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BOUNDARY METHODS, c-COMPLETE SYSTEMS FOR STOKES PROBLEMS

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A boundary method for solving Stokes problem is presented. This is based on the use of systems of solutions of the homogeneous equations, which are complete. A convenient criterion, for the completeness of such systems, is the notion of c-completeness. An apparently new representation of solutions of Stokes equations is derived and is used to develop a procedure for constructing a c-complete system. Examples of such systems are constructed.

1. Introduction

The numerical solution of Stokes and Navier–Stokes equations, is a problem of great practical interest at present. However, it is not our purpose to review it, since recent surveys are available [1–2]. Instead, our purpose here is to develop a procedure for solving such problems using a method studied recently by Herrera [3–5]. This method uses general results on the continuous dependence of solutions on boundary data in the theory of partial differential equations (for example, those given by Lions and Magenes [6], or more specifically for Stokes' problems, by Temam [2]), with a view to approximating any solution. A criterion for completeness [7] which is suitable for most applications is incorporated. Associated with the method is an efficient procedure for computing boundary values, which has been developed with complete generality using an abstract theory of boundary value problems [5, 8]. One of the important features of this boundary method, is that it can be applied to any linear partial differential equation, irrespective of its type. Applications to an important class of non-linear problems have previously been reported [4–5].

In addition to the general theory of partial differential equations [6], the method owes much to the work of Trefftz [9], and of the Italian mathematicians Amerio, Fichera and Picone [10].

The main result reported in this paper is a procedure for developing c-complete systems for Stokes equations. This is based on a representation theorem for solutions of Stokes equations, which also seems to be new. The representation theorem is given in Section 3. Then, Sections 4 and 5 are devoted to showing how it can be used to construct c-complete systems. An important example of such systems is given in Section 6.

2. Notations and preliminaries

In a bounded domain $\Omega \subset \mathbb{R}^2$ with a regular boundary $\partial\Omega$ of finite length (Fig. 1), we consider the Stokes problem

$$-\Delta \boldsymbol{u} + \operatorname{grad} p = 0 \quad \text{in } \Omega\,, \qquad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega\,,$$

for a function $\boldsymbol{u} : \Omega \to \mathbb{R}^2$, and where $p : \Omega \to \mathbb{R}$. The boundary conditions will be set up later on. Firstly some function spaces are defined.

$$\hat{\mathcal{U}}^r(\Omega) = \{(\boldsymbol{u}, p) \colon (\boldsymbol{u}, p) \in \boldsymbol{H}^{r+1/2}(\Omega) \times H^{r-1/2}(\Omega) \text{ that satisfy (2.1)}\}$$

REMARK 2.1. If $(\boldsymbol{u}, p)$ is an element of $\hat{\mathcal{U}}^r(\Omega)$, then $(\boldsymbol{u} + \boldsymbol{\lambda}, p + \mu)$ with $\lambda \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$ is also an element of $\hat{\mathcal{U}}^r(\Omega)$.

We also define two spaces on $\partial\Omega$,

$$\mathcal{U}_1^r(\partial\Omega) = \left\{ \boldsymbol{u} \in \boldsymbol{H}^r(\partial\Omega) \colon \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}x = 0 \right\},$$

$$\mathcal{U}_2^r(\partial\Omega) = \left\{ \boldsymbol{\phi} \in \boldsymbol{H}^{r-1}(\partial\Omega) \colon \int_{\partial\Omega} \boldsymbol{\phi} \, \mathrm{d}x = \boldsymbol{0} \right\}.$$

All of these spaces become Hilbert spaces when they are endowed with the usual inner product.

For every $(\boldsymbol{u}, p) \in \boldsymbol{H}^{r+1/2}(\Omega) \times H^{r-1/2}(\Omega)$ we write $\gamma_0 \boldsymbol{u} \in \boldsymbol{H}^r(\partial\Omega)$ for the trace of $\boldsymbol{u}$ on $\partial\Omega$, and $\gamma_1(\boldsymbol{u}, p) \in \boldsymbol{H}^{r-1}(\partial\Omega)$ for the trace of $\partial \boldsymbol{u}/\partial n - p\boldsymbol{n}$ on $\partial\Omega$. Two kinds of Stokes problems will be considered, namely those for which $\gamma_0 \boldsymbol{u}$ or, alternatively, $\gamma_1(\boldsymbol{u}, p)$ is prescribed. They will be referred to as Stokes problem No. 1 and No. 2, respectively.

Here we state some existence and continuity results.

THEOREM 2.2. For every $r \geq \frac{1}{2}$, the mapping

$$\gamma_0 : \hat{\mathcal{U}}^r(\Omega)/\{0\} \times \mathbb{R} \to \mathcal{U}_1^r(\partial\Omega)$$

is continuous, onto and one-to-one.

PROOF. The Stokes problem

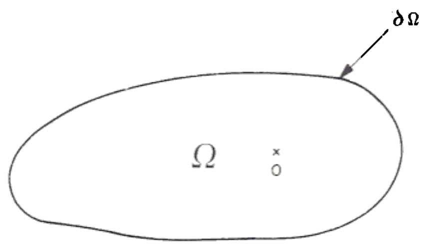

Fig. 1.

$$(\boldsymbol{u}, p) \text{ satisfies (2.1)},$$
$$\boldsymbol{u} = \boldsymbol{u}_\beta \quad \text{on } \partial\Omega,$$
$$\boldsymbol{u}_\beta \text{ given in } \mathscr{U}_1^r(\partial\Omega),$$

has a unique solution up to an additive constant in p; the continuity properties are presented in [2, Prop. 2.3].

This is the well-known Stokes problem. We wish to present the same result for a different problem.

THEOREM 2.3. For every $r \geq \frac{1}{2}$, the mapping

$$\gamma_1 : \hat{\mathscr{U}}^r(\Omega)/\mathbb{R}^2 \times \{0\} \to \mathscr{U}_2^r(\partial\Omega)$$

is continuous, onto and one-to-one.

PROOF. It must be shown that the problem

$$(\boldsymbol{u}, p) \text{ satisfies (2.1)},$$
$$\partial \boldsymbol{u}/\partial n - p\boldsymbol{n} = \boldsymbol{\phi} \quad \text{in } \partial\Omega,$$
$$\boldsymbol{\phi} \text{ given in } \mathscr{U}_2^r(\partial\Omega),$$

has a unique solution in $\mathscr{U}^r(\Omega)$ up to an additive constant function in $\boldsymbol{u}$, and that

$$\|\boldsymbol{u}\|_{H^{r+1/2}(\Omega)/\mathbb{R}^2} + \|p\|_{H^{r-1/2}(\Omega)} \leq C\|\boldsymbol{\phi}\|_{H^{r-1}(\partial\Omega)}$$

Notice that problem (2.8) is the adjoint boundary value problem of the preceding Stokes problem (2.6)

First, the proof is done for $r = \frac{1}{2}$. The space $\mathscr{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^2 : \nabla \cdot \boldsymbol{v} = 0\}$ is a Hilbert space. Then if $\boldsymbol{u}$ is solution of (2.8), it satisfies

$$\int_\Omega (-\Delta \boldsymbol{u} + \nabla p) \cdot \boldsymbol{v} \, dx = 0 \quad \forall \boldsymbol{v} \in \mathscr{V},$$

and by integration by parts

$$\sum_{i=1}^{2} \int_\Omega \nabla u_i \nabla v_i \, dx - \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial n} \boldsymbol{v} \, dx - \int_\Omega p \, \nabla \cdot \boldsymbol{v} \, dx + \int_{\partial\Omega} p \boldsymbol{v} \cdot \boldsymbol{n} = 0 .$$

Then, because of $\boldsymbol{v} \in \mathscr{V}$,

$$\sum_{i=1}^{2} \int \nabla u_i \nabla v_i \, dx - \int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{v} \, dx = 0 \quad \forall \boldsymbol{v} \in \mathscr{V} \tag{2.10}$$

Conversely, if $\boldsymbol{u} \in \mathcal{V}$ satisfies (2.10), we deduce

$$\int_\Omega \Delta \boldsymbol{u} \cdot \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \{[D(\Omega)]^2 \colon \nabla \cdot \boldsymbol{v} = 0\}$$

and, as for the Stokes problem [2, Prop. 1]

$$\exists p \in L^2(\Omega) \quad \text{such that } \Delta \boldsymbol{u} = \nabla p$$

This p is not uniquely determined, but two functions satisfying (2.11) only differ by a constant
Then integrating (2.10) by parts, we find that

$$\int_\Omega \Delta \boldsymbol{u}\boldsymbol{v}\, \mathrm{d}x + \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial n} \cdot \boldsymbol{v} - \int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{v}\, \mathrm{d}x = 0$$

and it follows, using (2.11) and $\nabla \cdot \boldsymbol{v} = 0$, that

$$\int_{\partial\Omega} \left(\frac{\partial \boldsymbol{u}}{\partial n} - p\boldsymbol{n}\right) \cdot \boldsymbol{v}\, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{v}\, \mathrm{d}x$$

Eq. (2.12) only means that $\partial \boldsymbol{u}/\partial n - p\boldsymbol{n} - \boldsymbol{\phi}$ is orthogonal to the traces of elements of $\mathcal{V}$, that is

$$\partial \boldsymbol{u}/\partial n - p\boldsymbol{n} - \boldsymbol{\phi} = \lambda \boldsymbol{n} \quad \text{on } H^{-1/2}(\partial\Omega) \text{ for some } \lambda \in \mathbb{R}$$

Recall that p in (2.11) was not uniquely determined, but that $p + \lambda$ is also a solution in (2.11). We can thus affirm that if $\boldsymbol{u}$ satisfies (2.10), there exists a unique p such that $(\boldsymbol{u}, p)$ satisfies (2.8).

Then the sets $\{(\boldsymbol{u}, p) \colon (\boldsymbol{u}, p) \in \hat{\mathcal{U}}^{1/2}(\Omega)$ and satisfying (2.8)$\}$ and $\{\boldsymbol{u} \colon \boldsymbol{u} \in \mathcal{V}$ and satisfying (2.10)$\}$ are equivalent.

The projection theorem ensures that problem (2.10) has a solution unique in $\mathcal{V}$ to within an additive constant. Then (2.8) has a unique solution in $(H^1(\Omega))^2/\mathbb{R}^2 \times H^0(\Omega)$.

The continuity follows from several known results. In what follows c_i will be constants which depend only of the geometry of the domain.

(1) $p \in L^2(\Omega)$ is decomposed in $L^2(\Omega)/\mathbb{R} \oplus \mathbb{R}$ by

$$p = p_0 + \lambda \quad \text{with} \int_\Omega p_0\, \mathrm{d}x = 0 \text{ and } \lambda = \frac{1}{\text{meas}\, \Omega} \int_\Omega p\, \mathrm{d}x \tag{2.13}$$

Then

$$\|p\|^2_{L^2(\Omega)} = \|p_0\|^2_{L^2(\Omega)} + \lambda^2(\text{meas}\, \Omega) \tag{2.14}$$

and

$$\|p\|^2_{L^2(\Omega)/\mathbb{R}} = \min_{\mu \in \mathbb{R}} \|p - \mu\|^2_{L^2(\Omega)} = \|p_0\|^2_{L^2(\Omega)} \tag{2.15}$$

(2) By Prop. 2.2 of [2]: if p is such that $\partial p/\partial x_i \in H^{-1}(\Omega)$, then

$$\|p\|^2_{L^2(\Omega)/\mathbb{R}} \leq c_1\|\nabla p\|_{(H^{-1}(\Omega))^2} \tag{2.16}$$

Combining (2.11), (2.15) and (2.16), we can write

$$\|p_0\|^2_{L^2(\Omega)} \leq c_1\|\Delta u\|^2_{(H^{-1}(\Omega))^2}$$

and recalling that Δ is a continuous embedding of $H^1(\Omega)$ on $H^{-1}(\Omega)$,

$$\|p_0\|^2_{L^2(\Omega)} \leq c_2\|u\|^2_{(H^1(\Omega)/\mathbb{R})^2}$$

(3) λ can be estimated by (2.8) or (2.13):

$$\lambda \boldsymbol{n} = \partial \boldsymbol{u}/\partial n - p_0\boldsymbol{n} - \boldsymbol{\phi} \quad \text{in } H^{-1/2}(\Omega)\,.$$

Taking norms,

$$|\lambda|\,\|\boldsymbol{n}\|_{H^{-1/2}(\Omega)} \leq \left\|\frac{\partial \boldsymbol{u}}{\partial n}\right\|_{H^{-1/2}(\Omega)} + \|p_0\|_{H^{-1/2}(\Omega)}\|\boldsymbol{n}\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{H^{-1/2}(\Omega)}$$

All of these numbers can be computed. Recall that the function $\boldsymbol{n}$ is

$$\boldsymbol{n} : \partial\Omega \to \mathbb{R}^2\,, \quad x \mapsto \boldsymbol{n}(x) = \begin{pmatrix} n_1(x) \\ n_2(x) \end{pmatrix},$$

from which

$$\|\boldsymbol{n}\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} (n_1^2(x) + n_2(x)^2)\,\mathrm{d}x$$

Since $\boldsymbol{n}$ is unitary,

$$\|\boldsymbol{n}\|^2_{L^2(\partial\Omega)}\ (= \text{meas}\ \partial\Omega) < +\infty$$

by hypothesis, on $\partial\Omega$, and $\|\boldsymbol{n}\|_{H^{-1/2}(\partial\Omega)} \leq c_3\|\boldsymbol{n}\|_{L^2(\partial\Omega)}$ by the embedding $L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$. Similarly, because of the continuous embedding

$$i_1 : L^2(\Omega) \to H^{-1/2}(\partial\Omega)\,, \quad p \mapsto p/\partial\Omega\,,$$

then

$$\|p_0\|_{H^{-1/2}(\partial\Omega)} \leq c_4\|p_0\|_{L^2(\Omega)}\,,$$

while the embedding

$$i_2 : (H^1(\Omega)/\mathbb{R})^2 \to \boldsymbol{H}^{-1/2}(\Omega)\,, \quad \boldsymbol{u} \mapsto \frac{\partial \boldsymbol{u}}{\partial n}/\partial\Omega$$

results in

$$\|\partial \boldsymbol{u}/\partial n\|_{\boldsymbol{H}^{-1/2}(\Omega)} \leq c_5 \|\boldsymbol{u}\|_{(H^1(\Omega)/\mathbb{R})^2}$$

So, using (2.17), (2.19), (2.20) and (2.21), the inequality (2.18) can be rewritten as

$$|\lambda| \leq c_6 \|\boldsymbol{u}\|_{(H^1(\Omega)/\mathbb{R})^2} + c_7 \|\phi\|_{H^{-1/2}(\Omega)} \tag{2.22}$$

Letting $v = u$ in (2.10),

$$\|\nabla u\|^2_{L^2(\Omega)} = \langle \boldsymbol{\phi}, \gamma_0 \boldsymbol{u}\rangle\,, \qquad \|\nabla u\|^2_{L^2(\Omega)} \leq \|\phi\|_{H^{-1/2}(\partial\Omega)} \|\boldsymbol{u}\|_{H^{1/2}(\partial\Omega)/\mathbb{R}^2} \tag{2.23}$$

because of $\langle \boldsymbol{\phi}, \boldsymbol{\lambda}\rangle = 0$ with $\boldsymbol{\lambda} \in \mathbb{R}^2$.

Poincaré's lemma gives another result:

$$c_8 \|u\|_{(H^1(\Omega)/\mathbb{R})^2} \leq \|\nabla u\|_{L^2(\Omega)}$$

Eqs. (2.23), (2.24) and the lifting of $\gamma_0 u$ in Ω give

$$c_8^2 \|\boldsymbol{u}\|^2_{(H^1(\Omega)/\mathbb{R})^2} \leq \|\phi\|_{H^{-1/2}(\partial\Omega)} \|u\|_{(H^1/\mathbb{R})^2}$$

or

$$\|u\|_{(H^1(\Omega)/\mathbb{R}^2)^2} \leq \frac{1}{c_8^2} \|\phi\|_{H^{-1/2}(\partial\Omega)}$$

Eqs. (2.14), (2.17), (2.22) and (2.25) finally give the result (2.9) for some constant C which only depends on the geometry of the domain but only for the case $r = \frac{1}{2}$. For regularity results with $r > \frac{1}{2}$, the techniques of [6] apply. For, the problem is then the adjoint boundary value problem of the elliptic problem of Stokes, for which regularity exists.

3. Representation of solutions

In this section we supply a useful representation of solutions.

Denote by $\mathcal{P}^r(\Omega)$, $(r \geq 0)$, the subspace of $\boldsymbol{H}^{r+1/2}(\Omega) \times H^{r+3/2}(\Omega)$ whose elements $\{\boldsymbol{H}, \phi\}$ satisfy

$$\Delta \boldsymbol{H} = 0 \quad \text{in } \Omega\,, \tag{3.1a}$$

$$\Delta\phi + \nabla \cdot \boldsymbol{H} = 0 \quad \text{in } \Omega \tag{3.1b}$$

Let

$$\mathcal{P}_0^r(\Omega) = \{\{\boldsymbol{H}, \phi\} \in \mathcal{P}^r(\Omega) \colon \partial\phi/\partial n = 0 \text{ in } \partial\Omega\}$$

Notice that for elements of $\mathcal{P}_0^r(\Omega)$ one has

$$\int_\Omega \nabla \cdot \boldsymbol{H} \, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{H} \cdot \boldsymbol{n} \, \mathrm{d}x = 0 \, .$$

For every $r \geq 0$, define a mapping

$$f : \mathcal{P}^r(\Omega) \to \hat{\mathcal{U}}^r(\Omega)$$

with the property that the correspondence $\{\boldsymbol{H}, \phi\} \to \{\boldsymbol{u}, p\}$ is given by

$$u = \frac{1}{\nu}(\nabla\phi + \boldsymbol{H}) \, , \qquad p = \Delta\phi \, ,$$

which is clearly continuous with respect to the metrics of these spaces. Stronger properties are enjoyed by the restriction

$$f_0 : \mathcal{P}_0^r(\Omega) \to \hat{\mathcal{U}}^r(\Omega)$$

of f to $\mathcal{P}_0^r(\Omega)$.

THEOREM 3.1. For every $r \geq 0$ the mapping f_0 is bijective (one-to-one and onto $\hat{\mathcal{U}}^r(\Omega)$), between $\mathcal{P}_0^r(\Omega)$ and $\hat{\mathcal{U}}^r(\Omega)$, except for an additive constant in the function $\phi \in H^{r+3/2}(\Omega)$.

PROOF. Let $\{\boldsymbol{u}, p\} \in \hat{\mathcal{U}}^r(\Omega)$ be given. Let $\phi \in H^{r+3/2}(\Omega)$ be such that the second part of (3.5) and the condition $\partial\phi/\partial n = 0$ on ∂R are satisfied. Then by well-known properties of elliptic differential equations [6] such a ϕ exists, is unique except for an additive constant, and depends continuously on p. Now define

$$\boldsymbol{H} = \nu \boldsymbol{u} - \nabla\phi \in H^{r+1/2}(\Omega)$$

Clearly (3.1) is satisfied because (3.7) implies

$$\Delta \boldsymbol{H} = \nu \, \Delta \boldsymbol{u} - \nabla p = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{H} = -\Delta\phi$$

It can be seen that the pair $\{\boldsymbol{H}, \phi\}$ is uniquely determined (except for an additive constant in ϕ) and depends continuously on the pair $\{\boldsymbol{u}, p\}$.

This representation of functions of $\hat{\mathcal{U}}(\Omega)$ is evidently independent of any boundary conditions. Nevertheless, in the next section it will be shown that such a representation can be characterized by the trace of $\boldsymbol{u}$ in $\partial\Omega$.

4. Construction of bases

In this section a method for constructing bases for Stokes problems is developed. It permits building them up from corresponding bases for Laplace's equation.

First, notice that the boundary values of elements $\{\boldsymbol{H}, \phi\} \in \mathcal{P}^r(\Omega)$ are restricted by the condition

$$\int_{\partial\Omega} \boldsymbol{H} \cdot \boldsymbol{n}\, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial \phi}{\partial n}\, \mathrm{d}x$$

by virtue of (3.1b). Hence we define

$$\mathcal{P}^r(\partial\Omega) = \{\{\boldsymbol{H}, \phi\} \in \boldsymbol{H}^r(\partial\Omega) \times H^r(\partial\Omega) \text{: (4.1) holds}\}$$

equipped with the metric of $\boldsymbol{H}^r(\partial\Omega) \times H^r(\partial\Omega)/\mathbb{R}$.

Define the trace mapping

$$g : \mathcal{P}^r(\Omega) \to \mathcal{P}^r(\partial\Omega)$$

as that which yields the correspondence

$$\{\boldsymbol{H}, \phi\} \to \{\gamma_0 \boldsymbol{H}, \gamma_1 \phi\}$$

Here

$$\gamma_0 \boldsymbol{H} = \boldsymbol{H}/\partial\Omega\,, \qquad \gamma_1 \phi = \frac{\partial \phi}{\partial n}/\partial\Omega$$

LEMMA 4.1. The mapping $g : \mathcal{P}^r(\Omega) \to \mathcal{P}^r(\partial\Omega)$ of (4.3) is a bijection and is bicontinuous.

PROOF. There exists a unique solution $\{\boldsymbol{H}, \phi\} \in \mathcal{P}^r(\Omega)$ of

$$\Delta \boldsymbol{H} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{H} = \gamma_0 \boldsymbol{H} \quad \text{on } \partial\Omega$$

and

$$\Delta \phi = -\nabla \cdot \boldsymbol{H} \quad \text{in } \Omega\,, \qquad \frac{\partial \phi}{\partial n} = \gamma_1 \phi \quad \text{on } \partial\Omega$$

which depends continuously on the data $\gamma_0 \boldsymbol{H}$ and $\gamma_1 \phi$. The converse statement is straightforward.

LEMMA 4.2. Define the mapping

$$\boldsymbol{k} : \mathcal{P}^r(\partial\Omega) \to \hat{\mathcal{U}}^r(\Omega)$$

as the composition $k = f \circ g$
Then k is continuous.

PROOF. k is the composition of two continuous mappings.

Consider a system

$$\{\phi_{\gamma\alpha}, H_{\gamma\alpha}\} \subset \mathcal{P}^r(\Omega), \tag{4.8}$$

where $\gamma = 0, 1$ while $\alpha = 1, 2,$, such that for every α,

$$H_{0\alpha} \equiv 0 \quad \text{in } \Omega \tag{4.9}$$

Recall that this implies

$$\Delta\phi_{0\alpha} \equiv 0 \quad \text{in } \Omega \tag{4.10}$$

Using the mapping (3.4) we can define $\hat{u}_{\gamma\alpha} = [u_{\gamma\alpha}, p_{\gamma\alpha}] \in \hat{\mathcal{U}}^r(\Omega)$ by

$$\hat{u}_{\gamma\alpha} = f(\phi_{\gamma\alpha}, H_{\gamma\alpha} \tag{4.11$$

Now let

$$\hat{\mathcal{B}} = \{\hat{u}_{\gamma\alpha} \in \hat{\mathcal{U}}^r(\Omega): \gamma = 0, \quad ; \alpha = 1, 2, \quad .\} \tag{4.12}$$

Using the trace mappings γ_0 and γ_1 given in Section 2, we define

$$\mathcal{B}_1 = \gamma_0\hat{\mathcal{B}}, \qquad \mathcal{B}_2 = \gamma_1\hat{\mathcal{B}} \tag{4.13}$$

This notation is now used in formulating the following theorem.

THEOREM 4.3. For $r \geq \frac{1}{2}$ let $\{\phi_{\gamma\alpha}, H_{\gamma\alpha}\} \subset \mathcal{P}^r(\Omega)$, $(\gamma = 0, 1;\ \alpha = 1, 2, \ldots)$, be such that (4.9) and (4.10) are satisfied. Assume $\{\partial\phi_{01}/\partial n, \partial\phi_{02}/\partial n, \ldots\} \subset H^r(\partial\Omega)$ to span the Hilbert subspace of $H^r(\partial\Omega)$ whose elements $v \in H^r(\partial\Omega)$ satisfy

$$\int_{\partial\Omega} v \, dx = 0 \tag{4.14}$$

Assume in addition the system of traces $\{H_{11}, H_{12},$ to span $H^r(\partial\Omega)$.
Then $\mathcal{B}_1 \subset H^r(\partial\Omega)$ spans $\mathcal{U}_1^r(\partial\Omega)$.

PROOF. We prove that $\mathcal{B}_1 \subset H^r(\partial\Omega) \subset H^0(\partial\Omega)$ spans $H^r(\partial\Omega)$. This clearly implies the desired result.

We shall establish that for any u in $\mathcal{U}_1^r(\partial\Omega)$ and any neighborhood of u, it is possible to construct a linear combination of elements of $\mathcal{B}_1$, belonging to that neighborhood.

Given $\boldsymbol{u}$ in $U_1^r(\partial\Omega)$, by Theorem 2.2 it can be extended in $\hat{U}^r(\Omega)$, uniquely except for an additive constant. Let $\{\boldsymbol{H}, \phi\}$ be the element of $\mathscr{P}_0^r(\Omega)$ related to $\boldsymbol{u}$ and p by (3.5), i.e. $\{\boldsymbol{H}, \phi\} = f_0^{-1}(\{\boldsymbol{u}, p\})$; the definition of $\{\boldsymbol{H}, \phi\}$ depends on the choice of the arbitrary constant in p. In view of Lemma 4.2 it will be enough to show that given $\varepsilon > 0$, it is possible to construct a linear combination $\{\boldsymbol{G}, \psi\}$ of elements of the family $\{\boldsymbol{H}_{\gamma\alpha}, \phi_{\gamma\alpha}\}$ such that

$$\|\boldsymbol{H} - \boldsymbol{G}\|_{\boldsymbol{H}^r(\partial\Omega)} < \varepsilon \quad \text{and} \quad \left\|\frac{\partial\psi}{\partial n}\right\|_{H^r(\partial\Omega)} < \varepsilon$$

Take $\theta \in H^r(\partial\Omega)$ such that

$$\int_{\partial\Omega} \theta \,\mathrm{d}x = 1$$

Given $\varepsilon > 0$ there exists $N_1 > 0$ such that there is a linear combination $\Sigma_{\alpha=1}^{N_1} a_\alpha \boldsymbol{H}_{1\alpha}$ satisfying

$$\left\|\boldsymbol{H} - \sum_{\alpha=1}^{N_1} a_\alpha \boldsymbol{H}_{1\alpha}\right\|_{\boldsymbol{H}^r(\partial\Omega)} < \varepsilon$$

while

$$\|\theta\|_{H^r(\partial\Omega)} \left| \int_{\partial\Omega} \sum_{\alpha=1}^{N_1} a_\alpha \frac{\partial\phi_{1\alpha}}{\partial n} \,\mathrm{d}x \right| < \frac{\varepsilon}{2} \tag{4.18}$$

Relation (4.17) is possible since $\{\boldsymbol{H}_{1\alpha}\}$ is complete in $\boldsymbol{H}^r(\partial\Omega)$; while (4.18) follows from (3.3) and (4.1), using the facts that

$$\sum_{\alpha=1}^{N_1} a_\alpha \int_{\partial\Omega} \boldsymbol{H}_\alpha \cdot \boldsymbol{n} \,\mathrm{d}x = -\sum_{\alpha=1}^{N_1} a_\alpha \int_{\partial\Omega} \frac{\partial\phi_{1\alpha}}{\partial n} \,\mathrm{d}x \quad \text{and} \quad \int_{\partial\Omega} \boldsymbol{H} \cdot \boldsymbol{n} \,\mathrm{d}x = 0$$

Once N_1 has been chosen, take N_2 such that there is a linear combination $\Sigma_{\alpha=1}^{N_2} b_\alpha \phi_{0\alpha}$ satisfying

$$\left\|\sum_{\alpha=1}^{N_1} a_\alpha \frac{\partial\phi_{1\alpha}}{\partial n} - c_{N_1}\theta + \sum_{\alpha=1}^{N_2} b_\alpha \frac{\partial\phi_{0\alpha}}{\partial n}\right\|_{H^r(\partial\Omega)} < \tfrac{1}{2}\varepsilon$$

where

$$c_{N_1} = \sum_{\alpha=1}^{N_1} a_\alpha \int_{\partial\Omega} \frac{\partial\phi_{1\alpha}}{\partial n} \,\mathrm{d}x$$

This choice of N_2 is possible because

$$\int_{\partial\Omega} \left\{ \sum_{\alpha=1}^{N_1} a_\alpha \frac{\partial\phi_{1\alpha}}{\partial n} - c_{N_1}\theta \right\} \mathrm{d}x = 0 \tag{4.21}$$

and $\{\partial\phi_{0\alpha}/\partial n\}_{\alpha=0}$ spans the subspace of $H^r(\partial\Omega)$ defined by (4.14).

So it is easy to see that

$$G = \sum_{\alpha=1}^{N_1} a_\alpha H_{1\alpha} \quad \text{and} \quad \psi = \sum_{\alpha=1}^{N_1} a_\alpha \phi_{1\alpha} + \sum_{\alpha=1}^{N_2} b_\alpha \phi_{0\alpha}$$

satisfy (4.15).

So the theorem is proved.

It means that the family $\{u_{\gamma\alpha}\}_{\gamma=0,1,\,\alpha\in N}$ defined by (4.11) is a complete system of solutions for Stokes problem No. 1, in the sense that any solution can be approximated by linear combinations of this family. This is clear because of Theorem 2.2, which ensures the bicontinuity of the trace operator γ_0.

5. More general boundary conditions

The results of Section 4 can be used to construct complete systems of solutions for many boundary value problems. This versatility is due to a theory recently developed by one of the authors [3, 8].

Let $\hat{D} = H^\circ(\partial\Omega) \oplus H^\circ(\partial\Omega)$ and denote the elements $\hat{u} \in \hat{D}$ by $\hat{u} = [u_1, u_2]$. Define a linear mapping $\mathscr{L} : H^{3/2}(\Omega) \oplus H^{1/2}(\Omega) \to \hat{D}$, such that for every $[u, p] \in H^{3/2}(\Omega) \oplus H^{1/2}(\Omega)$,

$$\mathscr{L}[u, p] = \hat{u} = [u_1, u_2] = [\gamma_0 u, \gamma_1(u, p)]$$

where the trace operators γ_0 and γ_1 of Section 2 have been used.

Let the bilinear form $A : \hat{D} \to \hat{D}^*$, where $\hat{D}^*$ is the algebraic dual of $\hat{D}$, be given by

$$\langle A\hat{u}, \hat{v}\rangle = (u_1, v_2) - (u_2, v_1)$$

Here (,) stands for the inner product in $H^\circ(\partial\Omega)$. Notice that when

$$\hat{u} = \mathscr{L}(u, p) \quad \text{and} \quad \hat{v} = \mathscr{L}(v, p)$$

there follows

$$\langle A\hat{u}, \hat{v}\rangle = \int_{\partial\Omega} \{u(\partial v/\partial n - qn) - v(\partial u/\partial n - pn)\}\, dx$$

Let

$$\hat{\mathscr{I}}_P = \{\mathscr{L}(u, p) \in \hat{D} : [u, p] \in \hat{\mathscr{U}}^1(\Omega)\}$$

More generally, define $\hat{I}^r_P$, $0 \leq r \leq \infty$ and replace $\hat{\mathscr{U}}^1(\Omega)$ by $\hat{\mathscr{U}}^{r+1}(\Omega)$ in (5.5). Adopting the notation $\hat{D}_r = H^r(\partial\Omega) \oplus H^r(\partial\Omega)$, it is easy to see that $\hat{I}^r_P \subset \hat{D}_r$.

Theorems 2.2 and 2.3 can be used to prove the following results

THEOREM 5.1. For every r, $0 \leq r \leq \infty$, one has

$$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{v} \in \hat{\mathscr{I}}^r_P \Leftrightarrow \hat{u} \in \hat{\mathscr{I}}^r_P$$

whenever $\hat{u} \in \hat{D}_r$.

PROOF. The implication from right to left states the fact that $\hat{I}^r_P$ is a commutative subspace for A. This follows from the fact that when $[\boldsymbol{u}, p]$ and $[\boldsymbol{v}, q]$ satisfy (2.1), one has

$$\int_\Omega \{(-\Delta \boldsymbol{u} + \operatorname{grad} p) \cdot \boldsymbol{v} + q \operatorname{div} \boldsymbol{u}\} \, \mathrm{d}x = 0$$

and simultaneously

$$\int_\Omega \{(-\Delta \boldsymbol{v} + \operatorname{grad} q) \cdot \boldsymbol{u} + p \operatorname{div} \boldsymbol{v}\} \, \mathrm{d}x = 0\,.$$

Subtracting (5.8) from (5.7) gives precisely the 'boundary' relation $\langle A\hat{u}, \hat{v}\rangle = 0$.

To deal with the implication from left to right, let the sign $\perp$ denote orthogonality in $H^\circ(\partial\Omega)$. Let us first remark that

$$\mathscr{U}^r_1 = \{\boldsymbol{n}\}^\perp \cap \boldsymbol{H}^r(\partial\Omega)$$

because the existence Theorem 2.2 proves precisely that any element of the space on the right is the trace of a solution of the Stokes problem.

Let $\hat{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2] \in \hat{D}_r$ and choose $\hat{v} = [\boldsymbol{0}, \boldsymbol{n}] \in \hat{\mathscr{I}}^r_P \subset \hat{D}_r$, where $\boldsymbol{n}$ is the unit vector normal to the boundary $\partial\Omega$; then the left-hand side of (5.6) becomes $(\boldsymbol{u}_1, \boldsymbol{n}) - (\boldsymbol{u}_2, \boldsymbol{0}) = 0$. This implies $\boldsymbol{u}_1 \in \{\boldsymbol{n}\}^\perp$. Hence $\boldsymbol{u}_1 \in \mathscr{U}^r_1$.

This result, together with Theorem 2.2, implies that there is a pair $[\boldsymbol{u}', p']$ which satisfies (2.1), and such that $\mathscr{L}[\boldsymbol{u}', p'] = [\boldsymbol{u}_1, \boldsymbol{u}'_2] \in \hat{\mathscr{I}}^r_P$. Therefore, for every $\hat{v} = [\boldsymbol{v}_1, \boldsymbol{v}_2] \in \hat{\mathscr{I}}^r_P$, one has $(\boldsymbol{u}_1, \boldsymbol{v}_2) - (\boldsymbol{u}'_2, \boldsymbol{v}_1) = 0$ and simultaneously $(\boldsymbol{u}_1, \boldsymbol{v}_2) - (\boldsymbol{u}_2, \boldsymbol{v}_1) = 0$. Hence, $\boldsymbol{u}_2 - \boldsymbol{u}'_2 \in \{\mathscr{U}_1\}^\perp = \{\boldsymbol{n}\}$. This means that there is a constant c, for which $\mathscr{L}[\boldsymbol{u}', p' + c] = [\boldsymbol{u}_1, \boldsymbol{u}_2]$. Therefore $[\boldsymbol{u}_1, \boldsymbol{u}_2] \in \hat{I}^r_P$.

A linear subspace of $\hat{D}_r$ enjoying property (5.6), is said to be completely regular with respect to $A_r : \hat{D}_r \to \hat{D}^*_r$ [3]. Here $A_r : \hat{D}_r \to \hat{D}^*_r$ is the restriction of $A : \hat{D} \to \hat{D}^*$ to $\hat{D}_r$. Therefore, Theorem 5.1 states that $\hat{\mathscr{I}}^r_P \subset \hat{D}_r$ is completely regular for $A_r : \hat{D}_r \to \hat{D}^*_r$, whenever $0 \leq r \leq \infty$. Clearly

$$\hat{\mathscr{I}}_P \supset \hat{\mathscr{I}}^1_P \supset \cdots \supset \hat{\mathscr{I}}^\infty_P$$

Let

$$\mathscr{I}^r_{P1} = \{u_1 \colon \exists \hat{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2] \in \hat{\mathscr{I}}^r_P\} \subset \boldsymbol{H}^r(\partial\Omega)\,, \tag{5.10a}$$

$$\mathscr{I}^r_{P2} = \{u_2 \colon \exists \hat{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2] \in \hat{\mathscr{I}}^r_P\} \subset \boldsymbol{H}^r(\partial\Omega)\,. \tag{5.10b}$$

It is easy to see that the stronger relation $\mathcal{I}^r_{P1} \subset H^{r+1}(\partial\Omega)$ holds; however, (5.10a) is the property we need for the discussion that follows. In view of the above and (2.3), (2.4), it is clear that

$$(\mathcal{I}^r_{P1})^\perp = \{\boldsymbol{n}\}, \qquad (\mathcal{I}^r_{P2})^\perp = \{\boldsymbol{\lambda}\} \tag{5.}$$

Here we have written $\{\boldsymbol{n}\}$ for the one-dimensional linear subspace of functions defined on $\partial\Omega$, generated by the function whose value is the unit normal vector $\boldsymbol{n}$; similarly, $\{\boldsymbol{\lambda}\}$ stands for the two-dimensional subspace of functions which are constant on $\partial\Omega$.

Therefore

$$\boldsymbol{H}^\circ(\partial\Omega) \supset \boldsymbol{H}^r(\partial\Omega) \supset (\mathcal{I}_{P1})^\perp + (\mathcal{I}_{P2})^\perp \tag{5.12}$$

where the orthogonal complements on the right are taken in $\boldsymbol{H}^\circ(\partial\Omega)$. Notice also that $\boldsymbol{H}^r(\partial\Omega)$ is dense in $\boldsymbol{H}^\circ(\partial\Omega)$. Due to these facts Theorem 10.1 of [5] is applicable; this implies that $\hat{\mathcal{I}}^r_P$ can be extended in a unique manner to become a linear subspace (not necessarily closed) which is completely regular for $A : \hat{D} \to \hat{D}^*$. It can be seen that this linear manifold is $\hat{\mathcal{I}}_P$.

The following concept will be useful for characterizing those systems of functions in terms of which any solution of (2.1) may be approximated in Ω.

DEFINITION 5.2. A set $\hat{\mathcal{B}} \subset \hat{\mathcal{I}}^r_P$ is *c-complete in* $\hat{\mathcal{I}}^r_P$ *with respect to* A_r $\hat{D}_r \to \hat{D}^*_r$, provided that

$$\langle A\hat{u}, \hat{w}\rangle = 0 \quad \forall \hat{w} \in \hat{\mathcal{B}} \Rightarrow \hat{u} \in \hat{\mathcal{I}}^r_P \tag{5.13}$$

whenever $\hat{u} \in \hat{D}_r$. $\hat{\mathcal{B}}$ is said to be a *connectivity basis* when, in addition, $A\hat{\mathcal{B}} \subset \hat{D}_r$ is linearly independent.

Given $\hat{\mathcal{B}} \subset \hat{D}$ it is convenient to introduce the notation

$$\mathcal{B}_1 = \{u_1 \in \boldsymbol{H}^\circ(\partial\Omega) : \exists [u_1, u_2] \in \hat{\mathcal{B}}\}, \qquad \mathcal{B}_2 = \{u_2 \in \boldsymbol{H}^\circ(\partial\Omega) : \exists [u_1, u_2] \in \hat{\mathcal{B}}\}. \tag{5.14}$$

A connection between c-complete systems and connectivity bases is supplied by the following result.

THEOREM 5.3. Let $r > 0$. Assume $\hat{\mathcal{B}} \subset \hat{\mathcal{I}}^r_P$. Then the following assertions are equivalent:

(i) $\hat{\mathcal{B}}$ *is c-complete for* $\hat{\mathcal{I}}^r_P$ *with respect to* $A_r : \hat{D}_r \to \hat{D}^*_r$;

(ii) $\hat{\mathcal{B}}$ *is c-complete for* $\hat{\mathcal{I}}_P$ *with respect to* $A : \hat{D} \to \hat{D}^*$;

(iii) span $\mathcal{B}_1 = \{\boldsymbol{n}\}^\perp$ *and* span $\mathcal{B}_2 \supset \{\boldsymbol{n}\}$; (5.15)

(iv) span $\mathcal{B}_2 = \{\boldsymbol{\lambda}\}^\perp$ *and* span $\mathcal{B}_1 \supset \{\boldsymbol{\lambda}\}$. (5.16)

PROOF. This theorem is implied by Theorem 10.2 [5].

For the application of this theorem to Stokes problems (2.6) and (2.8) of Section 2, it is recalled that

$$\bar{\mathscr{I}}_{P1} = \bar{\mathscr{U}}_1^\circ = \{\boldsymbol{n}\}, \qquad \mathscr{I}_{P2} = \bar{\mathscr{I}}_{P2} = \mathscr{U}_2^\circ(\partial\Omega) = \{\boldsymbol{\lambda}\}^\perp$$

Consider a connectivity basis $\hat{\mathscr{B}} = \{\hat{w}, \hat{w}_2, \quad\} \subset \hat{\mathscr{I}}_P$ where $\hat{w}_\alpha = [w_{\alpha 1}, w_{\alpha 2}]$, and let $[w,$ $\hat{\mathscr{U}}^{1/2}(\Omega)$ be such that

$$w_{\alpha 1} = w_\alpha \quad \text{on } \partial\Omega, \qquad w_{\alpha 2} = \partial w_\alpha/\partial n - p_\alpha \boldsymbol{n} \quad \text{on } \partial\Omega$$

By virtue of Theorem 5.3 we know that $\{w_{11}, w_{21}, \ldots\}$ spans $\bar{\mathscr{U}}_1^\circ(\partial\Omega)$, while $\{w_{12}, w_{22},$ $\mathscr{U}_2^\circ(\partial\Omega)$.

Let $u_1 \in \mathscr{U}_1^\circ(\partial\Omega)$ be given and consider Stokes problem No. We look for $[\boldsymbol{u}, p] \in \hat{\mathscr{U}}^{1/2}(\Omega)$ such that

$$\gamma_0 \boldsymbol{u} = u_1 \quad \text{on } \partial\Omega$$

Then we get the following approximating sequence

$$[\boldsymbol{u}^N, p^N] = \sum_{\alpha=1}^{N} a_\alpha^N [w_\alpha, p_\alpha] \tag{5.20}$$

where a_α^N are chosen so that

$$\|\gamma_0 \boldsymbol{u}^N - u_1\|^2 = (\gamma_0 \boldsymbol{u}^N - u_1, \gamma_0 \boldsymbol{u}^N - u_1) \tag{5.21}$$

is minimized. This choice is consistent with $\gamma_0 \boldsymbol{u}^N \to u_1$ on $\partial\Omega$, because $\{w_{11}, w_{21}, \ldots\}$ spans $\mathscr{U}_1^\circ(\partial\Omega)$, and therefore $[\boldsymbol{u}^N, p^N] \to [\boldsymbol{u}, p]$, by virtue of Theorem 2.2. Notice that the latter limit is taken in $\boldsymbol{H}^{1/2}(\Omega) \oplus H^{-1/2}(\Omega)/\mathbb{R}$.

Similarly, when Stokes problem (2.8) is considered and $u_2 \in \mathscr{U}_2^\circ(\partial\Omega)$, the approximating sequence is given again by (5.20), except that now a_α^N is chosen so that $\|\gamma_1(\boldsymbol{u}^N, p^N) - u_2\|^2$ is minimized. Again, this choice allows that $\gamma_1(\boldsymbol{u}^N, p^N) \to u_2$ on $\partial\Omega$, because $\{w_{12}, w_{22}, \ldots\}$ is complete in $\mathscr{U}_2^\circ(\partial\Omega)$; hence $[\boldsymbol{u}^N, p^N] \to [\boldsymbol{u}, p]$ by virtue of Theorem 2.3. The limit is taken in $(\boldsymbol{H}^{3/2}(\Omega)/\mathbb{R}^2) \oplus H^{1/2}(\Omega)$ and therefore also in $(\boldsymbol{H}^{1/2}(\Omega)/\mathbb{R}^2) \oplus H^{-1/2}(\Omega)$.

Notice that when Stokes problem (2.8) is considered, $\gamma_0 \boldsymbol{u}^N \to \gamma_0 \boldsymbol{u}$ in the $\boldsymbol{H}^1(\partial\Omega)$ norm and hence also in the $\boldsymbol{H}^\circ(\partial\Omega)$ norm. However, when problem (2.6) is considered, one cannot expect that $\gamma_1(\boldsymbol{u}^N, p) \to \gamma_1(\boldsymbol{u}, p)$, in the $\boldsymbol{H}^\circ(\partial\Omega)$ topology; indeed in general $\gamma_1(\boldsymbol{u}, p)$ may not exist in $\boldsymbol{H}^\circ(\partial\Omega)$. A necessary and sufficient condition for its existence in $\boldsymbol{H}^\circ(\partial\Omega)$, is that $u_1 \in \boldsymbol{H}^1(\partial\Omega)$. If the data satisfy this condition, then a sequence approximating to $\gamma_1(\boldsymbol{u}, p)$ is given by

$$u_2^N = \sum_{\alpha=1}^{N} b_\alpha^N w_{\alpha 1}$$

where b_α^N is chosen so that $\|\gamma_1 \boldsymbol{u}^N - u_2\|^2$ (with norm is in $\boldsymbol{H}^\circ(\partial\Omega)$) is minimized. This implies the following system of equations for b_α^N ($\alpha = 1, 2, \ldots, N$):

$$\sum_{n=1}^{N} b_\alpha^N (w_{\alpha 1} \; w_{\beta 1}) = (u_2, w_{\alpha 1}) = (u_1, w_{\alpha 2}). \tag{5.23}$$

Notice that the right-hand member of (5.23) is expressed in terms of the data of the problem only.

Many other problems can be treated with the same c-complete system $\hat{\mathscr{B}} \subset \hat{\mathscr{I}}_P$. Indeed, let $\mathscr{L} : \boldsymbol{H}^{3/2}(\Omega) \oplus H^{1/2}(\Omega) \to \hat{D} = \boldsymbol{H}^{\circ}(\partial\Omega) + \boldsymbol{H}^{\circ}(\partial\Omega)$ be any other linear mapping and retain the notation introduced in (5.1) and (5.2), except that the third equality of (5.1) is here left out; i.e. the notation to be used is defined by (5.2) and

$$\mathscr{L}[\boldsymbol{u}, p] = \hat{u} = [u_1, u_2] \tag{5.24}$$

In addition, it will be assumed that the mapping $\mathscr{L}$ is such that (5.4) holds.

There are many mappings satisfying these conditions. If we restrict attention to those which are point-wise, then it can be shown that the most general one is

$$u_1 = A(x)u + B(x)\left(\nu \frac{\partial u}{\partial n} - pn\right), \qquad u_2 = C(x)u + D(x)\left(\nu \frac{\partial u}{\partial n} - pn\right)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 square matrices, such that $\boldsymbol{AB}^{\mathrm{t}}$ and $\boldsymbol{CD}^{\mathrm{t}}$ are symmetric, while the 4×4 matrix

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

is non-singular. A special case of this situation is when

$$\boldsymbol{C} = \boldsymbol{B}^{\mathrm{t}}, \qquad \boldsymbol{D} = \boldsymbol{A}^{\mathrm{t}}$$

and the rank of the rectangular matrix $[\boldsymbol{A}, \boldsymbol{B}]$ is two.

Particular cases of special interest are

EXAMPLE 5.4. $\boldsymbol{A}$ and $\boldsymbol{B}$ are scalars; i.e. $\boldsymbol{A} = \alpha\boldsymbol{I}$, $\boldsymbol{B} = \beta\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix

EXAMPLE 5.5. There is a partition of the boundary into Γ_1 and Γ_2 such that $\boldsymbol{A}(x) = 0$ on Γ_1 and $\boldsymbol{B}(x) = 0$ on Γ_2.

6. Construction of c-complete systems of solutions

Theorem 5.3 exhibits a criterion that a set $\hat{\mathscr{B}} \subset \hat{\mathscr{I}}_P$ is a c-complete system of solutions for the Stokes equations (2.1); for example condition (iii) of that theorem. Assuming that $\hat{\mathscr{B}}$ is denumerable, write

$$\hat{\mathscr{B}} = \{\hat{w}_1, \hat{w}_2, \ldots\} = \{[w_{11}, w_{12}], [w_{21}, w_{22}], \ldots\},$$
$$\mathscr{B}_1 = \{w_{11}, w_{21}, \ldots\}, \qquad \mathscr{B}_2 = \{w_{21}, w_{22}, \ldots\}.$$

It is then required that

$$\text{span } \mathscr{B}_1 = \{\boldsymbol{n}\}^{\perp} \quad \text{and} \quad \text{span } \mathscr{B}_2 \supset \{\boldsymbol{n}\}$$

Noting that the pair $[\mathbf{0}, 1] \in \hat{\mathscr{U}}'(\Omega)$, and that, in view of (5.1),

$$\mathscr{L}[\mathbf{0}, 1] = [\mathbf{0}, \boldsymbol{n}] \subset \hat{\mathscr{I}}_P$$

it is natural to assume that span $\mathscr{B}_2 \supset \{\boldsymbol{n}\}$. Thus it will only be required to construct a system $\hat{\mathscr{B}}$, such that

$$\text{span } \mathscr{B}_1 \supset \{\boldsymbol{n}\}^{\perp}$$

Theorem 4.3 yields a procedure for constructing such a system starting from a corresponding set for Laplace's equation. Thus, recall that the system

$$\{\text{Re } z^{\alpha}, \text{Im } z^{\alpha}; \alpha = 0, 1, 2, \ \ .\}$$

is c-complete for any bounded region [11], and define

$$\psi_1 \quad 1\,, \quad \psi_{2\alpha} = \text{Re } z^{\alpha}\,, \quad \psi_{2\alpha+1} = \text{Im } z^{\alpha}\,, \qquad \alpha = 1, 2, \ldots$$

In view of the c-completeness of the system (6.5), it is clear that if we take in what follows $\gamma = 1, 2, \ldots, i = 1, 2$. Then the family $\{\phi_{0,\gamma}; \mathbf{0}\} \cup \{\phi_{1,\gamma}; \boldsymbol{H}_\gamma\}$, defined by

$$\phi_{0,\gamma} = \psi_{\gamma+1}\,, \qquad \phi_{1,\gamma} = -\sum_{i=1}^{2} \tfrac{1}{2} x_i H_{\gamma,i}$$

with

$$H_{\gamma,i} = 0 \quad \text{if } \gamma + i \text{ is odd}\,, \qquad H_{2p+i,i} = \psi_{p+1}\,, \quad p = 0, 1, 2,$$

is a c-complete system by (4.11).

In applications, solutions of Stokes equations (2.1) in an unbounded region which is not simply connected, deserve special attention, since they occur in the important problem of flow past a body. When the region considered is the complement of a bounded simply connected region (Fig. 2), we may formulate the homogeneous Stokes problem, supplementing either (2.6) or (2.8) with Sommerfeld's radiation conditions at infinity.

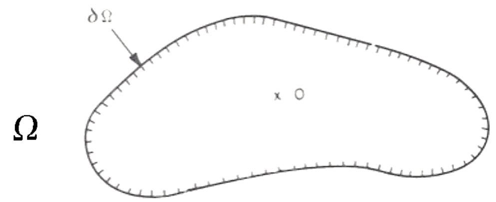

Fig.

A c-complete system for Laplace's equation in such region, satisfying radiation conditions, is [11]

$$\{\mathrm{Re}(\log z), \mathrm{Re}\, z^{-\alpha}, \mathrm{Im}\, z^{-\alpha}; \alpha = 1, 2, \ldots \tag{6.9}$$

Therefore the family $\{\phi_{0,\gamma}; \mathbf{0}\} \cup \{\phi_{1,\gamma}; \boldsymbol{H}_\gamma\}$ defined by

$$\phi_{0,\gamma} = \psi_\gamma, \qquad \phi_{1,\gamma} = -\sum_{i=1}^{2} \tfrac{1}{2} x_i H_{\gamma,i} \tag{6.10}$$

and (6.8), is again a c-complete system for this problem if we replace the definition (6.6) of ψ_α $(\alpha = 1, 2, \ldots)$ by

$$\psi_1 = \mathrm{Re}(\log z), \qquad \psi_{2\alpha} = \mathrm{Re}\, z^{-\alpha} \qquad \psi_{2\alpha+1} = \mathrm{Im}\, z^{-\alpha} \tag{6.11}$$

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