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ABSTRACT

A boundary method for solving the biharmonic equation is presented. It is based on the use of systems of solutions of the homogeneous equations, which are complete. A convenient criterium for the completeness of such systems, is the notion of c-completeness. Using a convenient representation of solutions for the biharmonic equation a procedure for constructing c-complete systems for this equation is developed. Examples of such systems are constructed.

1. INTRODUCTION

In recent years, by a boundary method, it is usually understood, a numerical procedure in which a subregion or the entire region, is left out of the numerical treatment, by use of available analytical solutions (or more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problem leading to considerable economy in the numerical work and constitute a very convenient manner of treating adequately unbounded regions by numerical means. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced [Zienkiewicz, 1977, Zienkiewicz, et al., 1977].

There are two main approaches for the formulation of boundary methods; one is based on the use of boundary integral equations and the other one, on the use of complete systems of solutions. In numerical applications, the first one of these methods has received most of the attention [Brebbia, 1978]. This is in spite of the fact that the use of complete systems of solutions presents important numerical advantages; e.g., it avoids the introduction of singular integral equations and it does not require the construction of a fundamental solution. The latter is especially relevant in connection with complicated problems, for which, it may be extremely laborious to build up a fundamental solution. This is illustrated by the fact that there are methods for synthetizing fundamental solutions starting from plane waves, which can be shown to be a complete system [Sánchez-Sesma, Herrera and Aviles, 1981].

One may advance some possible explanations for this situation. Although, the principle of superposition, is a standard procedure for building up solutions of linear equations, many of its applications have been based on the method of separation of variables; this has lead to the frequent, but false, belief that complete systems of solutions have to be constructed specifically for a given region. Of course, this is not the case; indeed, most frequently systems of solutions are complete independently of the detailed shape of the region considered [Herrera and Sabina, 1978], and the systems developed here for the biharmonic equation possess this property.

Also, in some fields of application, procedures which constitute particular cases of the approximation by complete systems of solutions, have presented severe restrictions and inconveniences. For the case of acoustics and electromagnetic field computations, a survey of such difficulties, was carried out by Bates [1975]. For this kind of studies, the so called "Rayleigh hypothesis", restricts drastically the applicability of the method. However, work by Millar [1973], implies that these difficulties are due, mainly, to lack of clarity, since he avoided Rayleigh hypothesis, altogether, by adopting a different point of view.

Motivated by this situation, one of the authors, started a systematic research of the subject [Herrera and Sabina, 1978; Herrera, 1977a, 1979b, 1980e], oriented to clarify the theoretical foundations of the method, allowing its systematic and reliable use. The aims of the research have been satisfactorily achieved to a large extent and have just been reported [Herrera, 1981b,c]. This has been possible due to the progress that has been made in the understanding of partial differential equations [Lions and Magenes, 1972; Temam, 1977]. The methodology also owes much to work of Amerio, Fichera, Picone, Kupradze and Trefftz [Miranda, 1955; Kupradze, 1967; Trefftz, 1926]. The systematic development of the procedure, in a manner which is applicable to any linear problem, was made possible, however, by an abstract theory that has been developed by Herrera [1979a,b, 1980b,c,d,e, 1981a].

The numerical solution of Stokes and Navier-Stokes equations, is a problem of great practical interest at present, and it is not our purpose to review it, since recent surveys are available [Glowinski and Pironneau, 1978; Temam, 1977]. Taking this interest for granted, we explain briefly the method mentioned before, in connection with the biharmonic equation and supply an efficient procedure for developing c-complete systems for this equation, starting from c-complete systems for Laplace equation.

2. THE BOUNDARY METHOD USED

Consider the biharmonic equation

$$\nabla^4 u = 0 \quad \text{in } \Omega \tag{2.1}$$

This equation must be satisfied in the sense of distributions by elements of some spaces of functions. In general, we ask u to be in a linear subspace $D \subset H^2(\Omega)$, so that the equation (2.1) is between elements of $H^{-2}(\Omega)$.†

On this assumption the biharmonic problem (2.1) is equivalent to the formulation

$$u \in \text{Ker } P \ , \ P: H^2(\Omega) \to (H^2)^*$$

defined by

$$\langle Pu, v\rangle = \int_\Omega \nabla^4 u \ v \ dx \quad \forall \ u \in H^2(\Omega) \quad \forall \ v \in H^2(\Omega) \tag{2.2}$$

Integration by parts gives

$$\langle Pu, v\rangle = \int_\Omega \nabla^2 u \ \nabla^2 v \ dx + \int_{\partial\Omega} \left\{\frac{\partial \Delta u}{\partial n} v - \Delta u \frac{\partial v}{\partial n}\right\} dx \tag{2.3}$$

In (2.3) four different boundary values occur. We note that [Lions and Magenes, 1972]

$$u \in H^2(\Omega) \Rightarrow \quad u \in H^{3/2}(\partial\Omega)$$
$$\frac{\partial u}{\partial n} \in H^{1/2}(\partial\Omega)$$
$$\Delta u \in H^{-1/2}(\partial\Omega)$$
$$\frac{\partial \Delta u}{\partial n} \in H^{-3/2}(\partial\Omega)$$

Let us associate with the operator P, an antisymmetric operator A by

$$A = P - P^*$$

$$\langle Au, v\rangle = \langle Pu, v\rangle - \langle Pv, u\rangle \tag{2.4}$$

$$\forall \ u, v \in D \quad \langle Au, v\rangle = \int_{\partial\Omega} \left\{v \frac{\partial}{\partial n} \Delta u - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n}\right\} dx \tag{2.5}$$

in which, only boundary values appear. Let us introduce the Boundary operators B and B':

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† We use the usual notation for Sobolev spaces.

$$\forall\, u,v \in D \quad \langle Bu,v\rangle = \int_{\partial\Omega} v \frac{\partial}{\partial n} \Delta u \, dx \tag{2.6}$$

$$\langle B'u,v\rangle = + \int_{\partial\Omega} \Delta v \frac{\partial u}{\partial n} \, dx \tag{2.7}$$

Then

$$A = B + B' - B'^* - B^* \tag{2.8}$$

In fact we can directly define A by (2.5) in a different space D:

$$D = \{u \in H^{1/2}(\Omega) \mid u \in H^{o}(\partial\Omega);\ \frac{\partial u}{\partial n} \in H^{o}(\partial\Omega)\ ;$$
$$\Delta u \in H^{o}(\partial\Omega)\ ;\ \frac{\partial \Delta u}{\partial n} \in H^{o}(\partial\Omega)\}$$

D is not a Sobolev space, but we note the inclusions:

$$H^{7/2}(\Omega) \subset D \subset H^{1/2}(\Omega) \tag{2.9}$$

and that $C^{\infty}(\Omega)$ is dense in D.

Define

$$I_1 = \text{Ker}(B + B') \text{ and } I_2 = \text{Ker}(B^* + B'^*) \tag{2.10}$$

Then it can easily be shown that $\{I_1, I_2\}$ is a canonical decomposition of D, in the sense defined in [Herrera, 1980b]; i.e., I_1, I_2 are completely regular:

$$\langle Au,v\rangle = 0 \quad \forall\, v \in I_1 \Leftrightarrow u \in I_1 \tag{2.11}$$

$$\langle Au,v\rangle = 0 \quad \forall\, v \in I_2 \Leftrightarrow u \in I_2 \tag{2.12}$$

and

$$I_1 + I_2 = D \qquad I_1 \cap I_2 = \text{Ker } A$$

This implies that every $u \in D$ can be written as $u = u_1 + u_2$, with $u_1 \in I_1$ and $u_2 \in I_2$, and this representation is unique, except for elements of Ker A.

Another canonical decomposition would be

$$I_1' = \text{Ker }(B - B'^*)\ ;\ I_2' = \text{Ker}(B^* - B')\ . \tag{2.13}$$

Notice that the boundary values of elements of D, can be characterized as follows:

$$u \in D \rightarrow [u_a, u_b, u_c, u_d] \in D/\text{Ker } A \subset [H_o(\partial\Omega)]^4$$

with

$$u_a = u \quad u_b = \frac{\partial u}{\partial n} \quad u_c = \Delta u \quad u_d = \frac{\partial}{\partial n} \Delta u \tag{2.14}$$

on $\partial\Omega$. Now, associated with the canonical decomposition $\{I_1, I_2\}$, we have

$$D/\text{Ker } A = I_1/\text{Ker } A \oplus I_2/\text{Ker } A$$

$\ldots \exists \{u_1, u_2\} \ni \; u = u_1 + u_2$

$$u_1 \in \mathrm{Ker}(B + B') \quad u_1 = [u_a , u_c] \tag{2.15}$$

$$u_2 \in \mathrm{Ker}(B^* + B'^*) \quad u_2 = [u_d, u_b] \tag{2.16}$$

with these notations then it is easy to exhibit the identity:

$$\langle Au,v\rangle = (u_2 , v_1) - (u_1 , v_2) \tag{2.17}$$

where it is understood that if $[a,b]$, $[c,d] \in [H_o(\partial\Omega)]^2$

$$([a,b] , [cd]) = \int_{\partial\Omega} (ac + bd)dx \tag{2.18}$$

Using the other decomposition we would have similarly

$$D/\mathrm{Ker}\, A = I_1'/\mathrm{Ker}\, A \oplus I_2'/\mathrm{Ker}\, A$$

$u \in D/\mathrm{Ker}\, A \qquad u = u_1' + u_2'$

$$u_1' = [u_a,u_b] \quad u_2' = [u_d,-u_c] \tag{2.19}$$

and the identity:

$$\langle Au,v\rangle = (u_2' , v_1') - (u_1' , v_2') \tag{2.20}$$

Let N_P be the subspace of solutions of the biharmonic equation (2.1) and define

$$I_P = N_P + N_A \tag{2.21}$$

Then I_P is completely regular; i.e.

$$\langle Au,v\rangle = 0 \quad \forall\, v \in I_P \Leftrightarrow u \in I_P \tag{2.22}$$

This comes straight forwardly from some results of existence of solution of the biharmonic equation with compatible boundary conditions, the density of $C^\infty(\Omega)$ in D and results reported previously [Herrera, 1980b]. Define $\mathcal{I}_P = I_P/\mathrm{Ker}\, A$. The results of uniqueness imply that $N_P \cap \mathrm{Ker}\, A = \{0\}$ so that $\mathcal{I}_P$ is naturally imbedded in D/Ker A, with the notations introduced. The following definition is relevant, for our discussion.

<u>Definition</u>: *A denumerable set* $B = \{w_1,w_2,\ldots\}$ *of* N_P *is* <u>*c-complete*</u> *(complete in connectivity), with respect to* A *if* $\langle Au,w_\alpha\rangle = 0 \quad \forall\, \alpha \in N \Leftrightarrow u \in \mathcal{I}_P$.

For any canonical decomposition $\{I_1,I_2\}$, $\mathcal{I}_P$ is decomposed as $\mathcal{I}_P = \mathcal{I}_{1P} \oplus \mathcal{I}_{2P}$, $\mathcal{I}_{1P} \subset I_1, \mathcal{I}_{2P} \subset I_2$

The following can be proved [Herrera, 1980e, 1981b,c].

<u>Proposition 1</u>. *The 3 statements are equivalent*

i) B is c-complete in I_P, with respect to ...

ii) $B_1 = \{w_{1\alpha}\}_{\alpha \in N}$ spans I_{1P}.

iii) $B_2 = \{w_{2\alpha}\}_{\alpha \in N}$ spans I_{2P}.

Let us suppose that we have a c-complete system

$$B = \{w_\alpha\}_{\alpha=1,\ldots} \subset N_P$$

and a canonical decomposition $\{I_1, I_2\}$. Consider the problem; find $u \in D$ such that

$$\nabla^4 u = 0 \quad \text{in} \quad \Omega$$

$$u_1 = u_\beta \text{ given on the boundary}$$

Assume $u_1 \in I_{1P}$, in order to have existence of a solution to the problem. In view of the Proposition, we know that $\{w_{1\alpha}\}_{\alpha \in N}$ spans I_{1P} so that any element of I_{1P} can be approximated by linear combinations of $\{w_{1\alpha}\}$; more precisely, one can choose coefficients $\{a_\alpha^N\}_{\alpha=1,\ldots N}$ such that

$$u_1^N = \sum_1^N a_\alpha^N w_{1\alpha} \tag{2.24}$$

has the property that $u_1^N \to u_1$ in $[H^0(\partial\Omega)]^2$.

Then

$$u^N = \sum_1^N a_\alpha^N w_\alpha \tag{2.25}$$

is the biharmonic function in Ω, such that the boundary value u_1^N approximates the data. Therefore, u^N is an approximation to a solution of the problem and as $N \to \infty$ one has $u^N \to u$ in the sense (at least) of $H^{1/2}(\Omega)$ [Lions and Magenes, 1972].

If the missing boundary value u_2 is required, and if it is known to be in $[H^0(\partial\Omega)]^2$ in I_{2P}, we can indeed approximate it by

$$u_2^N = \sum_{\alpha=1}^N a_\alpha^N w_{2\alpha} \tag{2.26}$$

but we can also compute it using (2.17). Indeed

$$(u_2, w_{1\alpha}) = (u_1, w_{2\alpha}) \quad \forall\, \alpha \in N$$

If $\{w_{1\alpha}\}$ is orthonormal (and if it is not, we can orthonormalize it by the well known Gram-Schmidt orthonormalization process), then, u_2 is given by

$$u_2 = \sum_{\alpha=1}^\infty (u_2, w_{1\alpha}) w_{1\alpha} = \sum_{\alpha=1}^\infty (u_1, w_{2\alpha}) w_{1\alpha} \tag{2.27}$$

3. C-COMPLETE SYSTEMS FOR THE BIHARMONIC EQUATION

In this section we give a general procedure for constructing c-complete systems for the biharmonic equation, whenever a c-complete system for Laplace's equation is known.

Proposition 2. *Let $\{\psi_1,\psi_2,\ldots\}$ be harmonic functions such that they are a c-complete system for Laplace equation in the region Ω. Assume $\{\phi_1,\phi_2,\ldots\}$ are also harmonic and such that*

$$\frac{\partial\phi_\alpha}{\partial x} = \psi_\alpha \quad ; \quad \alpha = 1,2,\ldots \tag{3.1}$$

Then the system $\{\psi_1,\psi_2,\ldots\} \cup \{x\phi_1,x\phi_2,\ldots\}$ are biharmonic and c-complete for equation (2.1).

Proof. Consider the canonical decomposition (2.15), (2.16), then

$$I_{1P} = \{u_1 = [u,\Delta u] \mid u \in I_P\} = [H^0(\partial\Omega)]^2 \tag{3.2}$$

Here, u, Δu refer to the boundary values on $\partial\Omega$. The biharmonic problem with u_1 given, is the biharmonic equation (2.1), subjected to the boundary conditions

$$u = f_1 \quad , \quad \text{on } \partial\Omega \tag{3.3a}$$

$$\Delta u = f_2 \quad , \quad \text{on } \partial\Omega \tag{3.3b}$$

where f_1 and f_2 are given functions of $H^0(\partial\Omega)$. Equivalently, one can solve

$$\Delta p = 0 \quad , \quad \text{in } \Omega \tag{3.4a}$$

$$p = f_2 \quad , \quad \text{on } \partial\Omega \tag{3.4b}$$

and

$$\Delta u = p \quad , \quad \text{in } \Omega \tag{3.5a}$$

$$u = f_1 \quad , \quad \text{on } \partial\Omega \tag{3.5b}$$

In view of Proposition 1, it is enough to prove that the system of boundary values $\{[\psi_1,\Delta\psi_1], [\psi_2,\Delta\psi_2],\ldots\} \cup \{[x\phi_1,\Delta x\phi_1], [x\phi_2,\Delta x\phi_2],\ldots\}$ spans $I_{1P} = [H_0(\partial\Omega)]^2$. To this end, notice that

$$\Delta\psi_\alpha = 0 \quad ; \quad \Delta(x\phi_\alpha) = 2\psi_\alpha \ , \quad \alpha=1,2,\ldots \tag{3.6}$$

Therefore, given $[f_1,f_2] \in [H^0(\partial\Omega)]^2$, consider the following approximating sequence

$$u^N \doteq \sum_{\alpha=1}^{N} a_\alpha^N \psi_\alpha + \sum_{\alpha=1}^{N} b_\alpha^N x\,\phi_\alpha \tag{3.7}$$

Define

$$p^N = \sum_{\alpha=1}^{N} b_\alpha^N \Delta x \phi_\alpha^N = 2 \sum_{\alpha=1}^{N} b_\alpha^N \psi_\alpha \tag{3.8}$$

and choose b_α^N, so that (as $N\to\infty$)

$$p^N \to f_2 \quad , \quad \text{on } H^0(\partial\Omega) \tag{3.9}$$

This is possible, because $\{\psi_1,\psi_2,\ldots\}$ is c-complete. Relation (3.9), implies that there exists $v\in H^{5/2}(\partial\Omega)$ such that as $N\to\infty$,

$$p^N \to v \quad ; \quad \text{in } H^{5/2}(\Omega) \tag{3.10}$$

Therefore

$$p^N \to v \quad , \quad \text{in } H^2(\partial\Omega)\subset H^0(\partial\Omega) \tag{3.11}$$

Choose now a_α^N so that

$$\sum_{\alpha=1}^{N} a_\alpha^N \psi_\alpha \to f_1 - v \quad , \quad \text{on } H^0(\partial\Omega) \tag{3.12}$$

This is again possible because $\{\psi_1,\psi_2,\ldots\}$ is c-complete. Hence, clearly

$$[u^N, \Delta u^N] \to [f_1,f_2] \quad , \quad [H^0(\partial\Omega)]^2 \tag{3.13}$$

and the proof of Proposition 2, is complete.

As an example of the application of Proposition 2, we exhibit a polynomial system which is c-complete for biharmonic equation in any bounded region Ω.

<u>Proposition 3</u>. *Let* $(\alpha=1,2,\ldots)$

$$\psi_\alpha = \text{Re } z^{(\alpha-1)/2} \quad \textit{when } \alpha \text{ is odd} \tag{3.14a}$$

$$\psi_\alpha = \text{Im } z^{\alpha/2} \quad \textit{when } \alpha \textit{ is even} \tag{3.14b}$$

Define

$$\phi_\alpha = \psi_{\alpha+2} \tag{3.15}$$

Then $\{\psi_1,\psi_2,\ldots\} \cup \{x\phi_1,x\phi_2,\ldots\}$ *is c-complete for the biharmonic equation, in any bounded region* Ω.

<u>Proof</u>. It has been shown [Herrera and Sabina, 1978], that $\{\psi_1,\psi_2,\ldots\}$ is c-complete for Laplace's equation in any bounded region. In addition, it is easy to see that equation (3.1) is satisfied.

We recall finally, that a c-complete can be used to approximate any other boundary value problem prescribed by means of regular subspace; this, by virtue of Proposition 1.

4. THE EXTERIOR DOMAIN

Let Ω be the exterior of a bounded domain. A c-complete system for Laplace's equation, which satisfies a radiation condition, in Ω, is given [Herrera and Sabina, 1978], by $\{\psi_1,\psi_2,\ldots\}$

$$\psi_1 = \text{Re Log } z \quad ; \qquad (4.1a)$$

$$\psi_\alpha = \text{Re } z^{-(\alpha-1)/2} \quad ; \ \alpha \text{ odd} \geq 3 \qquad (4.1b)$$

$$\psi_\alpha = \text{Im } z^{-\alpha/2} \quad ; \ \alpha \text{ even} \qquad (4.1c)$$

Applying Proposition 2, it can be seen that system $\{\psi_1,\psi_2,\ldots\} \cup \{x\phi_1,x\phi_2,\ldots\}$, where

$$\phi_1 = \text{Re}(z \log z - z) \ ; \ \phi_2 = \text{Im} \log z \qquad (4.2a)$$

$$\phi_\alpha = \psi_{\alpha-2} \quad ; \ \alpha \geq 3 \qquad (4.2b)$$

is a c-complete system for the exterior problem.

REFERENCES

Bates, R.H.T. (1975) Analytic Constraints on Electro-magnetic Field Computations. IEEE Trans. on Microwave Theory of Techniques. 23, 605-623.

Brebbia, C.A. (1978) The Boundary Element Method for Engineers. Pentech, Press. London.

Glowinski, R., and Pironneau, O. (1978) On Numerical Methods for the Stokes Problem. Chapter 13 of Energy Methods in Finite Element Analysis. R. Glowinski, E.Y. Rodin and O.C. Zienkiewicz, ed., Wiley and Sons.

Herrera, I. (1977a) General Variational Principles Applicable to the Hybrid Element Method. Proc. Nat. Acad. Sci. USA, 74, 7:2595-2597.

Herrera, I. (1979a) On the Variational Principles of Mechanics. Trends in Applications of Pure Mathematics to Mechanics. II, 115-128, H. Zorsky, ed., Pitman Publishing Limited (Invited general lecture)

Herrera, I. (1979b) Theory of Connectivity: A Systematic Formulation of Boundary Element Methods. Applied Mathematical Modelling. 3, 2:151-156.

Herrera, I. (1980b) Variational Principles for Problems with Linear Constraints. Prescribed Jumps and Continuation Type Restrictions. Jour. Inst. Maths. Applics. 25, 67-96.

Herrera, I. (1980c) Boundary Methods in Flow Problems. Proc. Third International Conference on Finite Elements in Flow Problems, Banff, Canada, 10-13 June, 30-42. (Invited general lecture).

Herrera, I. (1980d) Boundary Methods in Water Resources. Finite Elements in Water Resources, S.Y. Wang, et al., ed., The University of Mississippi, 58-71. (Invited general lecture).

Herrera, I. (1980e) Boundary Methods. A Criterion for Completeness. Proc. Nat. Acad. Sci. USA, 77, 8:4395-4398.

Herrera, I. (1981a) An Algebraic Theory of Boundary Value Problems. Comunicaciones Técnicas, IIMAS-UNAM. (In press).

Herrera, I. (1981b) Boundary Methods in Fluids. Finite Elements in Fluids, Volume IV, R.H. Gallagher, ed., John Wiley and Sons. (In press).

Herrera, I. (1981c) Boundary Methods. Theoretical Foundations for Numerical Applications of Complete Systems of Solutions. Comunicaciones Técnicas, IIMAS-UNAM.

Herrera, I., and Sabina, F.J. (1978) Connectivity as an Alternative to Boundary Integral Equations. Construction of bases. Proc. Nat. Acad. Sci. USA, 75, 5:2059-2063.

Kupradze, V.D. (1967) On the Approximate Solution of Problems in Mathematical Physics. Russian Math. Surveys. 22, 2:58-108. (Uspehi Mat. Nauk. 22, 2:59-107).

Lions, J.L., and Magenes, E. (1972) Non-homogeneous Boundary Value Problems and Applications. Springer-Verlag, New York.

Millar, R.F. (1973) The Rayleigh Hypothesis and a Related Least-Squares Solutions to Scattering Problems for Periodic Surfaces and other Scatterers. Radio Science. 8, 785-796.

Miranda, C. (1970) Partial Differential Equations of Elliptic Type. 2nd Ed., Springer-Verlag, New York. (Translation of Equazioni alle Derivate Parziali di Tipo Ellitico, 1955).

Sánchez-Sesma, F.J., Herrera, I., and Avilés, J. (1981) Boundary Methods for Elastic Wave Diffraction-Application to Scattering of SH Waves by Surface Irregularities. Comunicaciones Técnicas, IIMAS-UNAM. (In press).

Temam, R. (1977) Navier Stokes Equations: Theory and Numerical Analysis. North-Holland, Amsterdam.

Trefftz, E. (1926) Ein Gegenstruck zum Ritzschen Vergaren. Proc. 2nd. Int. Congress Appl. Mech. Zurich.

Zienkiewicz, O.C. (1977) The Finite Element Method in Engineering Science. Mc-Graw Hill, New York.

Zienkiewicz, O.C., Kelly, D.W., and Bettess P., (1977) The Coupling of the Finite Element Methods and Boundary Solution Procedures. Int. J. Num. Math. Eng. 11, 355-377.