given a system of functions $\mathcal{B} = \{w_1, w_2, \ldots\} \subset N(R)$, satisfying Equation (19.2.1), the following statements are equivalent.

(1) $\mathcal{B}$ is c-complete;
(2) $\mathcal{B}$ spans $N(R) = N^{3/2}(R) \subset H^{3/2}(R) \subset D_R$;
(3) The boundary values $\{w_1, w_2, \ldots\}$ span $H^0(\partial R)$ and simultaneously $\{(\partial w_1/\partial n), (\partial w_2/\partial n), \ldots\}$ span $\{1\}^\perp \subset H^0(\partial R)$.

Note, finally, that $\mathfrak{B}$ spans $N^{1/2}(R)$ whenever $\mathfrak{B}$ spans $N^{3/2}(R)$, which can easily be verified.

Assume that we look for a function $u \in H^{1/2}(R)$ which satisfies Equation (19.2.1) and is subjected to the boundary condition

$$u = f_{\partial R}, \qquad \text{on } \partial R \tag{19.2.12}$$

for some given function $f_{\partial R} \in H^0(\partial R)$. If a c-complete system $\mathcal{B} = \{w_1, w_2, \ldots\} \subset N^{3/2}(R)$ is given, then the desired approximating sequence (19.2.2), can be constructed so that on the boundary

$$u^N \to u, \qquad \text{on } H^0(\partial R) \tag{19.2.13}$$

The least-square condition on $H^0(\partial R)$ leads to the system of equations

$$\sum_{n=1}^{N} M_{nm} a_n^N = c_m \tag{19.2.14}$$

† This concept has some relation with ideas put forward by Trefftz (24).

where

$$M_{nm} = \int_{\partial R} w_n \bar{w}_m \, dx \qquad 19.2.15a)$$

and

$$c_m = \int_{\partial R} f_{\partial R} \bar{w}_m \, d\mathbf{x} \qquad (19.2.15b)$$

Here the bar refers to the complex conjugate. With this choice, $u^N \to u$ in $H^{1/2}(R)$ (19).

Similarly, for Neuman problems Equation (19.2.12) is replaced by

$$\frac{\partial u}{\partial n} = g_{\partial R}; \qquad \text{on } \partial R \qquad (19.2.16)$$

where the boundary values $g_{\partial R} \in \{1\}^{\perp} \subset H^0(\partial R)$. The previous argument still holds if Equations (19.2.15) are replaced by

$$M_{nm} = \int_{\partial R} \frac{\partial w_m}{\partial n} \frac{\partial \bar{w}_m}{\partial n} \, d\mathbf{x} \qquad (19.2.17a)$$

and

$$c_m = \int_{\partial R} g_{\partial R} \frac{\overline{\partial w_m}}{\partial n} \, d\mathbf{x} \qquad (19.2.17b)$$

In this case $u^N \to u$ in $H^{3/2}(R)$; therefore also $u^N \to u$ in $H^{1/2}(R)$ (19).

Computation of the boundary values when they are required may need a special method. In general, if the normal derivative $(\partial u^N/\partial n) \to g_{\partial R}$ in $H^0(\partial R)$, then $u^N \to u$ in $H^{3/2}(R)$. Hence on the boundary $u^N \to u$ in $H^1(R)$, which implies $u^N \to u$ in $H^0(\partial R)$. Thus in the case of Neuman problems, the unknown boundary values can be derived directly from the approximating sequence.

However, for Dirichlet problems the unknown normal derivatives cannot be derived from the approximating sequence u^N, because from $u^N \to f_{\partial R}$ on $H^0(\partial R)$ one can only grant that $(\partial u^N/\partial n) \to (\partial u/\partial n)$ in $H^{-1}(\partial R)$ (19). Thus $\partial u^N/\partial n$ diverges in $H^0(\partial R)$ in general. If the boundary data are sufficiently smooth, i.e. if $f_{\partial R} \in H^1(\partial R)$, it is known that $(\partial u/\partial n) \in H^0(\partial R)$ (19) and the following approximating sequence can be used:

$$\sum_{n=1}^{N} b_n^N w_n \to \frac{\partial u}{\partial n}, \qquad \text{in } H^0(\partial R) \qquad (19.2.18)$$

where the coefficients can be obtained from the system of equations†

$$\sum_{n=1}^{N} K_{nm} b_n^N = d_m \qquad (19.2.19)$$

† This procedure is based on a relationship derived by Italian mathematicians in the 1940s (29).

Here

$$K_{nm} = \int_{\partial R} w_n \bar{w}_m \, dx \qquad (19.2.20a)$$

where

$$d_m = \int_{\partial R} \frac{\partial u}{\partial n} \bar{w}_m \, d\mathbf{x} = \int_{\partial R} f_{\partial R} \frac{\overline{\partial w_m}}{\partial n} d\mathbf{x} \qquad (19.2.20b)$$

We recall that in Equation (19.2.18) the boundary values of the basic system of functions have been used to approximate the desired normal derivatives on the boundary. This is possible due to the reciprocity relation

$$\int_{\partial R} v \frac{\partial u}{\partial n} d\mathbf{x} = \int_{\partial R} u \frac{\partial v}{\partial n} d\mathbf{x} \qquad (19.2.21)$$

being satisfied by any pair of harmonic functions u and v (29).

Herrera and Sabina (14) have given the following c-complete systems.†

In Tables 19.1 and 19.2 $J_n(r)$ and $H_n^{(1)}(r)$ are the Bessel and Hankel functions of the first class (30, 31). P_n^q is the associated Legendre function while j_n and h_n^1 are the spherical Bessel and Hankel functions (30). We recall, in addition, that the c-complete systems given in Tables 19.1 and 19.2 for the Laplace equation in a bounded region are harmonic polynomials expressed in polar and spherical coordinates. Of special interest is the

† Some of the systems given here have been used in electromagnetic field computations

Table 19.1 c-complete systems in two dimensions

Bounded R	R = exterior of a bounded region
Laplace equation	
$\{1, r^n \cos n\theta, r^n \sin n\theta\}$	$\{\ln r, r^{-n} \cos n\theta, r^{-n} \sin n\theta\}$
Reduced wave equation $\Delta u + u = 0$	
$\{J_0(r), J_n(r) \cos n\theta, J_n(r) \sin n\theta\}$	$\{H_0^{(1)}(r), H_n^{(1)}(r) \cos n\theta, H^{(1)}(r) \sin n\theta\}$
$n = 1, 2, \ldots$	

Table 19.2 c-complete systems in three dimensions

Bounded R	R = exterior of a bounded region
Laplace equation	
$\{r^n P_n^q(\cos\theta) e^{iq\phi}\}$	$\{r^{-n-1} P_n^q(\cos\theta) e^{iq\phi}\}$
Reduced wave equation	
$\{j_n(r) P_n(\cos\theta) e^{iq\phi}\}$	$\{h_n^{(1)}(r) P_n^q(\cos\theta) e^{iq\phi}\}$
$n = 0, 1, 2, \ldots; -n \leq q \leq n$	

fact that these systems are c-complete independently of the particular region considered as long as the condition of being bounded or, alternatively, of being the exterior of a bounded region is fulfilled. This is a useful property that permits treating problems which satisfy a floating boundary condition. As an illustration a problem of seepage was previously solved by this method (16, 17).

For the reduced wave equation the author has exhibited systems of plane waves which are c-complete (4). For Stokes problems and the biharmonic equation they can be derived from corresponding systems for the Laplace equation (32, 33).

19.3 EXTENSION TO PROBLEMS WITH PRESCRIBED JUMPS

In many applications one has to deal with problems in which not only boundary conditions are prescribed but also jump conditions across surfaces on which discontinuities occur. For the Laplace equation, for example, one prescribes the jump of the function and its normal derivative; this corresponds to prescribing the jumps in the pressure (or piezometric head) and fluid velocity in problems of flow through porous media or potential flow.

This class of problems has been formulated systematically by the author (9, 11, 12, 27). The procedure is introduced here by means of an example. Consider the regions R and E illustrated in Figure 19.2. Take $D_R \subset H^{1/2}(R)$ and $N(R) = N^{3/2}(R)$ as in section 19.2. The definition of the linear subspace D_E becomes more involved when E is unbounded. Such technical difficulties can be avoided altogether if attention is restricted to boundary values. The details of such procedure have been given previously (4). For simplicity, here we illustrate the case when both R and E are bounded, although the results are valid generally when either R or E, or both, are unbounded.

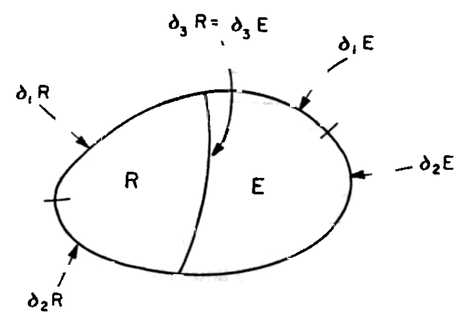

Figure 19.2 The region for the problem of matching

The problem we propose consists in finding a harmonic potential u in the region $R \cup E$, such that

$$u = f_1, \qquad \text{on } \partial_1(R \cup E) = \partial_1 R \cup \partial_1 E \tag{19.3.1a}$$

$$\frac{\partial u}{\partial n} = f_2, \qquad \text{on } \partial_2(R \cup E) = \partial_2 R \cup \partial_2 E \tag{19.3.1b}$$

while

$$[u] = j_1 \qquad \left[\frac{\partial u}{\partial n}\right] = j_2, \qquad \text{on } \partial_3 R = \partial_3 E \tag{19.3.2}$$

Here, f_1, f_2, j_1, and j_2 are prescribed functions, while [ ] represents the jump discontinuity across $\partial_3 R = \partial_3 E$.

In connection with this problem we consider pairs of functions $\{u_R, u_E\}$ such that $u_R \in D_R \subset H^{1/2}(R)$ and $u_E \in D_E \subset H^{1/2}(E)$; the linear space of such pairs will be denoted by $\hat{D} = D_R \oplus D_E$. Here, $D_E \subset H^{1/2}(E)$ is defined replacing R by E in the definition of D_R; a similar statement holds for the linear space $N(E) \subset D_E$ which is used in the following. The space $\hat{N} = N(R) \oplus N(E) \subset \hat{D}$ will be made by pairs $\{u_R, u_E\}$ such that $u_R \in N(R)$, while $u_E \in N(E)$. Note that these are harmonic functions on R and on E, separately. However, in general they are discontinuous across the common part of the boundary $\partial_3 R = \partial_3 E$. In a way similar to that in which I_R was introduced in section 19.2 the space $\hat{I}_P \subset \hat{D}$ will be defined. An element $\hat{u} = \{u_R, u_E\} \in \hat{D}$ belongs to $\hat{I}_P$ if and only if there exist harmonic functions $v_R \in N(R)$ and $v_E \in N(E)$ such that

$$u_R = v_R; \qquad \frac{\partial u_R}{\partial n} = \frac{\partial v_R}{\partial n} \qquad \text{on } \partial R \tag{19.3.3a}$$

and simultaneously

$$u_E = v_E; \qquad \frac{\partial u_E}{\partial n} = \frac{\partial v_E}{\partial n}, \qquad \text{on } \partial E \tag{19.3.3b}$$

Associated with every function $\hat{u} = \{u_R, u_E\} \in \hat{D}$, defined in $R \cup E$, there is a unique system of four functions

$$[u] \in H^0(\partial_3 R), \qquad \left[\frac{\partial u}{\partial n}\right] \in H^0(\partial_3 R), \qquad u \in H^0(\partial_1 R \cup E),$$
$$\frac{\partial u}{\partial n} \in H^0(\partial_2 R \cup E) \tag{19.3.4}$$

where the jump discontinuities are

$$[u] = u_E - u_R; \qquad \left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n}; \qquad \text{on } \partial_3 R \tag{19.3.5}$$

Here, for definiteness, it is assumed that the unit normal vector $\mathbf{n}$ is taken point-outwards from R in the common part of the boundary $\partial_3 R = \partial_3 E$.

In addition to the bilinear form A_R given by Equation 19.2.9) we introduce A_E by

$$\langle A_E u_E, v_E\rangle = \int_{\partial E} \left\{ v_E \frac{\partial u_E}{\partial n} - u_E \frac{\partial v_E}{\partial n} \right\} d\mathbf{x} \tag{19.3.6}$$

Define

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle A_R u_R, v_R\rangle + \langle A_E u_E, v_E\rangle \tag{19.3.7}$$

Then

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle \hat{J}\hat{u}, \hat{v}\rangle - \langle \hat{J}\hat{v}, \hat{u}\rangle + \langle \hat{B}\hat{u}, \hat{v}\rangle - \langle \hat{B}\hat{v}, \hat{u}\rangle \tag{19.3.8}$$

Where the jump operator $\hat{J}$ (27) is given by

$$\langle \hat{J}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R} \left\{ [\hat{u}]\left(\frac{\partial v}{\partial n}\right)_a - (v)_a \left[\frac{\partial \hat{u}}{\partial n}\right] \right\} d\mathbf{x} \tag{19.3.9a}$$

while

$$\langle \hat{B}\hat{u}, \hat{v}\rangle = \int_{\partial_2(R\cup E)} v\frac{\partial u}{\partial n} d\mathbf{x} - \int_{\partial_1(R\cup E)} u\frac{\partial v}{\partial n} d\mathbf{x} \tag{19.3.9b}$$

In Equation (19.3.9a) we use the notation

$$(v)_a = (v_R + v_E)/2; \qquad \left(\frac{\partial v}{\partial n}\right)_a = \left(\frac{\partial v_R}{\partial n} + \frac{\partial v_E}{\partial n}\right)\Big/2 \tag{19.3.10}$$

A system $\hat{\mathscr{B}} = \{\hat{w}_1, \hat{w}_2, \ldots\} \subset \hat{N}$ is said to be c-complete for the problem with prescribed jumps (Equations (19.3.1) and (19.3.2)) when for every $\hat{u} \in \hat{D}$ one has

$$\langle \hat{A}\hat{u}, \hat{w}\rangle = 0 \qquad \forall\, \hat{w} \in \hat{\mathscr{B}} \Rightarrow \hat{u} \in \hat{I}_P \tag{19.3.11}$$

At this point it is convenient to recall that with every function $\hat{u} \in \hat{D}$ one can associate in addition to the four functions given by Equation (19.3.4) four other functions

$$\begin{aligned} &(u)_a \in H^0(\partial_3 R) && \left(\frac{\partial u}{\partial n}\right)_a \in H^0(\partial_3 R), \\ &\frac{\partial u}{\partial n} \in H^0(\partial_1 R \cup E), && u \in H^0(\partial_2 R \cup E) \end{aligned} \tag{19.3.12}$$

where the notation introduced by means of Equation (19.3.10) is used.

It can be shown that $\hat{\mathscr{B}} \subset \hat{N}$ is c-complete if and only if $\{w_{R1}, w_{R2}, \ldots\} \subset N(R)$ spans $N(R)$ while $\{w_{E1}, w_{E2}, \ldots\} \subset N(E)$ spans $N(E)$. This in turn is equivalent (13, 28) to the condition that the system of quadruplets

$$\left\{ [w_\alpha], \left[\frac{\partial w_\alpha}{\partial n}\right], w_\alpha, \frac{\partial w_\alpha}{\partial n} \right\}, \qquad \alpha = 1, 2, \tag{19.3.13a}$$

defined by Equation (19.3.4) spans $H^0(\partial_3 R) \oplus H^0(\partial_3 R) \oplus H^0(\partial_1 R \cup E) \oplus H^0(\partial_2 R \cup E)$, and, simultaneously, the system of quadruplets

$$\left\{(w_\alpha)_a, \left(\frac{\partial w_\alpha}{\partial n}\right)_a, \frac{\partial w_\alpha}{\partial n} \quad w_\alpha\right\} \qquad \alpha = \quad 2, \tag{19.3.13b}$$

defined by Equation (19.3.12) spans $H^0(\partial_3 R) \oplus H^0(\partial_3 R) \oplus H^0(\partial_1 R \cup E) \oplus H^0(\partial_2 R \cup E)$.

Now every one of the elements $\hat{w}_\alpha$ has two components, $w_{R\alpha}$ defined on R and $w_{E\alpha}$ defined on E. The pair $\hat{w}_{R\alpha} = \{w_{R\alpha}, 0\}$ corresponds to a function which is harmonic in R and identically zero on E. Clearly, $\hat{w}_{R\alpha} \in \hat{N}$ for every $\alpha = 1, 2, \ldots$; similarly, $\hat{w}_{E\alpha} = \{0, w_{E\alpha}\} \in \hat{N}$ for $\alpha = 1, 2, \ldots$. In view of our previous discussion it is not difficult to see that when the system of harmonic functions $\{w_{R1}, w_{R2}, \ldots\} \subset N(R)$ is c-complete on R and simultaneously $\{w_{E1}, w_{E2}, \ldots\} \subset N(E)$ is c-complete on E, then the system

$$\hat{\mathscr{B}} = \hat{\mathscr{B}}_R \cup \hat{\mathscr{B}}_E \subset \hat{N} \tag{19.3.14}$$

is c-complete for the problem with prescribed jumps. In Equation (19.3.14) the notation

$$\hat{\mathscr{B}}_R = \{\hat{w}_{R1}, \hat{w}_{R2}, \quad .\} \tag{19.3.15a}$$

$$\hat{\mathscr{B}}_E = \{\hat{w}_{E1}, \hat{w}_{E2}, \tag{19.3.15b}$$

was used.

When $\{\hat{w}_1, \hat{w}_2, . \quad \subset \hat{N}$ is c-complete it is possible to construct approximating sequences

$$\hat{u}_N = \sum_{n=1}^{N} a_n^N \hat{w}_n; \qquad N = 1, 2, \tag{19.3.16}$$

such that $\hat{u}^N \to \hat{u}$ in the metric of $H^{1/2}(R) \oplus H^{1/2}(E)$. In this case the coefficients a_n^N again satisfy the system of Equations (19.2.14) if Equations (19.2.15) are replaced by

$$M_{nm} = \int_{\partial_3 R} \left\{[w_n][\bar{w}_m] + \left[\frac{\partial w_n}{\partial n}\right]\left[\overline{\frac{\partial w_n}{\partial n}}\right]\right\} d\mathbf{x} + \int_{\partial_1 R \cup E} w_n \bar{w}_m \, d\mathbf{x} + \int_{\partial_2 R \cup E} \frac{\partial w_n}{\partial n}\frac{\partial \bar{w}_m}{\partial n} d\mathbf{x} \tag{19.3.17a}$$

and

$$c_m = \int_{\partial_3 R} \left\{j_1[\bar{w}_m] + j_2\left[\overline{\frac{\partial w_m}{\partial n}}\right]\right\} d\mathbf{x} + \int_{\partial_1 (R \cup E)} f_1 \bar{w}_m \, d\mathbf{x} + \int_{\partial_2 (R \cup E)} f_2 \frac{\overline{\partial w_n}}{\partial n} d\mathbf{x} \qquad .3.17b)$$

Comparing Equations (19.3.1) and (19.3.2) with Equation (19.3.4) it is seen that in the problem with prescribed jumps considered here the system of four boundary functions (Equation (19.3.4)) is given as problem data. In some applications it may be required to evaluate the complementary system (19.3.12), from which the values of the functions and normal derivatives at the discontinuity surface $\partial_3 R = \partial_3 E$ can be derived. When this is the case the approximation of $(\partial u/\partial n) \in H^0(\partial_1 R \cup E)$ and $(\partial u/\partial n)_a \in H^0(\partial_3 R)$ is not directly possible. However, the approximating sequences

$$\sum_{n=1}^{N} b_n^N[w_n] \to \left(\frac{\partial u}{\partial n}\right)_a, \qquad \text{in } H^0(\partial_3 R) \qquad (19.3.18a)$$

$$\sum_{n=1}^{N} b_n^N w_n \to \frac{\partial u}{\partial n}, \qquad \text{in } H^0(\partial_1 R \cup E) \qquad (19.3.18b)$$

can be used. Here the coefficients b_n^N satisfy Equation (19.2.19), with Equations (19.2.20) replaced by

$$\mathrm{d}_m = \int_{\partial_3 R} \left\{ -[\bar{w}_m]\left(\frac{\partial u}{\partial n}\right)_a + (u)_a \left[\overline{\frac{\partial w_m}{\partial n}}\right] \mathrm{d}\mathbf{x} \right.$$

$$\int_{\partial_2} \quad u \frac{\quad\cdots}{\partial n} \qquad\qquad \bar{w}_m \frac{\partial u}{\partial n} \mathrm{d}\mathbf{x}$$

$$\int_{\partial_3} \qquad\qquad \left.\right\} \mathrm{d}\mathbf{x}$$

$$f_1 \overline{\frac{\partial w_m}{\partial n}} \mathrm{d}\mathbf{x} \qquad (19.3.19)$$

19.4 SCOPE OF THE THEORY

In order to apply the procedure presented in sections 19.2 and 19.3 to any linear problem, it will be necessary to have available a system of solutions, $\mathcal{B}$, of the homogeneous equations in terms of which any other solution can be approximated. If the solution of the problem depends continuously on the boundary values (and this is an assumption satisfied in most cases of practical interest), this will be granted if the system $\mathcal{B}$ is such that any boundary values can be approximated by using it. As a first step it is necessary to define the space of boundary values in which we will work. It has numerical advantages to remain in $\mathcal{L}^2(\partial R)$ for the boundary values, i.e. in $H^0(\partial R)$, if the notation for Sobolev spaces is used. In this connection, we have seen by means of an example that c-complete systems possess precisely this property, and in what follows we introduce a generalization of this concept applicable to any linear equation. It can be shown that in general c-complete systems enjoy this property (13).

Once the boundary values are properly approximated the actual convergence of the approximating sequence (Equation 19.2.2) in the region depends on the space of functions in which the problem is formulated. Much progress has been made in the study of partial differential equations, the existence of their solutions, and their continuity properties (19); in the study of fluids the results by Temam (34) have special interest. Of course, there remain open questions but many equations which occur in fluid problems are well understood by now. However, for the application of this knowledge to problems of practical interest it is necessary to introduce formulations which are suitable for numerical applications, and the notion of a c-complete system seems to be quite adequate for this purpose.

Computation of boundary information, complementary to boundary data is, most frequently, difficult. This was illustrated in the example given previously and by the fact that the approximating sequence for the normal derivative does not converge in the boundary; this is a general phenomenon observed in many problems. Fortunately, the procedure for computing such boundary values, as explained earlier, possesses complete generality when it is properly formulated.

In the following sections the concepts and procedures presented in sections 19.2 and 19.3 are generalized to make them applicable to any linear problem governed by partial differential equations.

There is an additional point that must be taken into account in order to enhance the versatility of the methods described here. It is necessary to develop general techniques for constructing complete systems of solutions. Generally, the completeness of a system may depend not only on the partial differential equation or system of partial differential equations considered but on the region and type of boundary conditions for which the problem is formulated.

An additional advantage of using c-complete system is that they can be applied irrespectively of the boundary conditions considered. For example, the c-complete systems given in Tables 19.1 and 19.2 can be applied not only to Dirichlet or Neuman problems but also when one prescribes other boundary conditions, such as $\alpha(\partial u/\partial n)+\beta u$, or u on $\partial_1 R$ and $(\partial u/\partial n)$ on $\partial_2 R$, where $\partial_1 R$ and $\partial_2 R$ is a partition of the boundary ∂R.

There are many examples of systems of functions which are c-complete independently of the detailed shape of the region considered (14). It seems that in general,† when a system is c-complete in a region R it also has this

† Professor S. Antman informed the author that this result has actually been shown. Unfortunately, we have not been able to locate the result and the precise conditions under which this proposition is true.

property in any subregion of R. This is a very useful result that permits treating, by these methods, non-linear problems in which the non-linearity is due to boundary conditions (16, 17), such as a floating boundary.

The adequacy of a c-complete system for the treatment of a problem depends not only on the completeness property but also on other characteristics, such as the stability of the numerical schemes to which they lead (35), and this is an additional reason for the usefulness of having available procedures for deriving c-complete systems which can supply alternative systems. Unfortunately, we will not be able to deal with this subject in detail, because it would mean going beyond the scope of this chapter. Therefore it seems appropriate to mention here some of the procedures available. A method of considerable generality was apparently originated by Italian mathematicians (29), applied to a good sample of problems by Kupradze (23), and modified and extended by Herrera and Sabina (14). Separation of variables also yields quite general systems of c-complete functions; of especial interest in connection with boundary methods are the biorthogonal systems obtained in this way (36). However, frequently it is not difficult to develop them in an *ad hoc* way. For example, we have given a general procedure for generating c-complete systems for Stokes problems and the biharmonic equation (32, 33). A system of plane waves which is c-complete for the reduced wave equation has been given by Sanchez-Sesma *et al.* (4).

The methodology explained here owes much to the progress that has taken place in the general theory of partial differential equations (19, 37). The notion of c-completeness bears some relation to ideas first presented by Trefftz (24). The systematic development of these procedures in a way which is applicable to any linear problem was made possible, however, by an abstract theory which has been developed by the author in a sequence of publications (see the references at the end of the chapter) and recently summarized (27, 28).

The situations occurring in applications can be classified into three general groups. First, there is the group with boundary values on a region R (Figure 19.1) on which a c-complete system is known, such as the example discussed in section 19.2. The other two are associated with the case when the problem is formulated in a region $R \cup E$ (Figure 19.2), consisting of two subregions R and E, and a solution to a boundary value problem satisfying prescribed jumps (possibly zero) across the common boundary $\partial_3 R = \partial_3 E$ is required. We will refer to this problem as the problem of connecting or matching. In general, two situations can occur for the problem of connecting; on one hand, c-complete sytems may be known both on R and on E, so that only the boundary of the region and the connecting boundary between R and E have to be treated numerically. Another variant of the problem of connecting is the case when the c-complete system of solution is only known in R, so that the other subregion E has to be treated numerically. In this last case, it is required to have efficient procedures for matching the analytical solution available for R with the numerical solution on E (1, 2); a possible method of achieving this is by means of variational principles.

Formulation of variational principles has been discussed extensively (38, 39). However, the abstract theory mentioned above provides variational principles of complete generality which can be applied to any linear problem. We prefer, however, to refrain from discussing these topics in detail here; a recent summary can be found in reference (27), where the problem of matching solutions in a region R with solutions in a region E was treated abstractly (the problem of connecting), with generality applicable to any linear problem.

In section 19.5 a classification of boundary values is introduced. When a boundary value problem is formulated only one part of this boundary information is prescribed and the other part must be derived after the solution has been obtained. This way of breaking the boundary information is associated, in section 19.6, with canonical decompositions. Every canonical decomposition is in turn associated in a one-to-one way with operators that decompose an antisymmetric bilinear form characteristic of each problem. This is explained in section 19.7, and the notion of c-complete system is introduced in section 19.8. This concept is incorporated in the Hilbert-space formulation in section 19.9. Finally, section 19.10 derives the general representation of solutions and the corresponding approximation for the complementary boundary information. This procedure is also applicable in obtaining such information at surfaces of discontinuity by using the general formulation of the problem of connecting (27), mentioned above; however, the details are not given here.

19.5 BOUNDARY VALUES

In order to develop boundary methods along the lines explained in sections 19.2 and 19.3 which are applicable to a wide range of boundary value problems it is necessary to focus attention on some properties which occur in all these problems. In this section we introduce a notation which is sufficiently general for our purposes. In addition, we present a general procedure for defining what can be considered as the relevant boundary values for each of these problems.

We consider a bilinear functional P defined on an arbitrary linear space D. This will be denoted by $P: D \rightarrow D^*$ because it can be thought as an operator defined on the linear space D and taking values on its algebraic dual D^* (this is the space of linear functionals defined on D) (25). The value of such bilinear functional at elements $u \in D$ and $v \in D$ will be denoted by $\langle Pu, v\rangle$. The transposed bilinear functional of $P: D \rightarrow D^*$ will be denoted by

$P^*: D \to D^*$; thus

$$\langle P^*u, v\rangle = \langle Pv, u\rangle \tag{19.5.1}$$

The theory is applicable to general non-symmetric linear operators (28), although its application to formally symmetric ones is simpler because it does not require the introduction of the formal adjoint. Given an operator $P: D \to D^*$, we define the antisymmetric bilinear form

$$A = P - P^* \tag{19.5.2}$$

The operator A given by Equation (19.5.2) plays a central role in the theory. First, we are going to use it to introduce a classification of boundary values. For this purpose, we consider the null subspace N_A of A; i.e.

$$N_A = \{u \in D \mid Au = 0\} \tag{19.5.3}$$

With reference to the reduced wave equation

$$\Delta u + k^2 u = 0, \qquad \text{on } R \tag{19.5.4}$$

as an example (recall that Equation (19.2.1) corresponds to the case $k = 0$) consider the bilinear functional $P: D \to D^*$ given by

$$\langle Pu, v\rangle = \int_R v(\Delta u + k^2 u)\, \mathrm{d}\mathbf{x} \tag{19.5.5}$$

Then $A = P - P^*$ is

$$\langle Au, v\rangle = \int_{\partial R} \left\{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right\} \mathrm{d}\mathbf{x} \tag{19.5.6}$$

The null subspace N_P is the linear subspace of functions that satisfy Equation (19.5.4).

There are many ways of defining the linear space D. A convenient one is

$$D = \left\{u \in H^{1/2}(R) \;\middle|\; \frac{\partial u}{\partial n} \in H^0(\partial R)\right\} \tag{19.5.7}$$

This defines a linear subspace of $H^{1/2}(R)$, which is not closed in the topology of $H^{1/2}(R)$. We note that the null subspace N_P is well defined if Equation (19.5.4) is interpreted in the sense of distributions (19). Also well defined is the bilinear form $A: D \to D^*$, given by Equation (19.5.6); however, the operator $P: D \to D^*$, given by Equation (19.5.5), is not. In order to avoid going into too much detail we will not make any further reference to the operator P.

It is easy to see that

$$N_A = \left\{u \in D \mid u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial R\right\} \tag{19.5.8}$$

Due to Equation (19.5.8) the relevant boundary values for Laplace and reduced wave equations (19.5.4) will be u and $\partial u/\partial n$, on ∂R. We note that, given $u \in D$ and $v \in D$,

$$u = v \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} \qquad \text{on } \partial R \tag{19.5.9}$$

if and only if $u - v \in N_A$.

The notions that we have formulated can be applied to any linear differential equation. Let us consider the biharmonic equation

$$\Delta^2 u = 0; \qquad \text{on } R \tag{19.5.10}$$

which occurs in connection with incompressible flows at low Reynolds numbers. Define

$$\langle Pu, v \rangle = \int_R v \Delta^2 u \, d\mathbf{x} \tag{19.5.11}$$

Then

$$\langle Au, v \rangle = \int_{\partial R} \left\{ v \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n} \right\} d\mathbf{x} \tag{19.5.12}$$

Again, a convenient definition of the space D is

$$D = \left\{ u \in H^{1/2}(R) \,\middle|\, \frac{\partial u}{\partial n} \in H^0(\partial R), \Delta u \in H^0(\partial R), \frac{\partial \Delta u}{\partial n} \in H^0(\partial R) \right\} \tag{19.5.13}$$

Then, A as given by Equation (19.5.12) is well defined, and N_P can be taken as the linear subspace of D which satisfies Equation (19.5.10) in the sense of distributions. The operator $P: D \rightarrow D^*$, given Equation (19.5.11), is not defined for this space D, and we omit it from our discussion.

The null subspace N_A is

$$N_A = \left\{ u \in D \mid u = \frac{\partial u}{\partial n} = \Delta u = \frac{\partial \Delta u}{\partial n} = 0, \qquad \text{on } \partial R \right\} \tag{19.5.14}$$

The classification of boundary values induced by Equation (19.5.14) is characterized by quadruplets of functions u, $\partial u/\partial n$, Δu, $\partial \Delta u/\partial n$; recall that these functions yield enough information to determine u and its derivatives up to order 3.

The homogeneous stationary Stokes equations are

$$\nu \Delta \mathbf{u} - \nabla p = 0 \tag{19.5.15a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{19.5.15b}$$

where ν is the viscosity. In this case it is conveneient to define the bilinear form $P:D \to D^*$ by

$$\langle P\hat{u}, \hat{v}\rangle = \int_R \{\mathbf{v}\cdot(\nu\Delta\mathbf{u}-\nabla p)+q\nabla\cdot\mathbf{u}\}\,d\mathbf{x} \tag{19.5.16}$$

Here $\hat{u}$ represents a pair of functions; $\mathbf{u}$, which is vector-valued and defined in R, and p, which is scalar-valued and also defined in R. With $\hat{v}$ we have associated the pair $\mathbf{v}$, q. Then

$$\langle A\hat{u}, \hat{v}\rangle = \int_{\partial R} \left\{\mathbf{v}\cdot\left(\nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}\right)-\mathbf{u}\cdot\left(\nu\frac{\partial\mathbf{v}}{\partial n}-q\mathbf{n}\right)\right\}d\mathbf{x} \tag{19.5.17}$$

Let us denote by $\mathbf{H}^{1/2}(R)\oplus H^{-1/2}(R)$ the linear space of pairs of functions $[\mathbf{u}, p]$ with the property that $\mathbf{u}\in\mathbf{H}^{1/2}(R)$, while $p\in H^{-1/2}(R)$. Then a convenient choice for the linear space D is

$$D=\left\{\hat{u}\in\mathbf{H}^{1/2}(R)\oplus H^{-1/2}(R) \mid \nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}\in\mathbf{H}^0(\partial R)\right\} \tag{19.5.18}$$

Use of this linear space (which is not closed) grants that A is well defined. Then $N_P\subset D$ is taken as the set of functions that satisfy Stokes Equations (19.5.15) in the sense of distributions in R. Again, $P:D\to D^*$ is not defined, and it will be left out of our discussion. The null subspace

$$N_A=\left\{\hat{u}\in D \mid \mathbf{u}=\nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}=0, \quad \text{on } \partial R\right\} \tag{19.5.19}$$

The classification of boundary values induced by Equation (19.5.19) is characterized by the values of $\mathbf{u}$ and $\nu(\partial\mathbf{u}/\partial n)-p\mathbf{n}$ on the boundary ∂R.

19.6 BOUNDARY DATA AND DERIVED BOUNDARY INFORMATION

A systematic discussion of boundary values and boundary conditions will facilitate the application of boundary methods to many problems. This section is devoted to presenting the corresponding theory.

In the definitions that follow it is assumed that there is available an antisymmetric bilinear form $A:D\to D^*$.

A subspace $I\subset D$ is said to be regular for A when

(1) For every $u\in I$ and $v\in I$,

$$\langle Au, v\rangle = 0 \tag{19.6.1}$$

i.e. I is a commutative subspace for A.

(2) $$I\supset N_A \tag{19.6.2}$$

In section 19.5 we have seen that the null subspace N_A induces a classification of D which defines what could be properly called the boundary values which are relevant for the differential equation considered. In the light of this, condition (2) implies that a regular subspace is characterized by boundary values only.

To illustrate this, assume $I \subset D$ is a regular subspace. In connection with the examples given in section 19.5 let $u \in D$ and $v \in D$ be such that

$$u = v; \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n}, \qquad \text{on } \partial R \tag{19.6.3}$$

when the reduced wave equation is considered; or

$$u = v, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} \qquad \Delta u = \Delta v \qquad \frac{\partial \Delta u}{\partial n} = \frac{\partial \Delta v}{\partial n} \qquad \text{on } \partial R \tag{19.6.4}$$

for the biharmonic equation. Then there are only two mutually exclusive possibilities:

(a) u and v belong to I; or
(b) neither u nor v belongs to I.

A corresponding proposition holds for $\hat{u} \in D$ and $\hat{v} \in D$, in connection with Stokes equations, when it is assumed that

$$\mathbf{u} = \mathbf{v}; \qquad \nu \frac{\partial \mathbf{u}}{\partial n} - p\mathbf{n} = \nu \frac{\partial \mathbf{v}}{\partial n} - q\mathbf{n}; \qquad \text{on } \partial R \tag{19.6.5}$$

To summarize this discussion: a regular subspace is a commutative subspace which is defined through boundary values only.

Examples of regular subspaces for the reduced wave equation are

$$I_1 = \{u \in D \mid u = 0, \qquad \text{on } \partial R\} \tag{19.6.6a}$$

$$I_2 = \left\{u \in D \,\middle|\, \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial R\right\} \tag{19.6.6b}$$

and

$$I_3 = \left\{u \in D \mid \alpha \frac{\partial u}{\partial n} + \beta u = 0, \qquad \text{on } \partial R\right\} \tag{19.6.6c}$$

where $\alpha^2 + \beta^2 \neq 0$.

Many examples of regular subspaces can be given for the biharmonic equation; an interesting set of such subspaces is

$$I_1 = \left\{u \in D \mid u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial R\right\} \tag{19.6.7a}$$

$$I_2 = \left\{u \in D \mid u = \Delta u = 0, \qquad \text{on } \partial R\right\} \tag{19.6.7b}$$

and

$$I_3=\left\{u\in D\ \frac{\partial u}{\partial n}=\frac{\partial\Delta u}{\partial n}=0, \qquad \text{on } \partial R\right\} \tag{19.6.7c}$$

These are particular cases of the following, which is more general. Let a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 be two linearly independent quadruplets of real numbers such that

$$a_1b_4+a_3b_2=a_2b_3+a_4b_1 \tag{19.6.8}$$

Consider the restriction

$$a_1u+a_2\frac{\partial u}{\partial n}+a_3\Delta u+a_4\frac{\partial\Delta u}{\partial n}=b_1u+b_2\frac{\partial u}{\partial n}+b_3\Delta u+b_4\frac{\partial\Delta u}{\partial n}=0 \tag{19.6.9}$$

Then the subspace

$$I=\{u\in D\mid u \text{ satisfies Equation (19.6.9)}\} \tag{19.6.10}$$

is regular for A, as given by Equation (19.5.12).

To see that these subspaces are regular it is enough to verify that they satisfy conditions (1) and (2), with $A:D\to D^*$ given by Equation (19.5.12) and N_A by Equation (19.5.14).

For Stokes problem we have the following regular subspaces

$$I_1=\{\hat{u}\in D\mid \mathbf{u}=0, \qquad \text{on } \partial R\} \tag{19.6.11a}$$

$$I_2=\left\{\hat{u}\in D\mid \nu\frac{\partial u}{\partial n}-pn=0, \qquad \text{on } \partial R\right\} \tag{19.6.11b}$$

Of course, many more can be given.

Of special interest is the case when a regular subspace $I\subset D$ has the following additional property.

(3) For every $u\in D$

$$\langle Au, v\rangle=0 \qquad \forall\, v\in I\Rightarrow u\in I \tag{19.6.12}$$

A regular subspace which enjoys condition (3) is called completely regular.

It is not difficult to verify that in all the examples given in Equations (19.6.6)–(19.6.11) the regular subspaces are actually completely regular.

Given an antisymmetric bilinear form $A:D\to D^*$ we say that a pair of subspaces $\{I_1, I_2\}$ is a canonical decomposition of D for A when

(1) I_1 and I_2 are regular subspaces; and
(2) $D=I_1+I_2$ (19.6.13)

It has been shown (27, 28) that when $\{I_1, I_2\}$ is a canonical decomposition of D then I_1 and I_2 are necessarily completely regular and

$$N_A=I_1\cap I_2 \tag{19.6.14}$$

Now, condition (19.6.13) is equivalent to the requirement that, given any $u \in D$, one can find elements $u_1 \in I_1$ and $u_2 \in I_2$ such that

$$u = u_1 + u_2 \tag{19.6.15}$$

In the presence of Equation (19.6.14) this representation of u is unique except for elements of the subspace N_A; more precisely, if $u_1' \in I_1'$ and $u_2' \in I_2$ are such that

$$u = u_1' + u_2' \tag{19.6.16}$$

then $u_1 - u_1' \in N_A$ and $u_2 - u_2' \in N_A$. Taking into account that N_A is the set of functions with vanishing boundary values, it is seen that the boundary values of u_1 and u_2 are uniquely defined. Thus, when a canonical decomposition $\{I_1, I_2\}$ is available, representation (19.6.15) supplies a convenient manner of dividing the information on the boundary values of the function u into two parts, $u_1 \in I_1$ and $u_2 \in I_2$, which is useful in the formulation of many boundary value problems.

For the reduced wave equation the pair $\{I_1, I_2\}$, defined by Equations (19.6.6a) and (19.6.6b) constitutes a canonical decomposition of the space D, as given by Equation (19.5.7), with respect to A, defined by Equation (19.5.6). In this case, representation (19.6.15) breaks the boundary information in the following manner:

$$u = u_2; \qquad \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n}, \qquad \text{on } \partial R \tag{19.6.17}$$

The pair $\{I_1, I_3\}$, given by Equations (19.6.6a) and (19.6.6c), is also a canonical decomposition whenever $\alpha \neq 0$. In this case, if $u = u_1 + u_3$, with $u_1 \in I_1$ and $u_3 \in I_3$, then the boundary values are given by

$$u = u_1 + u_3; \qquad \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n} + \frac{\partial u_3}{\partial n} \qquad \text{on } \partial R \tag{19.6.18}$$

If we define

$$I_4 = \left\{ u \in D \;\; \gamma \frac{\partial u}{\partial n} + \delta u = 0, \qquad \text{on } \partial R \right\} \tag{19.6.19}$$

it is easy to see that $\{I_3, I_4\}$ is a canonical decomposition whenever $\alpha\delta - \beta\gamma \neq 0$. Clearly, the previous examples are particular cases of this more general one.

For the biharmonic equation the following pair is a canonical decomposition:

$$I_1 = \left\{ u \in D \mid u = \frac{\partial u}{\partial n} = 0, \qquad \text{on } \partial R \right\} \tag{19.6.20a}$$

$$I_2 = \left\{ u \in D \mid \Delta u = \frac{\partial \Delta u}{\partial n} = 0, \qquad \text{on } \partial R \right\} \tag{19.6.20b}$$

Also

$$I_1 = \{u \in D \mid u = \Delta u = 0, \quad \text{on } \partial R\} \tag{19.6.21a}$$

$$I_2 = \left\{u \in D \,\middle|\, \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial R\right\} \tag{19.6.21b}$$

Finally, for Stokes problems, one has

$$I_1 = \{\hat{u} \in D \mid \mathbf{u} = 0, \quad \text{on } \partial R\} \tag{19.6.22a}$$

$$I_2 = \left\{\hat{u} \in D \mid \nu \frac{\partial \mathbf{u}}{\partial n} - p\mathbf{n} = 0, \quad \text{on } \partial R\right\} \tag{19.6.22b}$$

Of course, many more examples can be constructed.

In many boundary value problems the prescribed boundary data are given by means of one of the elements in Equation (19.6.15). For example u_1, and the complementary boundary information u_2, can only be obtained after the boundary value problem has been solved. In Dirichlet problems, for example, u is prescribed on ∂R and the derived boundary information $\partial u/\partial n$ on ∂R is obtained only after the problem has been solved. A group of Italian mathematicians, in connection with the representation of the solution by means of fundamental solutions, developed a procedure which permitted them to obtain directly, without solving the problem, the derived boundary information (29). Their procedure, which was applicable to a restricted class of partial differential equations, will be generalized in section 19.10, making it applicable to any linear partial differential equation. However, the method discussed here does not require a representation by means of fundamental solutions, as do their methods.

19.7 OPERATORS THAT DECOMPOSE A

The notion of canonical decomposition, introduced in section 19.6, is closely related to that of operators that decompose A; in fact, there is a one-to-one correspondence between them. This section is devoted to explain operators that decompose A because they will be used in further developments.

Let $B: D \to D^*$ be a bilinear form such that

$$\langle Au, v\rangle = \langle Bu, v\rangle - \langle Bv, u\rangle \tag{19.7.1}$$

We say that B decomposes A when, in addition,

$$D = N_B + N_{B^*} \tag{19.7.2}$$

When Equation (19.7.2) is satisfied one says that B and B^* can be varied independently.

There is a general result of the theory (18, 27, 28), according to which there is a one-to-one correspondence between canonical decompositions

$\{I_1, I_2\}$ and operators that decompose A. This is established as follows:

(1) Given $B: D \to D^*$ that decomposes A, define

$$I_1 = N_{B^*}; \qquad I_2 = N_B \tag{19.7.3}$$

then $\{I_1, I_2\}$ is a canonical decomposition

(2) Given a canonical decomposition $\{I_1, I_2\}$, let $B: D \to D^*$ be defined by

$$\langle Bu, v\rangle = \langle Au_1, v_2\rangle \tag{19.7.4}$$

Here, representation (19.6.15) of every element $u \in D$ of the space in terms of its components $u_1 \in I_1$ and $u_2 \in I_2$ has been used.

To illustrate these concepts in the case of the Laplace and reduced wave equations we note that if we define

$$\langle Bu, v\rangle = \int_{\partial R} v \frac{\partial u}{\partial n} \, d\mathbf{x} \tag{19.7.5}$$

then Equation (19.7.1) is fulfilled. Also, the canonical decomposition $\{I_1, I_2\}$, given by Equations (19.6.6a) and (19.6.6b), satisfies Equation (19.7.3).

In the case of the biharmonic equation the canonical decomposition (19.6.20) is associated with

$$\langle Bu, v\rangle = \int_{\partial R} \left\{ v \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} \right\} d\mathbf{x} \tag{19.7.6}$$

The canonical decomposition (19.6.21), on the other hand, yields

$$\langle Bu, v\rangle = \int_{\partial R} \left\{ v \frac{\partial \Delta u}{\partial n} + \Delta u \frac{\partial u}{\partial n} \right\} d\mathbf{x} \tag{19.7.7}$$

Finally, for Stokes equations the canonical decomposition (19.6.22) is associated with

$$\langle B\hat{u}, \hat{v}\rangle = \int_{\partial R} \mathbf{v} \cdot \left(\nu \frac{\partial \mathbf{u}}{\partial n} - p\mathbf{n} \right) d\mathbf{x} \tag{19.7.8}$$

19.8 COMPLETE SYSTEMS

With every operator $P: D \to D^*$ we can associate a linear subspace $I_P \subset D$ defined by

$$I_P = N_P + N_A \tag{19.8.1}$$

This equation implies that every element $u \in I_P$ can be written as

$$u = u_P + u_A \tag{19.8.2}$$

with $u_P \in N_P$ while $u_A \in N_A$. Since u_A vanishes on the boundary we see that a function u belongs to I_P if and only if the boundary values of u coincide with some solution u_P of the homogeneous partial differential equation.

For example, in the case of the reduced wave equation (19.5.4) a function $v \in I_P$ if and only if there is a solution $u \in D$ of Equation (19.5.4), such that $v = u$ and $(\partial v/\partial n) = (\partial u/\partial n)$ on the boundary ∂R.

It can be shown (28) that I_P, as defined by Equation (19.8.1), is always regular. Due to this fact the concept of c-complete systems will be useful. Let $I \subset D$ be regular and $\mathscr{B}$ be a subset of I; then we say that $\mathscr{B} \subset I$ is c-complete for I when for every $u \in D$

$$\langle Au, w\rangle = 0 \qquad \forall\, w \in \mathscr{B} \Rightarrow u \in I \tag{19.8.3}$$

It is easy to see that a c-complete system exists if and only if $I \subset D$ is completely regular. Therefore the existence of such systems for I_P is granted for most cases of interest because it has been shown (27) that, under very general conditions, I_P is completely regular. In this case, from its definition (Equation (19.8.1)), it is easy to see that N_P is necessarily c-complete for I_P. For the representation of solutions it is, however, of greater interest to have subsets $\mathscr{B} \subset N_P$, whose elements are solutions of the homogeneous equations especially when $\mathscr{B}$ is denumerable.

Examples of such systems were given in Tables 19.1 and 19.2 (p. 409).

19.9 RELATIONSHIPS WITH THE HILBERT SPACE FORMULATION

Most of the developments of the general theory of partial differential equations have been carried out in the setting of Hilbert spaces (19). It is therefore important to incorporate our discussion into that framework in order to be able to make use of the results of that theory.

For this purpose we focus our attention on boundary values; i.e. we identify functions possessing the same boundary values. More precisely, two functions u and v of D are identified whenever $u - v \in N_A$. The resulting space $\mathscr{D}$ is called the quotient space; i.e.

$$\mathscr{D} = D/N_A \tag{19.9.1}$$

Corresponding to each of the examples given in section 19.5 we obtain:

(1) For the Laplace and reduced wave equations $\mathscr{D}$ consists of pairs of functions u, $\partial u/\partial n$, defined on the boundary ∂R and square-integrable there. Thus

$$\mathscr{D} = \left\{\left[u, \frac{\partial u}{\partial n}\right] \quad u \in H^0(\partial R), \frac{\partial u}{\partial n} \in H^0(\partial R)\right\} \tag{19.9.2}$$

(2) The biharmonic equation

$$\mathscr{D}=\left\{\left[u,\frac{\partial u}{\partial n},\Delta u,\frac{\partial\Delta u}{\partial n}\right]\,\middle|\,\text{each one of } u,\frac{\partial u}{\partial n},\Delta u,\frac{\partial\Delta u}{\partial n}\in H^0(\partial R)\right\} \quad (19.9.3)$$

(3) The Stokes equation

$$\mathscr{D}=\left\{\left[\mathbf{u},\nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}\right]\,\middle|\,\mathbf{u}\in\mathbf{H}^0(\partial R),\ \nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}\in H^0(\partial R)\right\} \quad (19.9.4)$$

In each of these examples one can give to $\mathscr{D}$ the structure of a Hilbert space. The corresponding inner products are

1) $$\int_{\partial R}\left\{u\bar{v}+\frac{\partial u}{\partial n}\frac{\partial\bar{v}}{\partial n}\right\}\mathrm{d}\mathbf{x} \quad (19.9.5a)$$

(2) $$\int_{\partial R}\left\{u\bar{v}+\frac{\partial u}{\partial n}\frac{\partial\bar{v}}{\partial n}+\Delta u\,\Delta\bar{v}+\frac{\partial\Delta u}{\partial n}\frac{\partial\Delta\bar{v}}{\partial n}\right\}\mathrm{d}\mathbf{x} \quad (19.9.5b)$$

(3) $$\int_{\partial R}\left\{\mathbf{u}\cdot\bar{\mathbf{v}}+\left(\nu\frac{\partial\mathbf{u}}{\partial n}-p\mathbf{n}\right)\cdot\left(\nu\frac{\partial\bar{\mathbf{v}}}{\partial n}-\bar{q}\mathbf{n}\right)\right\}\mathrm{d}\mathbf{x} \quad (19.9.5c)$$

where the bar refers to the complex conjugate. With this inner product, the linear space $\mathscr{D}$ is isomorphic to the following Hilbert spaces:

(1) $H^0(\partial R)\oplus H^0(\partial R)$ (19.9.6a)

(2) $H^0(\partial R)\oplus H^0(\partial R)\oplus H^0(\partial R)\oplus H^0(\partial R)$ (19.9.6b)

(3) $\mathbf{H}^0(\partial R)\oplus\mathbf{H}^0(\partial R)$ (19.9.6c)

Thus in the following developments it will be assumed that $\mathscr{D}=\mathscr{H}$ possesses a Hilbert-space structure; this means that there is an inner product defined on $\mathscr{D}$, and $\mathscr{D}$ is complete with respect to the metric induced by this inner product (i.e. every Cauchy sequence converges).

Now, given any canonical decomposition $\{I_1, I_2\}$ of the space $\mathscr{D}=\mathscr{H}$, we know that there is an operator $B:D\to D^*$ such that

$$\langle Au, v\rangle=\langle Bu, v\rangle-\langle Bv, u\rangle \quad (19.9.7a)$$

and

$$I_1=N_{B^*} \qquad I_2=N_B \quad (19.9.7b)$$

In this case

$$\langle Bu, v\rangle=\langle Bu_1, v_2\rangle \quad (19.9.8)$$

In what follows we will be interested in the case in which there is a linear mapping $\beta : \mathscr{H} \to \mathscr{H}$ such that

$$\langle Bu, v\rangle = (v, \beta\bar{u}) = (v_2, \beta\bar{u}_1) \tag{19.9.9}$$

where $\bar{u}$ represents a unitary antilinear mapping.

A sufficient condition for Equation (19.9.9) to hold is that the bilinear form B be continuous with respect to the metric of $\mathscr{H}$.

As an illustration, when the operator $B : D \to D^*$ is given by Equation (19.7.5) then, given $u = [u, (\partial u/\partial n)] \in \mathscr{H}(\partial R)$,

$$\beta u = \left[\frac{\partial \bar{u}}{\partial n}, 0\right] \tag{19.9.10}$$

When $B : D \to D^*$ is given by Equation (19.7.6), given

$$u = \left[u, \frac{\partial u}{\partial n}, \Delta u, \frac{\partial \Delta u}{\partial n}\right] \in \mathscr{H}(\partial R)$$

then

$$\beta u = \left[\frac{\partial \overline{\Delta u}}{\partial n}, -\Delta u, 0, 0\right] \tag{19.9.11}$$

If Equation (19.7.7) holds, then

$$\beta u = \left[\frac{\partial \overline{\Delta u}}{\partial n}, 0, \frac{\partial \bar{u}}{\partial n}, 0\right] \tag{19.9.12}$$

Finally, for Stokes equations, when Equation (19.7.8) holds, given

$$\hat{u} = \left[\mathbf{u}, \nu\frac{\partial \mathbf{u}}{\partial n} - p\mathbf{n}\right] \in \hat{\mathscr{H}}(\partial R)$$

then

$$\beta \hat{u} = \left[\nu\frac{\partial \mathbf{u}}{\partial} - p\mathbf{n}, 0\right] \tag{19.9.13}$$

19.10 REPRESENTATION OF SOLUTIONS

For the formulation of the general boundary value problem to be considered here, we assume there is a canonical decomposition $\{I_1, I_2\}$ and an operator $B : D \to D^*$ that decomposes A, satisfying Equation (19.7.3). Using representation (19.6.15) we formulate the problem as follows. Find $u \in N_P$ such that

$$u_1 = U_1 \tag{19.10.1}$$

where U_1 is a given element of I_1.

Let $N_P = N_P/N_A \subset \mathfrak{D} = \hat{\mathscr{H}}$ be the linear space generated by the boundary values of solutions of the homogeneous equation. Then every $u \in N_P$ can be written as

$$u = u_1 + u_2 \tag{19.10.2}$$

where $u_1 \in \mathcal{I}_1 = I_1/N_A$ while $u_2 \in \mathcal{I}_2 = I_2/N_A$. Let $\mathcal{N}_1 \subset \mathcal{I}_1$ be the range of values taken by u_1 in Equation (19.10.2) when u ranges over $\mathcal{N}_P$. Similarly, let $\mathcal{N}_2 \subset \mathcal{I}_2$ be the range of values taken by u_2 in Equation (19.10.2) when u ranges over $\mathcal{N}_P$.

Given a system of functions $\mathcal{B} = \{w_1, w_2, \ldots\} \subset N_P$, using representation (19.6.15) we can write

$$w_\alpha = w_{\alpha 1} + w_{\alpha 2} \tag{19.10.3}$$

We denote

$$\mathcal{B}_1 = \{w_{11}, w_{21}, w_{31} \quad \subset \mathcal{I}_1; \qquad \mathcal{B}_2 = \{w_{12}, w_{22}, w_{32}, \quad .\} \subset \mathcal{I}_2 \tag{19.10.4}$$

Clearly, we will be able to approximate the boundary values of every solution of Equation (19.10.1) if and only if

$$\operatorname{span} \mathcal{B}_1 = \bar{\mathcal{N}}_1 \tag{19.10.5}$$

Here the bar refers to the closure of $\mathcal{N}_1$.

The following result (13, 28) supplies a criterion for completeness which possesses considerable generality and flexibility.

Assume $I_P = N_P + N_A$ is completely regular. Then the following statements are equivalent:

(1) $\mathcal{B} \subset N_P$ is c-complete for I_P; and
(2) span $\mathcal{B}_1 = \bar{\mathcal{N}}_1$ while span $\mathcal{B}_2 = \bar{\mathcal{N}}_2$ (19.10.6)

A proof with a more precise and elaborate form of this result is given in reference 28. Therefore when $\mathcal{B}$ is c-complete it is possible to construct approximating sequences

$$u^N = \sum_{n=1}^{N} a_n^N w_n; \qquad N = 1, 2, \ldots \tag{19.10.7}$$

such that $u_1^N \to U_1$ whenever $U_1 \in \bar{\mathcal{N}}_1$. Therefore if problem (19.10.1) has a solution u, then

$$u^N \to u \tag{19.10.8}$$

The convergence in Equation (19.10.5) is in any metric in which the solution of the problem depends continuously on the boundary data U_1.

The results of section 19.9 give an efficient procedure to compute the complementary boundary data. From Equations (19.10.1), (19.9.9), and

(19.7.1) it follows that for every $w_\alpha \in \mathscr{B}$, we have

$$(w_{\alpha 2}, \beta \bar{U}_1) = (w_{\alpha 2}, \beta \bar{u}_1) = (w_{\alpha 1}, \beta \bar{u}_2) \tag{19.10.9}$$

which gives $(w_{\alpha 1}, \beta \bar{u}_2)$ in terms of the boundary data $\bar{U}$ This gives the approximating sequence

$$\beta \bar{u}_2^N = \sum_{n=1}^{N} b_n^N \bar{w}_{n1} \qquad N = \qquad 19.10.10)$$

where the coefficients b_n^N satisfy, for every fixed N, the system of equations

$$(w_{m2}, \beta \bar{U}_1) = \sum_{n=1}^{N} b_n^N (w_{m1}, \bar{w}_{n1}) \tag{19.10.11}$$

The convergence of sequence (19.10.10) is assumed whenever the solution U_1 exists. A more detailed discussion of these points is given in reference 18.

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