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Kinam, Vol 3, 161-229 (1981) TESTIMONIO DE FISICA

An algebraic theory of boundary value problems

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(Recibido el 6 de junio de 1981)

An abstract theory for boundary value problems that has been recently developed by the author, is presented. It exhibits the algebraic structure possessed by linear problems. A characterization is given for complete systems of solutions in regions of arbitrary form. A systematic development of biorthogonal systems of solutions is presented and it is used to obtain generalized Fourier series. General variational principles for boundary value problems are formulated. They include problems with prescribed jumps and subjected to continuation type restrictions. Examples of applications are given for fluid and solid mechanics; among them, theory of plates, Stokes flows, diffraction problems in elastic solids, etc.

Note: By invitation, this paper was presented at a special session on Abstract Adjoints and Boundary Problems, at the 89th. Annual Meeting of AMS (Denver, January 1983).

I. INTRODUCTION

In a sequence of papers [1-28], the author has developed an abstract theory of boundary value problems, exhibiting an algebraic structure which systematically occurs in boundary value problems which are linear. This structure is in the first place, interesting in itself, because of its simplicity and beauty. But, more important, it has relevant applications. Thus far, these applications have been mainly along three different lines: variational principles, numerical solution of boundary value problems and development of biorthogonal systems of functions, to obtain generalized Fourier series developments.

The abstract theory is formulated for general functional-valued operators defined on arbitrary linear spaces which, generally, do not possess a metric or an inner product. This supplies greater flexibility, for applications of the theory, than standard approaches to this kind of problems.

The notions of boundary operator, formal adjoint, and formal symmetry are defined abstractly for these operators. This allows the introduction of abstract Green's formulas.

Regular and completely regular subspaces play an important role in the theory. By a canonical decomposition of the basic linear space D, it is meant a pair of regular subspaces $\{I_1, I_2\}$ which span the total space D.

An important result of the theory is a one-to-one correspondence between Green's formulas and canonical decompositions; furthermore, an explicit expression for the Green's formula associated with any canonical decomposition is supplied. Another important property is that the regular subspaces I_1 and I_2, occurring in any canonical decomposition, are necessarily completely regular. This is relevant for the representation of solutions, because completely regular subspaces are characterized by the fact that they possess connectivity bases.

Problems formulated in discontinuous fields, which satisfy prescribed jump conditions, occur frequently in applications; for example, diffraction problems in acoustics, electromagnetism (theory of antennas), and elasticity (seismology, seismic engineering, etc.). As a very general example of application, operators associated to such problems are developed abstractly and Green's formulas are given for them.

A general problem with linear restrictions, associated with any functional valued operator $P : D \to D^*$, is formulated. The general solution of the homogeneous equation ($Pu = 0$) is N_P; the space of functions which take the same boundary values as elements of N_P, is introduced abstractly, and it is denoted by I_P. For the representation of solutions the notion of c-complete system (complete in connectivity), is very important. As mentioned before, a regular subspace possesses a c-complete system, if and only if, it is completely regular. The subspace, I_P, is always regular and, under very general conditions, it is, in addition, completely regular. Thus, I_P and, even more, N_P, possess connectivity bases. This supplies the basis for the representation of solutions [22, 25].

Most of the developments of the general theory of partial differential equations have been carried out in the setting of Hilbert spaces [29, 30]. Thus, it was convenient to relate the algebraic theory with that framework. This is achieved by means of a theorem that shows that a system of solutions is c-complete, if and only if, it spans the same space as the boundary values of solutions $u \in N_P$, of the homogeneous equation.

A first line of applications of the theory is the development of variational principles. This is a subject which attracted much attention in recent years (see for example Refs., 31 and 32), because of its relevance in connection with the numerical treatment of partial differential equations. This is, at the same time, the most direct kind of application, because with every Green's formula, there is a variational principle associated. Also, a class of variational principles has been developed [19], which can be applied even when no Green's formula is available. Besides, if the functional is convex, the variational principle becomes a maximum principle; when it is saddle, dual extremal principles are obtained [3, 6, 8].

The second line of applications has been in connection with the numerical solution of boundary value problems. In recent years by a boundary method it is usually understood a numerical procedure in which a subregion or the entire region is left out of the numerical treatment by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problem leading to considerable economy in the numerical work and constitute a very convenient manner of treating adequately unbounded regions by numerical means. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced [33, 34].

There are two main approaches for the formulation of boundary methods; one is based on the use of boundary integral equations, and the other one on the use of complete systems of solutions. In numerical applications, the first one of these methods has received most of the attention [35]. This is in spite of the fact that the use of complete systems of solutions presents important numerical advantages; e.g., it avoids the introduction of singular integral equations and it does not require the construction of a fundamental solution. The latter is especially relevant in connection with complicated problems, for which it may be extremely laborious to build up a fundamental solution. This is illustrated by the fact that there are methods for synthetizing fundamental solutions starting from plane waves, which can be shown to be a complete system [28].

One may advance some possible explanations for this situation. Although the principle of superposition is a standard procedure for building up solutions of linear equations, many of its applications have been based on the method of separation of variables; this has lead to the frequent, but false, belief that complete systems of solutions have to be constructed specifically for a given region. Of course, this is not the case; most frequently, systems of solutions are complete independently of the detailed shape of the region considered. In Refs. 13, 25 and 28 we have exhibited systems which are complete for any bounded region, and other ones possessing the same property in the exterior of any bounded domain.

Also, in some fields of application, procedures which constitute particular cases of the approximation by complete systems of solutions have presented severe restrictions and inconveniences. For the case of acoustics and electromagnetic field computations, a survey of such difficulties was carried out by Bates [36]. For this kind of studies, the so called "Rayleigh hypothesis" restricts drastically the applicability of the method. However, work by Millar [37] implies that these difficulties are due, mainly, to lack of clarity, since he avoided Rayleigh hypothesis altogether, by adopting a different point of view. Work by other authors has similar implications (see for example Oliveira [38]).

Motivated by this situation, the author started a systematic research of the subject [9, 10, 12-14, 16, 17, 22]. The aim of the study was two-fold; firstly, to clarify the theoretical foundations required for using complete systems of solutions in a reliable manner, and secondly, to expand the versatility of such methods, making them applicable to any problem which is governed by partial differential equations that are linear.

The aims of that research were satisfactorily achieved to a large extent, and several reports have already appeared [20, 21]; in addition, two more complete ones are now in press [23, 24]. The task was facilitated by the progress that has been made in the understanding of partial differential equations [29]. In addition, the methodology bears some relation with ideas that had been advanced by Amerio, Fichera, Picone, Kupradze, and Trefftz, [39-44]. The systematic development of the procedures in a manner which is applicable to any linear problem was made possible, however, by the algebraic theory developed by the author.

The theory encompases the following aspects:

1. Development of algorithms for computing the solution in the region and on its boundary.

2. Conditions under which the convergence of the procedure can be granted.

3. Development of criteria for the completeness of a given system of solutions.

4. General methods for developing complete systems of solutions.

In addition, the variational principles mentioned before can be used to formulate these problems. This can be especially useful when part of the region is treated numerically [33, 34].

The third line of applications has been in the development of biorthogonal systems of functions to obtain generalized Fourier series. Biorthogonal systems of functions, which occur when applying the method of separation of variables to fourth order equations, such as the biharmonic equation, have received much attention in recent years (see for example Refs. 45-49).

In general, the procedure followed by those authors consists in exhibiting a differential equation satisfied by the boundary values of any solution. Then the adjoint of this differential equation is constructed, and it is shown that the eigenfunctions of these two systems are biorthogonal. In this manner, a formal expansion for any solution of the original differential equation is obtained, in which the coefficients are easily derived by means of the biorthogonality relation. Further analysis is required in order to establish the completeness of the system of biorthogonal functions and the convergence of the expansion. Apparently, Smith [50] was the first to deal with these problems successfully. Joseph [47] has exhibited in some recent work the considerable generality of the method by applying it to a good sample of different problems.

The procedure, however, is not completely satisfactory in some respects. In particular, the development of the differential equation for the boundary values, and its adjoint, has an *ad hoc* character which bears little, or no relation, with the physical situation at hand.

In geophysical studies an independent approach has been followed to obtain also biorthogonal systems of functions. Indeed, in this field, Herrera's [51] orthogonality relations for Rayleigh waves have been known since 1964 [52, 53]. The argument used by Herrera to derive such relations allows complete generality, if suitably formulated, but had remained unnoticed until recently by researchers working in other kinds of applications.

Herrera and Spence [27] have explained how the algebraic theory can be used to generalize Herrera's [51] procedure to obtain biorthogonal systems of solutions for a wide class of equations. Essentially, the method consists in considering the linear space of solutions N_P, given a partial differential equation which is linear and homogeneous. As it is shown in Sec. XI of this paper, under quite general conditions, the space N_P can be decomposed into two linear subspaces N_P^1 and N_P^2, which are commutative for an antisymmetric bilinear form A_0. Generally, product form solutions yield two families $\{w_1, w_2, \ldots\} \in N_P^1$ and $\{w_1^*, w_2^*, \ldots\} \in N_P^2$, which are necessarily biorthogonal with respect to the bilinear form A_0. When the systems of biorthogonal solutions are c-complete; i.e., when, for every $u \in N_P$

$$\langle A_0 u, w_\alpha \rangle = 0 \quad \forall \alpha = 1, 2, \quad \Rightarrow u \in N_P^1 \ ,$$

arbitrary solutions can be developed in a direct manner.

The algebraic theory supplies a general framework which places Fourier biorthogonal systems in a more clear perspective. The biorthogonal systems are associated with corresponding canonical decompositions of the space of solutions N_P. The bilinear functional A_0, which defines the biorthogonality relation, is clearly related to the partial differential equations considered, and is derivable from the corresponding operator by integration by parts; no auxilliary adjoint differential system is required. The relation between c-completeness and the notion of Hilbert-space bases is well established; this is discussed in Sec. X of the present article. The generality of the procedure is wide; it is not restricted by the order of the partial differential equations, and it is applicable to equations with variable coefficients, as it is the case of Herrera's [51] orthogonality relations for Rayleigh waves, which hold for waveguides with arbitrary transversal heterogeneity. Finally, the biorthogonality property appears as a relation that is satisfied by pairs of solutions in the whole region where they are defined, instead of satisfying it just at the boundary. This fact is specially useful when matching solutions in different regions or when modifying the region of definition.

In this paper the algebraic theory is developed systematically. The abstract framework in which the theory is developed, is presented in Sec. II. Formal adjoints and abstract Green's formulas are introduced in Sec. III. Section IV is devoted to regular subspaces and canonical decomposition, establishing their one-to-one correspondence with abstract Green's formulas, leaving for Sec. V the introduction of the general problem with linear restrictions considered by the theory. Generalizations of these notions and variational principles are discussed in sections VI and VII. The power of the methodology is exhibited in Sec. VIII, by formulating problems in discontinuous fields, with prescribed jump conditions, in an abstract manner which is applicable independently of the specific operator considered, as long as it is linear; thus, any linear partial differential (or system of differential equations) is included. From the point of view of continuum mechanics, this includes solids and liquids; even more, mixed systems in which part of the space is occupied by a liquid and another part by a solid (the details are given in Ref. 19). Results which are relevant for the representation of solutions are given in sections IX and X. In Secs. XI-XV, the application of the theory to biorthogonal Fourier series is presented.

II. PRELIMINARY NOTIONS AND NOTATIONS

Let D be a linear space over the field F of real or complex numbers. Elements of D will be denoted by $u, v, \ldots$. Write D^* for the linear space of linear functionals defined on D; i.e., D^* is the algebraic dual of D. Hence, any element $\alpha \in D^*$ is a function $\alpha : D \to F$ which is linear. Given $v \in D$, the value of the function α at v will be denoted by

$$\alpha(v) = \langle \alpha, v \rangle \in F$$

In this work, functional-valued operators

$$P : D \to D^*$$

will be considered. Given $u \in D$, the value $P(u) \in D^*$ is itself a linear functional. According with (2.1), given any $v \in D$, $\langle P(u), v\rangle \in F$ will be the value of this linear functional at v. When the operator P is itself linear, $\langle P(u), v\rangle$ is linear in u when v is kept fixed. Therefore, as it is costumary, we write

$$\langle Pu, v\rangle = \langle P(\mathrm{u}), v\rangle \in F$$

for this value. In this work we shall be concerned, exclusively, with functional valued operators that are linear.

On the other hand, let $D^2 = D \oplus D$ be the space of pairs (u, v) with $u \in D$ and $v \in D$. We may consider functions $\beta : D^2 \to F$. The values of such functions on a pair $(u, v) \in D^2$, will be written as $\beta(u, v)$. Such a function is said to be a bilinear functional if it is linear in u when $v \in D$ is kept fixed, and conversely.

Recall that given any functional valued operator $P : D \to D^*$ which is linear, one can define a bilinear functional $\beta : D^2 \to F$ by means of

$$\beta(u, v) = \langle Pu, v\rangle$$

Conversely, given any bilinear functional $\beta : D^2 \to F$, we can associate with it an operator $P : D \to D^*$ which is linear. Indeed, given any $u \in D$ let

$$P(u) = \alpha \in D^*$$

where $\alpha \in D^*$ is defined as the linear functional whose value at any $v \in D$ is

$$\langle \alpha, v\rangle = \beta(u, v) \tag{2.6}$$

Then $P : D \to D^*$ is linear. This establishes a one-to-one correspondance between bilinear functionals and operators $P : D \to D^*$ that are linear.

Given $P : D \to D^*$, take $\beta : D^2 \to F$ as the bilinear functional (2.4). Define the operator $P^* : D \to D^*$ as the one associated with the transposed β^* of the bilinear functional β; i.e.,

$$\langle P^*u, v\rangle = \beta^*(u, v) = \beta(v, u) = \langle Pv, u\rangle$$

When $P^* : D \to D^*$ satisfies (2.7), P^* will be called the adjoint of P. Notice that given $P : D \to D^*$, P^* always exists and it is a mapping of D into D^*.

For any operator $P : D \to D^*$, the null subspace of P will be denoted by N_P; i.e.,

$$N_P = \{u \in D \mid Pu = 0$$

A few relations between the null subspaces of functional valued operators will be used in the sequel.

Definition 2.1 *One says that the operators $P : D \to D^*$ and $Q : D \to D^*$ can be varied independently, when*

$$D = N_P + N_Q$$

Lemma 2.1. *Let $P : D \to D^*$ and $Q : D \to D^*$ be linear operators. Then the fellowing assertions are equivalent*

a) *P and Q can be varied independently.*

For every $U \in D$ and $V \in D$

$$\text{b)}\ \exists \quad u \in D_{-)} \quad Pu = PU \quad \text{and} \quad Qu = QV$$

$$\text{c)}\ \exists \quad u \in D_{-)} \quad Pu = PU \quad \text{and} \quad Qu = 0\ , \tag{2.11}$$

$$\text{d)}\ \exists \quad u \in D_{-)} \quad Pu = 0 \quad \text{and} \quad Qu = QV$$

Proof. Assume (2.9) holds; given $U \in D$ and $V \in D$, write $U = U_1 + U_2$ and $V = V_1 + V_2$ where $U_1, V_1 \in N_P$ while $U_2, V_2 \in N_Q$. Then $u = V_1 + U_2$ satisfies (2.10). Hence (a) implies (b). It is clear that (b) implies (c) and (d). In view of the symmetric role played by P and Q in lemma 2.1, to finish its proof it is enough to show that (c) implies (a). Assume (c), then given $v \in D$, take $v_2 \in D$ such that

$$Pv_2 = Pv \quad \text{and} \quad Qv_2 = 0\ . \tag{2.13}$$

Define $v_1 = v - v_2$. Then it is seen that $v = v_1 + v_2$, with $v_1 \in N_P$ while $v_2 \in N_Q$. This completes the proof of lemma 2.1.

Assume $P : D \to D^*$, $Q : D \to D^*$ and $R : D \to D^*$ are such that

$$P = Q + R \tag{2.14}$$

Then it is straightforward to see that

$$N_P \supset N_Q \cap N_R \tag{2.15}$$

A stronger result holds when Q^* and R^* can be varied independently.

Lemma 2.2 *Assume P, Q and R are such that $P = Q + R$, while Q^* and R^* can be varied independently. Then*

$$N_P = N_Q \cap N_R \ . \tag{2.16}$$

Proof. In view of (2.15), it is only necessary to prove that when Q^* and R^* can be varied independently, $N_P \subset N_Q \cap N_R$.

Assume $PU = 0$ and $QU \neq 0$. Then, $\exists\, V \in D_{-)} \langle QU, V\rangle \neq 0$. For such V, take $v \in D_{-)} Q^*v = Q^*V$ while $R^*v = 0$. Recall

$$0 \neq \langle QU, V\rangle = \langle Q^*V, U\rangle = \langle Q^*v, U\rangle + \langle R^*v, U\rangle$$

$$\langle QU, v\rangle + \langle RU, v\rangle = \langle PU, v\rangle = 0 \ . \tag{2.17}$$

Hence, $U \in N_P \Rightarrow U \in N_Q$. A similar argument replacing Q by R, yields $U \in N_P \Rightarrow U \in N_R$. Therefore, $N_P \subset N_Q \cap N_R$ and the lemma is established.

III. FORMAL ADJOINTS AND ABSTRACT GREEN'S FORMULAS

Definition 3.1. *$B : D \to D^*$ is a boundary operator for $P : D \to D^*$ when*

$$\langle Pu, v\rangle = 0 \ \forall\, v \in N_B \Rightarrow Pu = 0 \tag{3.1}$$

Definition 3.2. *Given $P : D \to D^*$* and *$Q : D \to D^*$, define $S = P - Q^*$. Then P and Q are formal adjoints if S is a boundary operator for Q while S^* is a boundary operator for P.*

Notice that this is a symmetric relation between P and Q as can be easily verified.

Definition 3.3. *An operator $P : D \to D^*$ is said to be formally symmetric when P is a formal adjoint of itself.*

Theorem 3.1. *Given $P : D \to D^*$ define $A = P - P^*$. Then P is formally symmetric if and only if*

$$\langle Pu, v\rangle = 0 \quad \forall\, v \in N_A \Rightarrow Pu = 0$$

Proof. Let $S = P - P^* = A$. Then P is formally symmetric, if and only if S and $S^* = -S$ are boundary operators for P. Clearly this is equivalent to (3.2).

Theorem 3.2. *Given $P : D \to D^*$ and $Q : D \to D^*$, define $\hat{P} : \hat{D} \to \hat{D}^*$ by*

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = \langle Pu_1, v_2\rangle + \langle Qu_2, v_1\rangle$$

where $\hat{D} = D \oplus D$. Then P and Q are formal adjoints, if and only if, $\hat{P}$ is formally symmetric.

Proof. The following lemma can be used to prove this theorem.

Lemma 3.1. *Let*

$$\hat{A} = \hat{P} - \hat{P}^*$$

Then

$$\hat{N}_A = N_S \oplus N_{S^*}$$

where $S = P - Q^*$.

Proof. For every $\hat{u} \in \hat{D}$ and $\hat{v} \in \hat{D}$, one has

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle Su_1, v_2\rangle - \langle S^* u_2, v_1\rangle$$

Thus $\hat{u} \in \hat{N}_A \Leftrightarrow u_1 \in N_S$ while $u_2 \in N_{S^*}$. This completes the proof of the lemma.

Using this lemma and (3.3) it is clear that

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = 0 \quad \forall\, \hat{v} \in \hat{N}_{\hat{A}} \Rightarrow \hat{P}\hat{u} = 0$$

if and only if

$$\langle Pu_1, v_2\rangle = 0 \quad \forall\, v_2 \in N_{S^*} \Rightarrow Pu_1 = 0$$

and simultaneously

$$\langle Qu_2, v_1\rangle = 0 \qquad \forall\, v_1 \in N_S \Rightarrow Qu_2 = 0 \ . \tag{3.9}$$

Clearly, (3.8) means that S^* is a boundary operator for P, while (3.9) means that S is a boundary operator for Q. Application of definition 3.2, yields the desired result.

Definition 3.4. *When $P : D \to D^*$ is formally symmetric, a relation*

$$P - P^* = B - B^* \tag{3.10}$$

where $B : D \to D^*$ and B^* are boundary operators for P, is said to be a Green's formula.

Definition 3.5. *An operator $B : D \to D^*$ is said to decompose A, when B and B^* can be varied independently and simultaneously*

$$A = B - B^* \tag{3.11}$$

Remark. When B decomposes A, $-B^*$ also decomposes A.

Theorem 3.3. *If $B : D \to D^*$ decomposes A, then*

$$N_A = N_B \cap N_{B^*} \tag{3.12}$$

Proof. A straightforward application of lemma 2.2, using the facts that $A = B - B^*$ and $D = N_B + N_{B^*}$, yields this theorem.

Corollary 3.1. *When $P : D \to D^*$ is formally symmetric and B decomposes A, B and B^* are boundary operators.*

Proof. This follows from definition 3.1, because A is a boundary operator and $N_B \supset N_A$ as well as $N_{B^*} \supset N_A$.

Corollary 3.2. *Under the hypotheses of corollary 3.1,*

$$A = B - B^* \tag{3.13}$$

is a Green' formula.

Proof. Because both B and B^* are boundary operators.

Definition 3.6. *Relation (3.10) is said to be a Green's formula in the strict sense, when B decomposes $A = P - P^*$.*

In what follows we shall be concerned only with Green's formulas in the strict sense. Thus, frequently, we will delete the latter designation. Also, it will be assumed that an operator $P : D \rightarrow D^*$ is given and that, associated with it, we have $A = P - P^*$.

IV. REGULAR SUBSPACES AND CANONICAL DECOMPOSITIONS

Green's formulas in the strict sense can be characterized by properties of the null subspaces N_B *and* N_{B^*}. The following discussion is oriented to supply such characterization.

Definition 4.1. *A linear subspace $I \subset D$, is said to be regular for P, when* $P : D \rightarrow D^*$, $A = P - P^*$

a) $$I \supset N_A \ ,$$

b) $$\langle Au, v \rangle = 0 \qquad \forall \ u \in I \text{ and } v \in I$$

Definition 4.2. *A linear subspace $I \subset D$ is said to be completely regular for P, when*

$$\langle Au, v \rangle = 0 \qquad \forall \ v \in I \Leftrightarrow u \in I \tag{4.3}$$

An alternative manner of defining completely regular subspaces is as a commutative subspace that is largest. The precise meaning of this statement is given next.

Lemma 4.1. *A linear subspace I, which is commutative, is completely regular, if an only if, for every commutative subspace I', one has*

$$I' \supset I \Rightarrow I' = I$$

Proof. It is easy to verify.

Lemma 4.2. *$I \subset D$ is completely regular, if and only if, I is a regular subspace and*

$$\langle Au, v \rangle = 0 \qquad \forall \ v \in I \Rightarrow u \in I$$

Proof. The equivalence statement (4.3) is the conjunction of (4.2) and

(4.5); thus, any regular subspace satisfying (4.5) is completely regular. To prove the converse, it is only necessary to show that when $I \subset D$ is completely regular, (4.1) holds. But this is immediate because, if $u \in N_A$ and I is completely regular, then $\langle Au, v\rangle = 0 \;\forall\, v \in D \supset I$. Hence, $u \in I$ by virtue of (4.3).

Definition 4.3. *An ordered pair* $\{I_1, I_2\}$ *of regular subspaces such that*

$$D = I_1 + I_2$$

is said to be a canonical decomposition of D, with respect to P.

Lemma 4.3. *Assume* $B : D \to D^*$ *decomposes A. Define*

$$I_1 = N_B \;; \qquad I_2 = N_{B^*}$$

Then the pair $\{I_1, I_2\}$ *is a canonical decomposition of D.*

Proof. The fact that B and B^* can be varied independently, implies (4.6). Assume $u \in I_1$ and $v \in I_1$, then

$$\langle Au, v\rangle = \langle Bu, v\rangle - \langle Bv, u\rangle = 0 \tag{4.8}$$

Also, $I_1 = N_B \supset N_A$ by virtue of (3.12). This shows that I_1 is regular. In a similar fashion it can be shown that I_2 is regular.

Theorem 4.1. *A pair of subspaces* $\{I_1, I_2\}$ *is a canonical decomposition of D, if and only if,* I_1 *and* I_2 *are completely regular*

$$D = I_1 + I_2 \quad \text{and} \quad N_A = I_1 \cap I_2$$

Proof. It is clear that any pair of regular subspaces $\{I_1, I_2\}$ that satisfy (4.9) is a canonical decomposition. Thus, only the converse statement need to be proved. The following lemma will be useful for this purpose.

Lemma 4.4. *Assume* $\{I_1, I_2\}$ *is a canonical decomposition of D. Then,*

$$u_1 \in I_1 \;\; \textit{and} \;\; \langle Au_1, v_2\rangle = 0 \quad \forall\; v_2 \in I_2 \Rightarrow u_1 \in N_A \tag{4.10}$$

Similarly

$$u_2 \in I_2 \quad \text{and} \quad \langle Au_2, v_1 \rangle = 0 \qquad \forall v_1 \in I_1 \Rightarrow u_2 \in N_A \tag{4.11}$$

Proof. To prove (4.10), assume $u_1 \in I_1$ is such that

$$\langle Au_1, v_2 \rangle = 0 \qquad \forall v_2 \in I_2 \ . \tag{4.12}$$

Then, given $W \in D$, let $W_1 \in I_1$ and $W_2 \in I_2$ be such that $W = W_1 + W_2$. Clearly.

$$\langle Au_1, W \rangle = \langle Au_1, W_1 \rangle + \langle Au_1, W_2 \rangle = 0 \ . \tag{4.13}$$

Hence $u_1 \in N_A$. That (4.11) also holds is clear, by virtue of the symmetric roles played by I_1 and I_2, in lemma 4.4 and definition 4.3.

An immediate corollary of lemma 4.4., is that when $\{I_1, I_2\}$ is a canonical decomposition

$$I_1 \cap I_2 = N_A \tag{4.14}$$

Hence, in order to complete the proof of theorem 4.1, it remains to prove that I_1 and I_2 are completely regular. To this end, given $u \in D$, write $u = u_1 + u_2$ with $u_1 \in I_1$ and $u_2 \in I_2$. Then, if for every $v_1 \in I_1$,

$$\langle Au, v_1 \rangle = \langle Au_2, v_1 \rangle = 0 \tag{4.15}$$

one has $u_2 \in N_A \subset I_1$ by (4.12). Hence, $u = u_1 + u_2 \in I_1$. This shows that I_1 is completely regular and a similar argument yields the corresponding result for I_2.

Theorem 4.2. *With every operator $B : D \to D^*$ that decomposes A associate a canonical decomposition $\{I_1, I_2\}$ by means of (4.7). Then, such correspondence between operators that decompose A and canonical decompositions is one-to-one and covers the set of canonical decompositions of D. Under this mapping, any canonical decomposition $\{I_1, I_2\}$ is the image of a unique operator $B : D \to D^*$ given by*

$$\langle Bu, v \rangle = \langle Au_2, v_1 \rangle \tag{4.16}$$

where u_2 and v_1 are the components of u and v on I_2 and I_1, respectively.

Proof. In order to show that such correspondence covers the set of canonical decompositions, given any canonical decomposition $\{I_1, I_2\}$, we exhibit an operator $B: D \to D^*$ that decomposes A and such that satisfies (4.7). To this end define B by (4.16). Notice that B is well defined by virtue of the second of equations (4.9). Equation (3.11) is satisfied, because

$$\langle Au, v\rangle = \langle Au_2, v_1\rangle + \langle Au_1, v_2\rangle = \langle Au_2, v_1\rangle - \langle Av_2, u_1\rangle \qquad (4.17)$$

Next, we prove that equations (4.7) hold. It is straightforward to see that $I_1 \subset N_B$; to show that $I_1 \supset N_B$, assume that $u \in N_B$ (i.e., $Bu = 0$), then

$$\langle Au, v_1\rangle = \langle Au_2, v_1\rangle = \langle Bu, v_1\rangle = 0 \quad \forall v_1 \in I_1 \ . \qquad (4.18)$$

Hence, $u \in I_1$ because I_1 is completely regular. This completes the proof of the first of equations (4.7). The proof of the second one is similar. The fact that B and B^* can be varied independently, follows now from (4.6), in view of (4.7) and definition 2.1. After this has been shown, only the assertion about the one-to-one character of the mapping remains to be proved. The following chain of equilities

$$\langle Au_2, v_1\rangle = \langle (B - B^*)u_2, v_1\rangle = \langle Bu_2, v_1\rangle = \langle Bu, v_1\rangle$$

$$= \langle B^*v_1, u\rangle = \langle B^*v, u\rangle = \langle Bu, v\rangle \qquad (4.19)$$

shows that $B: D \to D^*$, given by (4.16), is the only operator that decomposes A and satisfies (4.7), for a given canonical decomposition $\{I_1, I_2\}$. To establish (4.19) the relations $B^*u_2 = Bu_1 = B^*v_2 = 0$ were used.

V. THE PROBLEM WITH LINEAR RESTRICTIONS AND THE SUBSPACE I_P

The results of the abstract theory of boundary operators can be applied to discuss the representation of solutions of a wide class of problems with linear restrictions which include boundary value problems for partial differential equations. In this section the space I_P which characterizes the boundary values of the homogeneous equation $Pu = 0$, is introduced and under general conditions it is shown to be completely regular. This result is used in further sections to develop procedures for representing solutions. Later variational principles for such problems will also be developed.

Definition 5.1. *Consider $P : D \to D^*$ and a subspace $I \subset D$. Given $U \in D$ and $V \in D$, an element $u \in D$ is said to be a solution of the problem with linear restrictions or constraints, when*

$$Pu = PU \quad \textit{and} \quad u - V \in I$$

With every linear operator $P : D \to D^*$, it is possible to associate a subspace I_P that is regular for P. It is defined by

$$I_P = N_A + N_P$$

where N_A and N_P are the null subspaces of A and P, respectively.

Lemma 5.1. *The linear space I_P defined by equation (5.2) is a regular subspace for P.*

Proof. Condition (4.1) is clearly satisfied by I_P. In order to show that (4.2) is also satisfied, given any $u \in I_P$ and $v \in I_P$, write $u = u_P + u_A$ and $v = v_P + v_A$, where $u_P, v_P \in N_P$ while $u_A, v_A \in N_A$. Then

$$\langle Au, v\rangle = \langle Au_P, v_P\rangle = \langle Pu_P, v_P\rangle - \langle Pv_P, u_P\rangle = 0$$

Definition 5.2. *The problem with linear restrictions (5.1), satisfies*

a) *Existence, when there is at least one solution for every $U \in D$ and $V \in D$;*

b) *Uniqueness, when $U = 0$ and $V = 0 \Rightarrow u = 0$;*

c) *Uniqueness on the boundary, when*

$$U = 0 \textit{ and } V = 0 \Rightarrow u \in N_A$$

By a boundary solution it is meant an element $u \in D$ such that $u - U \in I_P$ while $u - V \in I$.

Clearly any strict solution is a boundary solution.

Lemma 5.2. *The problem with linear restrictions satisfies existence, uniqueness or uniqueness on the boundary, if and only if, the problem*

$$Pu = PU \quad \text{and} \quad u \in I$$

or alternatively

$$Pu = 0 \quad \text{and} \quad u - V \in I \tag{5.5}$$

enjoy corresponding properties.

Proof. Let us prove the assertion of the lemma with respect to (5.4). This follows from the fact that if $w \in D$ is defined by $w = u - V$, with $V \in D$ fixed, then

$$Pu = PU \quad \text{and} \quad u - V \in I \Leftrightarrow Pw = P(U - V) \quad \text{and} \quad w \in I \ . \tag{5.6}$$

A similar argument with $u - U$ yields the other part.

Theorem 5.1. *Let $I \subset D$ be a regular subspace for P. If the problem with linear restrictions satisfies existence, then the pair $\{I, I_P\}$ constitutes a canonical decomposition of D.*

Proof. In view of definition 4.3, it is only necessary to prove that

$$D = I + I_P$$

because both I and I_P are regular subspaces. This is immediate, because given $u \in D$, take $u_1 \in D$, such that

$$Pu_1 = Pu \ ; \quad u_1 \in I$$

and write $u_2 = u - u_1$. Then $u = u_1 + u_2$, with $u_1 \in I$ and $u_2 \in I_P$.

Corollary 5.1. *Under the assumptions of theorem 5.1, I and I_P are completely regular subspaces.*

Proof. In view of theorems 5.1 and 4.1.

Corollary 5.2. *Under the assumptions of theorem 5.1, when the problem with linear restrictions satisfies existence, it also satisfies uniqueness on the boundary.*

Proof. Because

$$I \cap I_P = N_A$$

Corollary 5.3. *If the problem with linear restrictions satisfies existence, there exists a Green's formula in the strict sense*

$$P - P^* = B - B^* \tag{5.10}$$

such that $u \in D$ is a boundary solution of the problem with linear restrictions, if and only if

$$Bu = BV \quad \textit{while} \quad B^*u = B^*U \tag{5.11}$$

Proof. This result is implied by theorem 5.1 in view of theorem 4.2 and definition 3.6.

VI. CANONICAL DECOMPOSITIONS IN SUBSPACES

Let $D_1 \subset D$ be any subspace of D. Consider $P_1 \quad D_1 \to D_1^*$, defined by

$$\langle P_1 u, v\rangle = \langle Pu, v\rangle \ , \quad \forall u \in D_1 \quad \text{and} \quad v \in D_1 \tag{6.1}$$

Definition 6.1. *A subspace $I \subset D_1 \subset D$ is said to be regular or completely regular for P in D_1, when it is completely regular for P_1.*

Theorem 6.1. *Let $I \subset D$ be completely regular for P. Let $D_1 \subset D$ be any subspace of D. Then I is completely regular for P in D_1, whenever $D_1 \supset I$.*

Proof. Because when property (4.3) is satisfied for every $u \in D$, then it is also satisfied for every $u \in D_1 \subset D$.

The subspaces of D which contain I, constitute an algebra with respect to the operations $D_1 + D_2$; $D_1 \cap D_2$. Theorem 6.1 shows that when I is completely regular for $P : D \to D^*$ in D, then it is also completely regular for P in this algebra of subspaces. When on the contrary it is known that $I \subset D$ is completely regular in subspaces D_1 and D_2 separately, it has interest to establish sufficient conditions for the complete regularity of I in $D_1 \cap D_2$ and $D_1 + D_2$. The first of these questions is easily answered.

Theorem 6.2. *Let $I \subset D_1$ and $I \subset D_2$. Assume I is completely regular in D_1 or alternatively, in D_2. Then I is completely regular in $D_1 \cap D_2$*

Proof. Notice that $D_1 \cap D_2 \subset D_1$ and $D_1 \cap D_2 \subset D_2$. Hence, this theorem is a corollary of theorem 6.1.

The following theorem supplies an answer to the second question.

Theorem 6.3. *Let I be completely regular for P in subspaces $D_1 \supset I$ and $D_2 \supset I$. Then I is completely regular for P in $D_1 + D_2$ if and only if, for every $u_1 \in D_1$ and $u_2 \in D_2$*

$$\langle Au_1, v\rangle = \langle Au_2, v\rangle \quad \forall v \in I \Rightarrow u_1 \in D_2$$

<u>Proof.</u> Assume $I \subset D_1 + D_2$ is completely regular in $D_1 + D_2$. Then if

$$\langle A(u_1 - u_2), v\rangle = 0 \quad \forall v \in I \Rightarrow u_1 - u_2 \in I \subset D_2 \ . \tag{6.3}$$

Therefore $u_1 = (u_1 - u_2) + u_2 \in D_2$. Conversely, if (6.2) holds, then given any $u \in D_1 + D_2$ write $u = u_1 - u_2$ with $u_1 \in D_1$ and $u_2 \in D_2$. Now

$$\langle Au, v\rangle = 0 \quad \forall v \in I \Rightarrow u_1 \in D_2 \Rightarrow u \in D_2$$

Since I is completely regular in D_2 and $u \in D_2$, it is clear that

$$\langle Au, v\rangle = 0 \quad \forall v \in I \Rightarrow u \in I \ . \tag{6.4}$$

This establishes the implication (4.3) in the right-hand sense. The implication in the opposite sense is clear, because I is a commutative subspace. This completes the proof of the theorem.

A result that is useful in many applications, is given next. Let $A : D \to D^*$, $A' : D \to D^*$ and $A'' : D \to D^*$ be antisymmetric operators, while $I_1' \subset D, I_2' \subset D, I_1'' \subset D, I_2'' \subset D$ are linear subspaces such that

a) $A = A' + A''$ (6.5)
b) A' and A'' can be varied independently.
c) $\{I_1', I_2'\}$ is a canonical decomposition of D with respect to A'.
d) $\{I_1'', I_2''\}$ is a canonical decomposition of D with respect to A''.
e) Define

$$I_1 = I_1' \cap I_1'' \ ; \quad I_2 = I_2' \cap I_2'' \tag{6.6}$$

Theorem 6.4. *When hypotheses (a) to (e) are satisfied, the pair $\{I_1, I_2\}$ is a canonical decomposition of D with respect to A.*

Proof. This theorem can be established using the following result.

<u>Lemma 6.1.</u> *The pair $\{I_1, I_2\}$, satisfies*

$$D = I_1 + I_2 \ . \tag{6.7}$$

Proof. This lemma can be shown observing that

$$I_1'' \cap N_{A'} + I_2'' \cap N_{A'} + I_1' \cap N_{A'} + I_2' \cap N_{A''} = D \ , \tag{6.8}$$

$$I_1 = I_1' \cap I_1'' \supset I_1'' \cap N_{A'} + I_1' \cap N_{A''} \tag{6.9a}$$

and

$$I_2 = I_2' \cap I_2'' \cap N_{A'} + I_2' \cap N_{A''} \tag{6.9b}$$

To prove (6.8), we show*

$$I_1'' \cap N_{A'} + I_2'' \cap N_{A'} = N_{A'} \tag{6.10}$$

and we observe that for any $v \in I_1''$ there are $x \in N_{A'}$ and $y \in N_{A''}$, such that

$$v = x + y \tag{6.11}$$

because A' and A'' can be varied independently. Then $y \in I_1''$, since $I_1'' \supset N_{A''}$. This shows that $x \in I_1'' \cap N_{A'}$. Given any $u' \in N_{A'}$, we have

$$u' = v + w \tag{6.12}$$

with $v \in I_1''$ and $w \in I_2''$; i.e.,

$$u' = x + y + r + s \ , \tag{6.13}$$

where $x \in N_{A'} \cap I_1''$, $y \in N_{A''}$, $r \in N_{A'} \cap I_2''$ and $s \in N_{A''}$. This shows that $(y + s) \in N_{A'} \cap N_{A''} \subset N_{A'} \cap I_1''$. Hence $u' \in I_1'' \cap N_{A'} + I_2'' \cap N_{A'}$. In a similar fashion we can show that

$$I_1' \cap N_{A''} + I_2' \cap N_{A''} = N_{A''} \tag{6.14}$$

* The proof of (6.8) here supplied is due to Prof. John Evans, from the Univ. of California at San Diego, whose contribution is here gratefully acknowledged.

Equations (6.10) and (6.14) together, imply (6.8), because $N_{A'} + N_{A''} = D$, since A' and A'' can be varied independently. The theorem follows from the fact that I_1 and I_2 are regular subspaces for A, as it is not difficult to verify.

VII. VARIATIONAL PRINCIPLES

There is a very straightforward result that will be used when formulating variational principles. Let $S : D \to D^*$ be symmetric and $f \in D^*$; then

$$Su = f \Leftrightarrow \Omega'(u) = 0 ,$$

where

$$\Omega(u) = \frac{1}{2} \langle Su, u \rangle - \langle f, u \rangle$$

Here, the derivative Ω' of $\Omega : D \to F$ is taken in the sense of additive Gateaux variation [54], which is probably the weakest definition of derivative. Relation (7.1) was given in Ref. 3, has been used in previous work [6, 8], an a related result has been given by Oden and Reddy [31]; it follows from the fact that when S is symmetric

$$\Omega'(u) = Su - f$$

The theory developed in this paper will be used in this section to formulate two types of variational principles for problems with linear restrictions.

The first one applies when there is available a canonical decomposition $\{I_1, I_2\}$ one of whose elements (the first one, to be definite) is the linear subspace I which specifies the restriction in problem (5.1). In this case $P - B$, where $B : D \to D^*$ is the operator associated with the canonical decomposition by means of (4.16), is symmetric; by its use one obtains variational principles for which the variations need not be restricted. However, it must be observed that the mere existence of such canonical decompsoition is not sufficient to permit the formulation of these variational principles; it is required, in addition, that the actual decomposition of every vector $u \in D$ in terms of its components u_1 and u_2 can be carried out easily, because this is necessary in order to construct B by means of (4.16). It will be shown that there are cases, such as when the problems are subjected to restrictions of continuation type, to be discussed later, which do not fulfill this requirement in spite of the fact that for them the pair $\{I_1, I_P\}$, constitutes a canonical decomposition whenever the hypotheses of theorem 5.1 are satisfied. In such cases in order to obtain the components u_1, u_2 of any $u \in D$, it would be required to solve the problem with linear restrictions (5.1).

When the operator B cannot be constructed, the second type of variational principle can be applied. It is associated with the operator $2P - A$, which is always symmetric and can be used if variations are restricted to be in the regular subspace I; the results are enhanced when the subspace is completely regular.

The following lemmas lead to the desired variational principles.

Lemma 7.1. *Let $I \subset D$ be a completely regular subspace for P, then given $U \in D$ and $V \in D$, an element $u \in D$ is solution of the problem with linear constraints (5.1), if and only if*

$$Pu = PU$$

and

$$\langle A(u - V), v\rangle = 0 \quad \forall\, v \in I$$

When I is regular, but not completely regular, the above assertion holds for elements $u \in V + I$.

Proof. The mere regularity of $I \subset D$, is enough to guarantee that Eq. (5.1) implies (7.4) and (7.5). When, in addition $I \subset D$ is completely regular, conversely, (7.5) implies that $u - V \in I$; hence, Eq. (5.1) follows from (7.4) and (7.5), in this case. The second part of the lemma is now straightforward.

Lemma 7.2. *Assume $\{I', I_c\}$ constitutes a canonical decomposition of D with respect to P, and let $B : D \to D^*$ be defined by (4.16), taking u_2 and v_1 as components of vectors on $\{I, I_c\}$. Then $u \in D$ is a solution of the problem with linear constraints (5.1), if and only if*

$$Pu = PU \quad \text{and} \quad Bu = BV. \tag{7.6}$$

Proof. By theorem 4.2, B satisfies (4.7). Hence, $u - V \in I$ if and only if $B(u - V) = 0$.

Lemma 7.3. *Assume $P : D \to D^*$ is formally symmetric and $I \subset D$ is regular for P. Then*

a) *(7.4) and (7.5) hold simultaneously if and only if*

$$\langle (2P - A)u, v\rangle = \langle 2PU - AV, v\rangle \quad \forall\, v \in I \;.$$

b) *When B decomposes A, and $I = N_B$, Eq. (7.6) holds, if and only if*

$$(P - B)u = PU - BV$$

Proof. Rearranging, equation (7.7) becomes

$$\langle 2P(u - U), v\rangle = \langle A(u - V), v\rangle \quad \forall\, v \in I$$

Clearly, (7.4) and (7.5) imply (7.9). Conversely, (7.9) implies

$$\langle 2P(u - U), v\rangle = 0 \quad \forall\, v \in N_A \subset I \;, \tag{7.10}$$

which in turn implies (7.4), by virtue of theorem 3.1, because P is formally symmetric. Once this has been shown, (7.9) reduces to (7.5). This proves (a).

Equation (7.8) can be obtained substracting one of equations (7.6) from the other. Conversely, (7.8) implies

$$\langle P(u - U), v\rangle = \langle B(u - V), v\rangle = 0 \quad \forall\, v \in N_A \subset D \;, \tag{7.11}$$

because $N_{B^*} \supset N_A$. The first of equations (7.6) follows from (7.11), because P is formally symmetric. Once that equation has been shown, (7.8) reduces to the second equation in (7.6)

Theorem 7.1. *Assume $P : D \to D^*$ is formally symmetric and $\{I, I_c\}$ constitutes a canonical decomposition of D. Then $u \in D$ is a solution of the problem with linear restrictions (5.1), if and only if*

$$\Omega'(u) = 0 \;, \tag{7.12}$$

where

$$\Omega(u) = \frac{1}{2}\langle (P - B)u, u\rangle - \langle PU - BV, u\rangle \;. \tag{7.13}$$

Here $B : D \to D^*$ is the operator associated with $\{I, I_c\}$ by means of (4.16).

Proof. Recall that $P - P^* = A = B - B^*$; hence, $P - B$ is symmetric. Applying (7.2) to this symmetric operator, theorem 7.1 follows from lemmas 7.2 and 7.3.

Theorem 7.2. *Assume P is formally symmetric and $I \subset D$ is a completely regular subspace for P. Define*

$$X(u) = \langle Pu, u\rangle - \langle 2PU - AV, u\rangle \ . \qquad (7.14)$$

Then $u \in D$ is a solution of the problem with linear restrictions (5.1), if and only if

$$\langle X'(u), v\rangle = 0 \quad \forall\, v \in I \ . \qquad (7.15)$$

When I is regular but not completely regular, an element $u \in V + I$ is a solution of (5.1), if and only if (7.15) holds.

Proof. $2P - A$ is symmetric with quadratic form $\langle 2Pu, u\rangle$, because A is antisymmetric. From (7.14), if follows that

$$X'(u) = (2P - A)u - (2PU - AV) \ . \qquad (7.16)$$

Theorem 7.2, follows from lemmas 7.4 and 7.5, by virtue of (7.16).

VIII. THE PROBLEM OF CONNECTING

An advantage of introducing abstract boundary operators is the large class of problems that can be formulated using them; a rather general example is the problem of connecting. This is an abstract version of problems formulated in discontinuous fields with prescribed jump conditions.

Consider two neighboring regions R and E (figure 1) with boundaries ∂R and ∂E, respectively. For simplicity R and E are illustrated as bounded; however, the theory can be applied even if they are unbounded. By reasons that will become apparent in some of the examples to be given, the common boundary between R and E will be denoted by $\partial_3 R = \partial_3 E$. The general problem to be discussed consists in finding solutions to specific partial differential equations on $R \cup E$ subjected to a given smoothness condition or more generally, to a jump condition across the connecting boundary $\partial_3 R = \partial_3 E$. Problems of this kind occur frequently in applications; the smoothness criterion may be in potential theory, for example, that u and $\partial u/\partial n$ be continuous across $\partial_3 R$, or in elasticity, that displacements and tractions be continuous across that part of the boundary. However, more complicated criteria may be included in the theory; this is the case for example, when R is occupied by an inviscid liquid while in E there is an elastic solid.

In general we consider two linear spaces D_R and D_E which may be associated with functions defined on R and on E, respectively. The linear space $\hat{D} = D_R \oplus D_E$ is made of elements $\hat{u} \in \hat{D}$ that can be thought as pairs $[u_R, u_E]$, where $u_R \in D_R$ while $u_E \in D_E$. An operator $\hat{P} : \hat{D} \to \hat{D}^*$ possessing the additive property

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = \langle \hat{P}u_R, v_R\rangle + \langle \hat{P}u_E, v_E\rangle \qquad (8.$$

will be considered. If the operators $\hat{P}_R : \hat{D} \to \hat{D}^*$ and $\hat{P}_E : \hat{D} \to \hat{D}^*$ are defined by

$$\langle \hat{P}_R \hat{u}, \hat{v}\rangle = \langle \hat{P}u_R, v_R\rangle \ ; \quad \langle \hat{P}_E \hat{u}, v\rangle = \langle \hat{P}u_E, v_E\rangle \ ,$$

then

$$\hat{P} = \hat{P}_R + \hat{P}_E$$

Operators $P_R : D_R \to D_R^*$ and $P_E : D_E \to D_E^*$ can also be defined, they are given by

$$\langle P_R u_R, v_R\rangle = \langle \hat{P}u_R, v_R\rangle \ ; \quad \langle P_E u_E, v_E\rangle = \langle \hat{P}u_E, v_E\rangle$$

Then

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = \langle P_R u_R, v_R\rangle + \langle P_E u_E, v_E\rangle$$

Using these operators, the following can be defined

$$\hat{A} = \hat{P} - \hat{P}^*, \ \hat{A}_R = \hat{P}_R - \hat{P}_R^* ; \quad A_R = P_R - P_R^* ;$$

$$\hat{A}_E = \hat{P}_E - \hat{P}_E^* ; \quad A_E = P_E - P_E^*$$

They satisfy

$$\hat{A} = \hat{A}_R + \hat{A}_E \tag{8.7}$$

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle A_R u_R, v_R\rangle + \langle A_E u_E, v_E\rangle \tag{8.8}$$

The null subspaces of $\hat{A}$, $\hat{A}_R$, $\hat{A}_E$, A_R and A_E will be denoted by $\hat{N}$, $\hat{N}_R$, $\hat{N}_E$, N_R and N_E, respectively. The relation

$$\hat{N} = N_R \oplus N_E \tag{8.9}$$

will be used later; it is equivalent to

$$\hat{u} = (u_R, u_E) \in \hat{N} \Leftrightarrow u_R \in N_R \quad \text{and} \quad u_E \in N_E \tag{8.10}$$

This latter relation follows from (8.8).

It will be assumed that there is a linear subspace $\hat{S} \subset \hat{D}$ of smooth elements $\hat{u} = (u_R, u_E)$. When $\hat{u} = (u_R, u_E) \in \hat{S}$, $u_R \in D_R$ and $u_E \in D_E$ will be said to be smooth extensions of each other.

Definition 8.1. *Let $\hat{S} \subset \hat{D} = D_R \oplus D_E$ be a linear subspace. Then $\hat{S}$ will be said to be a smoothness condition or relation if every $u_R \in D_R$ possesses at least one smooth extension $u_E \in D_E$ and conversely.*

The smoothness relation $\hat{S}$ will be said to be regular and completely regular for $\hat{P}$, when as a subspace, it is regular and completely regular for $\hat{P}$, respectively.

Lemma 8.1. *A smoothness condition $\hat{S} \subset \hat{D}$ is regular for $\hat{P}$, if and only if*

a) $$A_R u_R = 0 \quad \text{and} \quad A_E u_E = 0 \Rightarrow [u_R, u_E] \in \hat{S} \tag{8. 1a}$$

b) *For every $\hat{u} = [u_R, u_E] \in \hat{S}$ and $\hat{v} = [v_R, v_E] \in \hat{S}$* (8.11b)

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle A_R u_R, v_R\rangle + \langle A_E u_E, v_E\rangle = 0 \ .$$

In addition, a smoothness condition $\hat{S} \subset \hat{D}$ is completely regular for $\hat{P}$, if and only if

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \langle A_R u_R, v_R\rangle + \langle A_E u_E, v_E\rangle = 0 \quad \forall\, \hat{v}\in\hat{S} \Leftrightarrow \hat{u}\in\hat{S}\ . \quad (8.11c)$$

Proof. This lemma follows from definitions 4.1 and 4.2.

As an example, assume each of the boundaries ∂R and ∂E of regions R and E (figure 1) is divided into three parts ∂_iR and $\partial_i E$ (i = 1, 2, 3), where $\partial_3 R = \partial_3 E$ is the common boundary between R and E. Let n be the unit normal vector on these boundaries, which will be taken pointing outwards from R and from E. On the common boundary $\partial_3 R = \partial_3 E$, there are defined two unit normal vectors which have opposite senses, one associated with R and the other one with E. Some times they will be represented by n_R and n_E; more often, however, the ambiguity will be resolved by the suffix used under the integral sign. Take

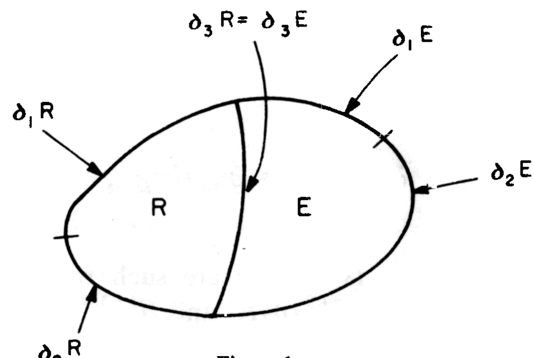

Figure 1

$$D_R = \{u_R \in H^s(R)\} \quad (8.12a)$$

$$D_E = \{u_E \in H^s(R)\} \quad (8.12b)$$

Define $P_R \quad D_R \to D_R^*$ by

$$\langle P_R u_R, v_R\rangle = \int_R v_R \nabla^2 u_R dx + \int_{\partial_1 R} u_R \frac{\partial v_R}{\partial n} dx - \int_{\partial_2 R} v_R \frac{\partial u_R}{\partial n} dx \quad (8.13)$$

and let $P_E : D_E \to D_E^*$ satisfy the equation that is obtained when R is replaced by E in (8.13). Then

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \int_{\partial_3 E} \left\{ v_E \frac{\partial u_E}{\partial n} \quad u_E \frac{\partial v_E}{\partial n} \right\} dx$$

$$+ \int_{\partial_3 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} dx \ , \qquad (8.14)$$

while

$$\hat{N} = \{\hat{u} \in \hat{D} | u_R = u_E = \partial u_R/\partial n$$

$$\partial u_E/\partial n = 0 \ , \quad \text{on } \partial_3 R\} \ . \qquad (8.15)$$

Let

$$\hat{S} = \{\hat{u} \in \hat{D} | u_R = u_E \ ; \ \partial u_R/\partial n_R = \partial u_E/\partial n_R \ , \ \text{on } \partial_3 R\} \ . \qquad (8.16)$$

Functions $u_R \in D_R \subset H^s(R)$ ($s \geqslant 3/2$) are such that their boundary values u_R, $\partial u_R/\partial n$ belong to $H^{s-1/2}(\partial_3 R)$ and $H^{s-3/2}(\partial_3 R)$, respectively (see for example Lions and Magenes [29]). A corresponding result holds for functions $u_E \in D_E = H^s(E)$. This shows that every $u_R \in D_R$ can be extended smoothly into a function $u_E \in D_E$, and conversely. Thus $\hat{S}$ is a smoothness relation.

When $\hat{v} = (v_R, v_E) \in \hat{S}$,

$$\langle A_R u_R, v_R\rangle + \langle A_E uE, v_E\rangle$$

$$= \int_{\partial_3 R} \left\{ v_R \left[\frac{\partial u_R}{\partial n} - \frac{\partial u_E}{\partial n} \right] - (u_R - u_E) \frac{\partial v_R}{\partial n} \right\} dx \qquad (8.17)$$

for arbitrary $\hat{u} = (u_R, u_E) \in \hat{D}$. Using (8.17) it can be seen that condition (8.11) is satisfied by $\hat{S} \subset \hat{D}$; this shows that $\hat{S}$ is regular for $\hat{P}$.

When $\hat{S}$ is regular, it is easy to construct a regular subspace which together with $\hat{S}$ constitutes a canonical decomposition of $\hat{S}$, for the operator $\hat{P}$.

Definition 8.2. *An element $\hat{u} = [u_R, u_E] \in \hat{D}$ is said to have zero mean when $[u_R, -u_E] \in \hat{S}$. The collection of elements of $\hat{D}$ with zero mean will be denoted by $\hat{M}$.*

Lemma 8.2. *The set $\hat{M}$ of zero mean elements is regular or completely regular, if and only if, $\hat{S}$ is regular or completely regular.*

Proof. Clearly $\hat{M}$ is a regular subspace of D, if and only if, so is $\hat{S}$. This lemma follows from lemma 8.1, because conditions (8.11) are invariant under the change of sign implied by the definition 8.2 of $\hat{M}$.

Theorem 8.1. *When the smoothness relation $\hat{S}$ is regular for $\hat{P}$, the pair $\{\hat{S}, \hat{M}\}$ constitutes a canonical decomposition of $\hat{D}$.*

Proof. In view of lemma 8.2, $\hat{M}$ is regular. Therefore, in order to verify definition 4.3, it remains to prove that

$$\hat{D} = \hat{S} + \hat{M} \tag{8.18}$$

To show (8.18), given any $\hat{u} = (u_R, u_E) \in \hat{D}$, choose smooth extensions $u'_R \in D_R$ and $u'_E \in D_E$ of $u_E \in D_E$ and $u_R \in D_R$, respectively. Then

$$\hat{u} = \overline{u} - \frac{1}{2}[\hat{u}] \ , \tag{8.19}$$

where $\overline{u} \in \hat{S}$ and $[\hat{u}] \in \hat{M}$ are

$$\overline{u} = \frac{1}{2}(u'_R + u_R, \ u'_E + u_E \tag{8.20a}$$

$$[u] = (u'_R - u_R, u'_E - u_E) \tag{8.20b}$$

The fact that the pair $\{\hat{S}, \hat{M}\}$ constitutes a canonical decomposition of $\hat{D}$, implies that given any $\hat{u} \in \hat{D}$, the elements $\overline{u} \in \hat{S}$ and $[\hat{u}] \in \hat{M}$ are defined up to elements of $\hat{N}_A$. Elements $\overline{u}$ and $[\hat{u}]$ satisfying (8.20) will be called the average and the jump of $\hat{u}$, respectively.

By means of (4.16), it is possible now to define an operator $\hat{B} : \hat{D} \to \hat{D}$ that decomposes $\hat{A}$ and satisfies (4.7) with $I_1 = \hat{S}$ and $I_2 = \hat{M}$. Such operator will be called the jump operator and will be denoted by $\hat{J}$. It is defined by

$$2\langle \hat{J}\hat{u}, \hat{v}\rangle = 2\langle \hat{A}\hat{u}_2, \hat{v}_1\rangle = -\langle \hat{A}[\hat{u}], \overline{v}\rangle \tag{8.21}$$

Notice that

$$\hat{J}\hat{u} = 0 \Leftrightarrow \hat{u} \in \hat{S} \ , \tag{8.22}$$

which motivates the terminology.

Equation (8.21) will be used extensively when formulating variational principles for problems with prescribed jumps in discontinuous fields, and it is worthwhile to ellaborate it further. Let $\hat{u} = \hat{u}_1 + \hat{u}_2 ; \hat{v} = \hat{v}_1 + \hat{v}_2$, where $\hat{u}_1 = (u_{1R}, u_{1E}) \in \hat{S}$, $\hat{u}_2 = (u_{2R}, u_{2E}) \in \hat{M}$ and similarly for $\hat{v}$. Then

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \langle \hat{A}\hat{u}_2, \hat{v}_1 \rangle = \langle A_R u_{2R}, v_{1R} \rangle + \langle A_E u_{2E}, v_{1E} \rangle$$

$$= 2\langle A_R u_{2R}, v_{1R} \rangle = 2\langle \hat{A}_R \hat{u}_2, \hat{v}_1 \rangle = 2\langle \hat{A}_E \hat{u}_2, \hat{v}_1 \rangle \ , \tag{8.23}$$

where (8.8), (8.11c), and the definition 8.2 of $\hat{M}$ have been used. Hence

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \langle \hat{A}_R [\hat{u}], \overline{v} \rangle = -\langle \hat{A}_E [\hat{u}], \overline{v} \rangle \tag{8.24}$$

In addition

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle \hat{A}_R [\hat{v}], \overline{u} \rangle - \langle \hat{A}_R [\hat{u}], \overline{v} \rangle \tag{8.25}$$

because $\hat{A} = \mathcal{J} - \mathcal{J}^*$.

The use of formulas (8.24) and (8.25), will be illustrated applying them to the previous example. In view of (8.16), the smooth extensions $u_R' \in D_R$ and $u_E' \in D_E$ of u_R and u_E, respectively, satisfy

$$u_R' = u_E \ ; \quad \partial u_R' / \partial n_R = \partial u_E / \partial n_R , \quad \text{on } \partial_3 R \tag{8.26}$$

In addition

$$\langle A_R u_R, v_R \rangle = \int_{\partial_3 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} dx \tag{8.27}$$

Applying

$$\langle \hat{J}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R} \left\{ (\bar{v})_R \frac{\partial [\hat{u}]_R}{\partial n} - [\hat{u}]_R \frac{\partial (\bar{v})_R}{\partial n} \right\} \tag{8.28}$$

Equations (8.20) yield

$$[\hat{u}]_R = u_E - u_R \, ; \qquad \frac{\partial [\hat{u}]_R}{\partial n} = \frac{\partial u_E}{\hat{}} - \frac{\partial u_R}{\partial n} \, ; \quad \text{on } \partial_3 R \tag{8.29a}$$

$$(\bar{v})_R = \frac{1}{2}(v_E + v_R) \, ; \qquad \frac{\partial (\bar{v})_R}{\partial n} = \frac{1}{2}\left(\frac{\partial v_E}{\partial n} + \frac{\partial v_R}{\partial n}\right) \quad \text{on } \partial_3 R \tag{8.29b}$$

by virtue of (8.16). Equation (8.28) can be simplified if the component to be used is indicated by the index under the integral sign; thus

$$\langle \hat{J}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R} \left\{ [\hat{u}] \frac{\partial \bar{v}}{\partial n} - \bar{v} \left[\frac{\partial \hat{u}}{\partial n}\right] \right\} dx$$

$$= \int_{\partial_3 E} \left\{ [\hat{u}] \frac{\partial \bar{v}}{\partial n} - \bar{v} \left[\frac{\partial \hat{u}}{\partial n}\right] \right\} dx \, , \tag{8.30}$$

where $[\partial \hat{u}/\partial n]_R = \partial u_E/\partial n - \partial u_R/\partial n$, on $\partial_3 R$. The last equality in (8.30) follows from the second equation in (8.24), but can also be seen because there is a double change of signs on each term appearing in the integrals; one due to the change in the sense of the unit normal and the other one due to the change of sign of the jump of $\hat{u}$. Equation (8.25) yields

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \int_{\partial_3 R} \left\{ [\hat{u}] \frac{\overline{\partial v}}{\partial n} + \bar{u} \left[\frac{\partial \hat{v}}{\partial n}\right] - \bar{v} \left[\frac{\partial \hat{u}}{\partial n}\right] - [\hat{v}] \frac{\overline{\partial u}}{\partial n} \right\} dx \tag{8.31}$$

IX. BOUNDARY VALUES AND CONNECTIVITY BASES

In Sec. V, it was shown that generally (precisely, under the assumptions of theorem 5.1), the subspace I_P is completely regular. As it has been illustrated by means of examples [13, 19, 22, 23] in applications to boundary value problems the characteristic feature of elements belonging to I_P is that they assume boundary values corresponding to solutions of the homogeneous equation $Pu = 0$. Taking this into account, the property

$$\langle Au, v \rangle = 0 \quad \forall v \in I_P \Rightarrow u \in I_P ,$$

which is satisfied when I_P is completely regular, can be interpreted as a purely algebraic characterization of the boundary values of solutions of $Pu = 0$. Such characterization in connection with some specific problems apparently was originated by Trefftz [43].

Frequently, it is preferable to restrict attention to spaces of boundary values

$$\mathcal{D} = D/N_A , \tag{9.2a}$$

$$\mathcal{I} = I/N_A \tag{9.2b}$$

$$\mathcal{I}_P = I_P/N_A \tag{9.2c}$$

A corresponding notation will be used when a canonical decomposition $\{I_1, I_2\}$ is available. Given $u \in D$ there is a unique element of $\mathcal{D}$ associated with u; this will be represented by the same symbol, unless such ambiguity leads to confusion. The same usage will be followed in connection with boundary operators.

In applications it is preferable not to use the whole set I_P in order to characterize boundary solutions. This can be achieved by means of c-complete subsets.

Definition 9.1. *A subset $E \subset I$ is said to be c-complete (complete in connectivity) when*

$$\langle Au, w \rangle = 0 \quad \forall w \in E \Rightarrow u \in I$$

When in addition, for every finite subset $\{w_1, w_2, ..., w_n\} \subset E$, the functionals $\{Aw_1, Aw_2, ..., Aw_n\}$ are linearly independent, E is said to be a connectivity basis.

The following lemma is straightforward, but will be useful in applications.

Lemma 9.1. *Let $I \subset D$ be a commutative subspace for P. Then I is completely regular, if and only if, it possesses a c-complete subset.*

Proof. Because when $E \subset I$ is c-complete, one has that

$$\langle Au, w\rangle = 0 \quad \forall\, w \in I \Rightarrow \langle Au, w\rangle = 0 \;\; \forall w \in E \Rightarrow u \in I \tag{9.4}$$

Clearly (4.3) follows from (9.1) and the fact that I is commutative subspace for P.

X. CONNECTIVITY AND HILBERT-SPACE BASES

The concepts of c-complete subset and connectivity bases are purely algebraic. A connection between these notions and Hilbert-space bases is given in this section.

Given a separable Hilbert-space H, we consider the space $\hat{D} = H \oplus H$. Elements of $\hat{D}$ will be written as $\hat{u} = [u_1, u_2]$, with $u_1 \in H$ and $u_2 \in H$. With every subset $\hat{B} \subset \hat{D}$, we associate two sets

$$B_1 = \{u_1 \in H \quad \exists\; \hat{u} = [u_1, u_2] \in \hat{B}\} \subset H \tag{10.1a}$$

$$B_2 = \{u_2 \in H \quad \exists\; \hat{u} = [u_1, u_2] \in \hat{B}\} \subset H \tag{10.1b}$$

The notation $A : D \to D^*$ will be used to represent an antisymmetric operator given for every $\hat{u} \in \hat{D}$ and $\hat{v} \in \hat{D}$ by

$$\langle A\hat{u}, \hat{v}\rangle = (u_1, v_2) - (u_2, v_1) \;: \tag{10.2}$$

In what follows $\hat{I}_P \subset \hat{D}$ will be a commutative linear subspace for A; i.e.,

$$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall\; \hat{u} \in \hat{I}_P \;\text{ and }\; \hat{v} \in \hat{I}_P \;. \tag{10.3}$$

The subsets $I_{P_1} \subset H$ and $I_{P_2} \subset H$ are defined replacing $\hat{B}$ by $\hat{I}_P$ in

(10.1). When $\hat{I}_P \subset \hat{D}$ is a linear subspace, both $I_{P_1} \subset H$ and $I_{P_2} \subset H$, are linear subspaces, but they are not necessarily closed.

Assume

$$I_{P1} \supset I_{P2}^{\perp};\quad I_{P2} \supset I_{P1}^{\perp} \tag{10.4}$$

Define

$$H_o = \bar{I}_{P1} \cap \bar{I}_{P2} \tag{10.5}$$

Clearly $H_o \subset H$ is a closed Hilbert subspace and

$$H_o^{\perp} = I_{P1}^{\perp} + I_{P2}^{\perp} \ . \tag{10.6}$$

Define

$$I_{P10} = \text{proj}_0\ I_{P1}\ ;\quad I_{P20} = \text{proj}_0\ I_{P2} \tag{10.7}$$

Here proj_0 stands for the projection on the subspace H_0.

Lemma 10.1. *Let $\hat{I}_P \subset \hat{D}$ be a linear subspace for which (10.4) holds, then*

$$I_{P10} = I_{P1} \cap H_0\ ;\quad I_{P20} = I_{P2} \cap H_0. \tag{10.8}$$

$$\bar{I}_{P1} = H_o + I_{P2}\ ;\quad \bar{I}_{P2} = H_o + I_{P1} \tag{10.9}$$

$$H_o = \bar{I}_{P10} = \bar{I}_{P20}\ . \tag{10.10}$$

Proof. Recall, when (10.4) holds, $I_{P1}^{\perp}$ and $I_{P2}^{\perp}$ are orthogonal subspaces of H, because $\bar{I}_{P_1} \supset I_{P2}^{\perp}$. Equation (10.6) implies

$$H = H_o + I_{P1}^{\perp} + I_{P2}^{\perp} \tag{10.11}$$

This equation exhibits H as a sum of orthogonal subspaces. Equations (10.8) to (10.10) easily follow from equation (10.11).

Define

$$\hat{N}_1 = \{[u_1, 0] \in \hat{D} | u_1 \in I_{P2}^{\perp}\} ;$$

$$\hat{N}_2 = \{[0, u_2 \in \hat{D} | u_2 \in I_{P1}^{\perp} \{ \} . \tag{10.12}$$

Theorem 10.1. *Let $\hat{I}_P \subset \hat{D}$ be such that:*
i) *There is a dense linear subspace $E \subset H$ such that*

$$E \supset I_{P1}^{\perp} + I_{P2}^{\perp} \tag{10.13a}$$

ii) *For every $\hat{u} \in \hat{E} = E \oplus E$, one has*

$$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{v} \in \hat{I}_P \Leftrightarrow \hat{u} \in \hat{I}_P . \tag{10.13b}$$

Then, there is a unique subspace $\hat{I}_P^c \subset \hat{D}$, which is completely regular for $A : \hat{D} \to \hat{D}^*$ and such that

$$\hat{I}_P \subset \hat{I}_P^c \tag{10.14}$$

Even more, if $\hat{B} \subset \hat{I}_p$ is such that

$$\text{span}\, \hat{B}_1 = \bar{I}_{P1} \quad \text{and} \quad \text{span}\, \hat{B}_2 = \bar{I}_{P2} , \tag{10.15}$$

then $\hat{B} \subset \hat{I}_P^c$ is c-complete for $\hat{I}_P^c$ with respect to $A : \hat{D} \to \hat{D}^*$.

Proof. Notice that in the presence of assumption (ii), (i) implies

$$\hat{I}_P \supset \hat{N}_1 + \hat{N}_2 \tag{10.16}$$

This in turn implies that (10.4) is satisfied. Let $\hat{B} = \{\hat{w}_1, \hat{w}_2, ...\} \subset \hat{I}_P$ be any denumerable subset which satisfies the first of equations (10.15). Such choice is clearly possible, because H is separable. Write $\hat{w}_\alpha = [w_{\alpha 1}, w_{\alpha 2}]$, $\alpha = 1, 2, ...$. There is no lack of generality by assuming that such subset is chosen so that $\{w_{11}, w_{21}, w_{31},...$ is orthonormal. Define $I^c_{P1} \subset H$ by the condition

$$u_1 \in I^c_{P1} \Leftrightarrow u_1 \in \bar{I}_{P1} \quad \text{and} \quad \sum_{\alpha=1}^{\infty} a_\alpha^2 < \infty \tag{10.17}$$

where

$$a_\alpha = (u_1, w_{\alpha 2}) \; . \tag{10.18}$$

Let the mapping $\tau_2 : I^c_{P1} \rightarrow H$ be given by

$$\tau_2(u_1) = \sum_{\alpha=1}^{\infty} a_\alpha w_{\alpha 1} \tag{10.19}$$

for every $u_1 \in I^c_{P1}$ Clearly τ_2 is well defined by virtue of (10.17). Write

$$\hat{I}^c_P = \{\hat{u} = [u_1, \tau_2(u_1)] \in \hat{D} \mid u_1 \in I^c_{P1}\} + \hat{N}_2 \; . \tag{10.20}$$

With this definition $\hat{I}^c_P$ and I^c_{P1} satisfy a relation similar to (10.1a).
Next, we proceed to prove that indeed $\hat{I}^c_P$ possesses the properties asserted in theorem 10.1. These will follow from

a) $$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{u} \in \hat{I}^c_P \quad \text{and} \quad \hat{v} \in \hat{I}^c_P \tag{10.21}$$

b) For every $\hat{u} \in \hat{D}$ one has

$$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{v} \in \hat{I}_P \Rightarrow \hat{u} \in \hat{I}^c_P \; . \tag{10.22}$$

In order to prove (a), notice that

$$(w_{\alpha 2}, w_{\beta 1}) = (w_{\alpha 1}, w_{\beta 2}) \quad \forall \alpha = 1, 2, ... \quad \text{and} \quad \beta = 1, 2, ... \tag{10.23}$$

For every $v_1 \in I^c_{P1}$

$$(v_1, w_{\alpha 2}) = \sum_{\beta=1}^{\infty} (w_{\alpha 2}, w_{\beta 1})(v_1, w_{\beta 1}) =$$

$$\sum_{\beta=1}^{\infty} (w_{\alpha 1}, w_{\beta 2})(v_1, w_{\beta 1}) \ . \tag{10.24}$$

When $\hat{u} = [u_1, u_2] \in \hat{I}^c_P$ and $\hat{v} = [v_1, v_2] \in \hat{I}^c_P$, then

$$(u_1, v_2) = \sum_{\alpha=1}^{\infty} (v_1, w_{\alpha 2})(u_1, w_{\alpha 1})$$

$$= \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (w_{\alpha 1}, w_{\beta 2})(v_1, w_{\beta 1})(u_1, w_{\alpha 1}) \tag{10.25}$$

Equation (10.25) together with (10.23) imply (10.21).

To prove (b) recall that when the first of equations (10.15) holds, for any $\hat{u} = [u_1, u_2] \in \hat{D}$, one has that

$$u_1 \in \overline{I}_{P1} \quad \text{and} \quad \langle A\hat{u}, \hat{w}\rangle = 0 \quad \forall \hat{w} \in \hat{B} \subset \hat{I}_P \Rightarrow \hat{u} \in \hat{I}^c_P \ . \tag{10.26}$$

This can be seen by noticing that the premise in (10.26) implies

$$a_\alpha = (u_1, w_{\alpha 2}) = (u_2, w_{\alpha 1}) \tag{10.27}$$

so that

$$u_2 = \sum_{\alpha=1}^{\infty} a_\alpha w_{\alpha 1} + q_2 \ ; \quad q_2 \in I^{\perp}_{P1} \tag{10.28}$$

Therefore $\sum_{\alpha=1}^{\infty} a_\alpha^2 < \infty$, necessarily. Equation (10.28), permits writing

$$\hat{u} = [u_1, \tau_2(u_1)] + \hat{q} \in \hat{I}_P^c \ , \tag{10.29}$$

where $\hat{q} = [0, q_2] \in \hat{N}_2$.

Therefore, in order to prove (b), it is only necessary to prove that the premise in (10.22) implies that $u_1 \in \bar{I}_{P1}$. This is straightforward, because the premise in (10.22) implies by (10.16) that

$$\langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{v} \in \hat{N}_2 \ ; \tag{10.30}$$

i.e. that

$$(u_1, v_2) = 0 \quad \forall v_2 \in I_{P1}^{\perp} \tag{10.31}$$

Hence $u_1 \in I_{P1}$.

Once (b) has been shown, it is seen that $\hat{I}_P^c \supset \hat{I}_P$, because any element $\hat{u} \in \hat{I}_P$ satisfies the premise in (b). Hence, $\hat{I}_P^c$ is completely regular by virtue of lemma 9.1, because $\hat{I}_P$ is a c-complete system for $\hat{I}_P^c$.

In order to show that $\hat{I}_P^c$ is the only completely regular subspace such that $\hat{I}_P^c \supset \hat{I}_P$, recall that if $\hat{I}' \supset \hat{I}_P$ is completely regular, then $\hat{I}_P^c \supset \hat{I}'$ by virtue of (10.22). When $\hat{I}'$ is completely regular, $\hat{I}'$ is a largest commutative subspace, thus $\hat{I}' \supset \hat{I}^c$. Hence $\hat{I}' = \hat{I}^c$.

In the previous proof, $\hat{B}$ satisfied the first of equations (10.15) but was otherwise arbitrary. However, the fact that $\hat{I}_P^c$ is unique implies that the linear space $\hat{I}_P^c$ constructed in the manner explained before, is the same independently of the particular system $\hat{B}$ chosen. In particular, the same mapping $\tau_2 : I_{P1}^c \to H_0$ is defined independently of the particular $\hat{B}$ used, as long as the first of equations (10.15) is satisfied.

In connection with the mapping τ_2, there are two points which are worth observing. In the previous construction, we could have started with a system $\hat{B}$ which satisfied the second of equations (10.15) and such that $\{w_{12}, w_{22}, \ldots\}$ is orthonormal and we could have defined $I_{P2}^c \subset H$ replacing (10.17) by the condition

$$u_2 \in I_{P2}^c \Leftrightarrow u_2 \in \bar{I}_{P2} \quad \text{and} \quad \sum_{\alpha=1}^{\infty} b_\alpha^2 < \infty \tag{10.32}$$

where

$$b_\alpha = (u_2, w_{\alpha 1}) \tag{10.33}$$

Then, define the mapping $\tau_1 \quad \mathcal{I}_{P2}^c \to H$ by

$$\tau_1(u_2) = \sum_{\alpha=1}^{\infty} b_\alpha w_{\alpha_2}. \tag{10.34}$$

This leads to replace (10.20) by

$$\hat{\mathcal{I}}_P^c = \{\hat{u} = [\tau_1(u_2), u_2] \in \hat{\mathcal{D}} \mid u_2 \in \mathcal{I}_{P2}^c\} + \hat{N}_1 \tag{10.35}$$

However, definitions (10.20) and (10.35) are equivalent, because one can show that $\hat{I}_P^c$ as defined by (10.35) is completely regular for $A : \hat{\mathcal{D}} \to \hat{\mathcal{D}}^*$ and $\hat{\mathcal{I}}_P^c \supset \hat{\mathcal{I}}_P$. This shows by the way that

$$\tau_2(\mathcal{I}_{P1}^c) = \mathcal{I}_{P2}^c \cap H_o \quad ; \quad \tau_1(\mathcal{I}_{P1}^c) = \mathcal{I}_{P1}^c \cap H_o \tag{10.36}$$

Therefore, one can write

$$\tau_1 \quad \mathcal{I}_{P2}^c \to H_o \quad ; \quad \tau_2 \quad \mathcal{I}_{P1}^c \to H_o \tag{10.37}$$

By virtue of (10.26), in order to prove the second part of theorem 10.1, it is only necessary to prove that when equations (10.15) hold, for every $\hat{u} \in \hat{\mathcal{D}}$, one has

$$\langle A\hat{u}, \hat{w}\rangle = 0 \quad \forall\, \hat{w} \in \hat{\mathcal{B}} \Rightarrow u_1 \in \overline{\mathcal{I}}_{P1} \tag{10.38}$$

Assume $\hat{\mathcal{B}}$ satisfies (10.15), then without lack of generality, we can choose $\mathcal{B}_2 = \{w_{12}, w_{22}, w_{32}, \ldots\}$ as orthonormal and spaning $\overline{\mathcal{I}}_{P2}$. If $\hat{u} = [u_1, u_2] \in \hat{\mathcal{D}}$ satisfies the premise in (10.38), then

$$(u_1, w_{\alpha 2}) = (u_2, w_{\alpha 1} \qquad \forall\, \alpha = \ , 2, \tag{10.39}$$

Using (10.39), a direct computation shows that

$$(u_1, v_2) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (w_{\alpha 1}, w_{\beta 2})(u_2, w_{\beta 2})(v_2, w_{\alpha 2}) \tag{10.40a}$$

and

$$(u_2, v_1) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} (w_{\alpha 2}, w_{\beta 1})(u_2, w_{\beta 2})(v_2, w_{\alpha 2}) \qquad (10.40b)$$

whenever $\hat{v} = [v_1, v_2] \in \hat{I}_P^c$ and $v_1 \in H_0$. Taking $v_1 = 0$, equations (10.40) together imply

$$(u_1, v_2) = (u_2, v_1 = 0 \quad \forall\, v_2 \in I_{P1}^{\perp} \; ; \qquad (10.41)$$

i.e., $u_1 \in I_{P1}$. This shows (10.38). Hence, theorem 10.1.

There is a corollary of theorem10.1, that will be used later.

Corollary 10.1. *Under the assumptions of theorem 10.1, let* $\hat{u} = [u_1, u_2] \in H_0 \oplus H_0$ *and* $\hat{v} = [v_1, v_2] \in H_0 \oplus H_0$, *be such that* $\hat{u} \in \hat{I}_P^c$ *and* $\hat{v} \in \hat{I}_P^c$. *Then*

$$u_1 = v_1 \Leftrightarrow u_2 = v_2 \qquad (10.42)$$

Proof. When $u_1 = v_1$, then $[0, u_2 - v_2] \in \hat{I}_P^c$; $u_2 - v_2 \in I_{P1}^{\perp}$. This implies $u_2 - v_2 = 0$ because $u_2 - v_2 \in H_0$. A similar argument shows the converse.

The following general result will also be used in the sequel.

Lemma 10.2. *Let* $E_1 \subset H$ *be a dense linear subspace of* H. *Assume* $H' \subset H$ *is a closed Hilbert subspace of* H. *Then if the mapping* $\mu : E_1 \rightarrow H'$ *defined for every* $u_1 \subset E_1$ *by*

$$\mu(u_1) = \text{proj}\,(u_1) \qquad (10.43)$$

is one-to-one, one necessarily has $H' = H$. *In equation (10.43),* proj (u_1) *stands for the projection of* u_1 *on* H'.

Proof. Assume H' is a proper subspace of H. Let $\xi_0 \in H$, be a unit vector orthogonal to H' and let H'' be the orthogonal complement of ξ_0: then, $H'' \supset H'$. Let μ'' be the projection on H''. Then $\mu'' : E_1 \rightarrow H''$ is also one-to-one, because of the fact that $H'' \supset H'$ implies that for every $x \in E_1$ one has $\|\mu(x)\| \leqslant \|\mu''(x)\|$. Let $\{x_1, x_2, ...\} \subset E_1$, be an orthonormal system that spans H. Under the assumptions of the lemma, there exists a sequence of real numbers a_i such that

$$x_i = \mu''(x_i) + a_i\xi_0 \ , \qquad i = 1, 2, \dots \ . \tag{10.44}$$

Multiplication of (10.44) by ξ_0, yields

$$(\xi_0, x_i) = a_i \tag{10.45}$$

because $\mu''(x_i) \in H''$. Hence

$$\sum_{i=1}^{\infty} a_i^2 = ||\xi_0||^2 = 1 \ . \tag{10.46}$$

This shows that $a_i \to 0$ as $i \to \infty$; therefore, a_i is bounded. At the same time, by virtue of Pitagoras' theorem

$$0 \neq ||\mu''(x_i)||^2 = \ \ -a_i^2 \tag{10.47}$$

This implies that $a_i \neq 1$ for every $i = 1, 2, \dots$ and also that the mapping μ'' is bounded away from zero; i.e., the mapping is bicontinuous since μ'' is also bounded.

However, if $\{y_1, y_2, \dots\} \subset E_1$ is such that $y_n \xrightarrow[n\to\infty]{} \xi_0$ then $\mu''(y_n) \to 0$ while $\|y_n\| \to 1$. This implies that $(\mu'')^{-1}$ is unbounded, which in turn contradicts the fact that μ'' is bicontinuous.

Theorem 10.2. *Let assumptions of theorem 10.1 hold. Then, for any subset $\hat{B} \subset \hat{I}_P$ the following assertions are equivalent*

i) *For every $\hat{u} \in \hat{E}$, one has*

$$\langle A\hat{u}, \hat{w}\rangle = 0 \quad \forall \hat{w} \in \hat{B} \Rightarrow \hat{u} \in \hat{I}_P \tag{10.48}$$

ii) *$\hat{B}$ is c-complete in $\hat{I}_P^c$ with respect to $A \quad \hat{D} \to \hat{D}^*$.*

$$\text{iii) span } B_1 = \overline{I}_{P1} \quad \text{and} \quad \text{span } B_2 \supset I_{P1}^{\perp} \ . \tag{10.49}$$

$$\text{iv) span } B_2 = \overline{I}_{P2} \quad \text{and} \quad \text{span } B_1 \supset I_{P2}^{\perp} \ . \tag{10.50}$$

Proof. We prove first that (iii) ⇒ (iv). To this end we show that when (iii) holds, then

$$\text{span}\,\{\text{proj}_0\, \mathcal{B}_2\} = \mathcal{H}_0 \;, \tag{10.51}$$

where proj_0 stands for the projection on $\mathcal{H}_0$. Indeed, if $u \in \mathcal{H}_0 = \overline{\mathcal{I}}_{P1} \cap \overline{\mathcal{I}}_{P2}$ is such that

$$a_\alpha = (u, w_{\alpha 2}) = 0 \quad \forall\, \alpha = \;, 2, \ldots \;, \tag{10.52}$$

then $u \in \mathcal{I}^c_{P1}$ and

$$\tau_2(u) = \sum_{\alpha=1}^{\infty} a_\alpha w_{\alpha 1} = 0 \;; \tag{10.53}$$

i.e., $[u, 0] \in \hat{\mathcal{I}}^c_P$. This implies $u \in \mathcal{I}^\perp_{P2} \cap \mathcal{H}_0$. Hence, $u = 0$ by virtue of lemma 10.1, and (10.51) is established. Here, it was assumed that $\{w_{11}, w_{21}, \ldots\} = \mathcal{B}_1$ were orthonormal.

Equation (10.51) together with span $\mathcal{B}_2 \supset \mathcal{I}^\perp_{P1}$, imply span $\mathcal{B}_2 = \overline{\mathcal{I}}_{P2} = \mathcal{H}_0 + \mathcal{I}^\perp_{P1}$. Hence (iii) ⇒ (iv). The converse can be shown in a similar fashion.

To prove that (i) ⇒ (iii), we start showing the first of equations (10.49). Let $\mu : \mathcal{I}_{P1} \to \text{span}\; \mathcal{B}_1$, be defined for every $u_1 \in \mathcal{I}_{P1}$ by

$$\mu(u_1) = \sum_{\alpha=1}^{\infty} c_\alpha w_{\alpha 1} \;, \tag{10.54}$$

where

$$c_\alpha = (u_1, w_{\alpha 1}) \tag{10.55}$$

and $\{w_{11}, w_{21}, \ldots\}$ are again assumed to be orthonormal. Clearly μ is the projection of the linear subspace $\mathcal{I}_{P1}$ on the span $\mathcal{B}_1 \subset \overline{\mathcal{I}}_{P1}$ and recall that $\mathcal{I}_{P1}$ is dense in $\overline{\mathcal{I}}_{P1}$. Now, this mapping is one-to-one, because if $u_1' \in \mathcal{I}_{P1}$ then

$$\mu(u_1) = \mu(u_1') \Rightarrow (u_1 - u_1', w_{\alpha 1}) = 0 \quad \forall \alpha = 1, 2, \ldots , \qquad (10.56)$$

and therefore $[0, u_1 - u_1'] \in \hat{I}_P$ by (i). This implies $u_1 - u_1' \in I_{P2}$; i.e., $u_1 - u_1' \in I_{P1} \cap I_{P2} \subset H_0$. Hence, $u_1 = u_1'$ by corollary 10.1. Application of lemma 10.2, yields the first of equations (10.49). In a similar fashion one can obtain the first of equations (10.50). Form these two equations the desired result follows.

Either (iii) or (iv) imply equations (10.15), because they are equivalent. Hence, (iii) $\Rightarrow$ (ii) or equivalently (iv) $\Rightarrow$ (ii), by virtue of theorem 10.1

To show that (ii) $\Rightarrow$ (i), recall that $\hat{I}_P \subset \hat{I}_P^c$. Therefore, when (ii) holds, for any $\hat{u} \in \hat{E} \subset \hat{D}$ one has that

$$\langle A\hat{u}, \hat{w}\rangle = 0 \quad \forall \hat{w} \in \hat{B} \Rightarrow \langle A\hat{u}, \hat{v}\rangle = 0 \quad \forall \hat{v} \in \hat{I}_P \subset \hat{I}_P^c \ . \qquad (10.57)$$

This shows that the premise in (10.57) implies the premise in (10.13b); hence, $u \in \hat{I}_P$. This proves that $\hat{B} \subset \hat{I}_P$ satisfies (i).

XI. AN INTRODUCTORY EXAMPLE

To fix ideas let us consider a simple example. Let $u(x, y)$ and $v(x \cdot y)$ be solutions of the biharmonic equation in a horizontal strip (Fig. 2); i.e.

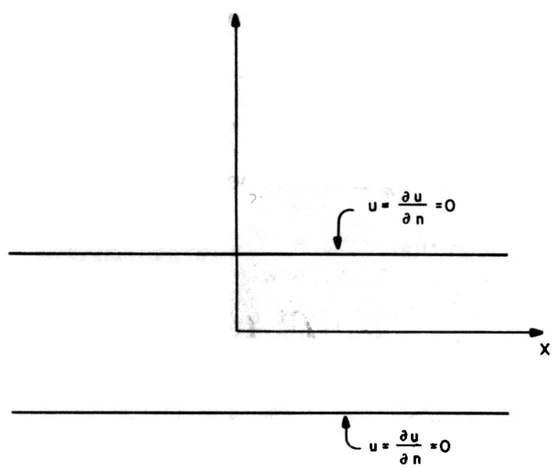

Figure 2

$$\Delta^2 u = \Delta^2 v = 0, \qquad 1 < y < 1, \qquad \infty < x < \infty \tag{11.1a}$$

such that

$$u = v = 0 \; ; \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} = 0, \quad \text{at } y = \pm 1 \; . \tag{11.1b}$$

Then, one can define an antisymmetric bilinear functional A_0, by

$$\langle A_o u, v\rangle = \int_{-1}^{1} \left\{ v\frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial v}{\partial x} + \Delta v \frac{\partial u}{\partial x} - u\frac{\partial \Delta v}{\partial x} \right\}_{x=\xi} dy \tag{11.2}$$

where $-\infty < \xi < +\infty$. Well known reciprocity relations for the biharmonic equation imply that the expression for A_0 given by (11.2) is independent of ξ whenever equations (11.1) are satisfied.

Separable solutions satisfy [46, 47]

$$\phi_n(x, y) = f_n(y) e^{-\lambda_n x} \tag{11.3}$$

where

$$\sin^2 2\lambda_n - 4\lambda_n^2 = 0 \; .$$

It can be shown [47] that for everyone of the roots λ_n of (11.4), one has that

i) $\operatorname{Re} \lambda_n \neq 0$

ii) If λ_n is a root, then $-\lambda_n$ is also a root.

Using (1 2), it is seen that

$$\langle A_o \phi_n, \phi_m\rangle = e^{-(\lambda_n + \lambda_m)\xi} \int_{-1}^{1} \left\{ \phi_m \frac{\partial \Delta \phi_n}{\partial x} - \Delta \phi_n \frac{\partial \phi_m}{\partial x} \right.$$
$$\left. + \Delta \phi_m \frac{\partial \phi_n}{\partial x} - \phi_n \frac{\partial \Delta \phi_m}{\partial x} \right\}_{x=0} dy \tag{11.6}$$

which holds for every ξ. Hence

$$\lambda_n + \lambda_m \neq 0 \Rightarrow \langle A_o \phi_n, \phi_m \rangle = 0 \ . \tag{11.7}$$

Let us restrict the definition (11.3) by the condition Re $\lambda_n \geqslant 0$ and introduce the notation ($n \geqslant 1$)

$$\phi_n^*(x,y) = f_n^*(y) e^{\lambda_n x} \tag{11.8}$$

Then it can be shown that

$$\langle A_0 \phi_n, \phi_n^* \rangle \neq 0 \ . \tag{11.9}$$

The notation established by Eqs. (11.3) and (11.8) classifies separable solutions into two disjoint groups. Define N_P^1 as the linear manifold of functions spanned by the system $\{\phi_1, \phi_2, \ldots\}$; while N_P^2 is defined correspondingly, using $\{\phi_1^*, \phi_2^*, \ldots\}$ instead. Let

$$N_P = N_P^1 + N_P^2 \ . \tag{11.10}$$

The properties characterizing the subspaces N_P^1 and N_P^2 are

$$u \to 0 \quad \text{as } x \to +\infty \quad \text{whenever } u \in N_P^1 \ , \tag{11.11a}$$

$$u \to 0 \quad \text{as } x \to -\infty \quad \text{whenever } u \in N_P^2 \ . \tag{11.11b}$$

The null subspace N_{A0} of A_0, will be needed. This is

$$N_{A0} = \{u \in N_P \mid \langle A_0 u, v \rangle = 0 \quad \forall v \in N_P\} \tag{11.12}$$

By virtue of (11.7) and (11.9), the only function belonging to this space is the zero function, i.e.

$$N_{A0} = \{0\} \ . \tag{11.13}$$

We notice the following properties of these spaces:

a) N_P^1 and N_P^2 are commutative subspaces; i.e.,

$$\langle A_0 u, v \rangle = 0 \quad \forall u \in N_P^\alpha \quad \text{and} \quad v \in N_P^\alpha \;;\; \alpha = \;\;, 2 \qquad (1\;\;14)$$

b) $N_P^\alpha \supset N_{A0}$; $\alpha = 1, 2$ (11.15)

c) Given $u \in N_P$

$$\langle A_0 u, v \rangle = 0\,, \quad \forall v \in N_P^\alpha \Rightarrow u \in N_P^\alpha \cdot \;\; \alpha = \;\;, 2 \qquad (11.16)$$

d) For every $u \in N_P$, there exist elements $u_1 \in N_P^1$ and $u_2 \in N_P^2$ such that

$$u = u_1 + u_2 \;. \qquad (11.17)$$

e) $N_{A0} = N_P^1 \cap N_P^2$. (11.18)

Notice that property (11.15) is trivially satisfied in this case. However, in further applications of the theory this will not be so.

We recall that the families

$$\mathcal{B} = \{\phi_1, \phi_2, .. \;\; \subset N_P^1 \qquad (11.19a)$$

and

$$\mathcal{B}^* = \{\phi_1^*, \phi_2^*, ... \;\; \subset N_P^2 \qquad (11.19b)$$

have the following properties given any $u \in N_P$, one has

$$\langle A_0 u, \phi_n \rangle = 0 \quad n = \;\;, 2, ... \Rightarrow u \in N_P^1 \qquad (11.20a)$$

and

$$\langle A_0 u, \phi_n^* \rangle = 0 \quad n = 1, 2, \ldots \Rightarrow u \in N_P^2 \tag{11.20b}$$

From equations (11.7) to (1 9), it follows that

$$\langle A_0 \phi_n, \phi_m^* \rangle = 0 \quad \text{if } n \neq m \ , \tag{11.21}$$

and

$$\langle A_0 \phi_n, \phi_n^* \rangle \neq 0 \quad \text{if } n = 1, 2, \ldots \ . \tag{11.22}$$

Hence, multiplying each of the functions of the families of separable solutions by suitable constants, one can assume that

$$\langle A_0 \phi_n, \phi_m^* \rangle = \delta_{nm} \qquad \text{(1 23)}$$

Clearly, properties (a) and (b) show that N_P^1 and N_P^2 are regular subspaces for $A_0 : N_P \to N_P^*$. Then, (d) imply that $\{N_P^1, N_P^2\}$ is a canonical decomposition of N_P. Properties (e) and (c) follow from theorem 4.1. In view of definition 9.1, equations (2.20) show that $\mathcal{B} \subset N_P^1$ and $\mathcal{B}^* \subset N_P^2$ are c-complete for N_P^1 and N_P^2, respectively. In addition, the families $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal by virtue of (11.21) and by multiplying them by suitable constants they can be transformed into biorthonormal families.

Now, any function $u \in N_P$ can be written as

$$u = \sum_{n=1}^{\infty} a_n \phi_n + \sum_{n=1}^{\infty} b_n \phi_n^* \ . \tag{11.24}$$

It is convenient to recall that each of the systems of constants a_n, b_n $(n = 1, 2, \ldots)$ possess only a finite number of non-vanishing elements because N_P^1 and N_P^2 were defined as the linear mani-folds spanned by separable solutions. Later actual infinite series will be considered, but this has been here avoided to keep this introductory example sufficiently simple. When the systems $\{\phi_1, \phi_2, \ldots\}$ and $\{\phi_1^*, \phi_2^*, \ldots\}$ are biorthonormal, it is straightforward to verify that

$$a_n = \langle A_0 u, \phi_n^* \rangle \tag{11.25a}$$

$$b_n = \langle A_0 \phi_n, u\rangle \ . \tag{11.25b}$$

Using the notions thus far introduced we can summarize our results as follows:

"The space of biharmonic functions N_P defined in a horizontal strip admits a canonical decomposition into two-subspaces $\{N_P^1, N_P^2\}$, such that the families of separable solutions $\{\phi_1, \phi_2, \ldots\} \subset N_P^1$ and $\{\phi_1^*, \phi_2^*, \ldots\} \subset N_P^2$ are c-complete for N_P^1 and N_P^2, respectively. Even more, these two systems are biorthogonal and by a suitable choice they can be taken to be biorthonormal. In this case any function of the space N_P can be represented by means of (11.24), where the coefficients a_n and b_n (n = 1, 2, ...) are given by (11.25)."

This part is devoted to explain how this simple scheme can be formulated in a manner that can be applied to a very general class of partial differential equations relevant in continuum mechanics and other fields of application. In the very simple introductory example given here, the space N_P is not equipped with a topological structure. However, in more general situations to be treated later, topological considerations will have to be included.

XII. CANONICAL DECOMPOSITIONS OF THE SPACE OF SOLUTIONS

When $P : D \to D^*$ is associated with a differential equation, the homogeneous equation is

$$Pu = 0 \ . \tag{12.1}$$

Thus, the space of solutions of the homogeneous equation is the null subspace N_P, of P.

Let $A = P - P^*$ and assume $A_1 : D \to D^*$ and $A_2 : D \to D^*$ are antisymmetric operators such that

a) $$A = A_1 \dotplus A_2 \qquad 12.2a)$$

b) A_1 and A_2 can be varied independently; i.e.,

$$D = N_{A1} + N_{A2} \ , \tag{12.2b}$$

it is possible to construct canonical decompositions of the space of solutions N_P. The corresponding theory has been developed systematically in the appendix. Here, we only recall a few results and give some examples.

When $u \in N_P$ and $v \in N_P$, ones has

$$\langle A_1 u, v\rangle + \langle A_2 u, v\rangle = \langle Au, v\rangle = 0 \tag{12.3}$$

This shows

$$\langle A_1 u, v\rangle = -\langle A_2 u, v\rangle \ , \quad \forall u \in N_P \ , \quad v \in N_P \tag{12.4}$$

In view of (12.4), one can define an operator $A_0 : N_P \to N_P^*$, given by

$$\langle A_0 u, v\rangle = \langle A_1 u, v\rangle = -\langle A_2 u, v\rangle \quad \forall u \in N_P \ , \quad v \in N_P \ . \tag{12.5}$$

Let $N_P^1 \subset N_P$ and $N_P^2 \subset N_P$ be two commutative subspaces of solutions, which span the space N_P; i.e.,

i) $$N_P = N_P^1 + N_P^2 \tag{12.6}$$

ii) For every $u_1 \in N_P^1$ and $v_1 \in N_P^1$, one has

$$\langle A_0 u_1, v_1\rangle = 0 \tag{12.7}$$

iii) For every $u_2 \in N_P^2$ and $v_2 \in N_P^2$, one has

$$\langle A_0 u_2, v_2\rangle = 0 \tag{12.8}$$

When (i) to (iii) are satisfied, given any solution $u \in N_P$, one can write $u = u_1 + u_2$, with $u_1 \in N_P^1$ and $u_2 \in N_P^2$, by virtue of (12.6). Therefore,

$$\langle A_0 u, v\rangle = \langle A_0 u_1, v_2\rangle + \langle A_0 u_2, v_1\rangle \quad \forall u \in N_P \ , \ v \in N_P \tag{12.9}$$

where (12.7) and (12.8) have been used. In view of the above, it is not difficult to establish the theorem that follows.

Theorem 12.1. *Given $P : D \to D^*$, A_1 and A_2 satisfying equations (12.2), let $N_P^1 \subset N_P$ and $N_P^2 \subset N_P$ be linear subspaces for which (i) to (iii) hold. Define*

$$I_P = N_P + N_A \tag{12.10}$$

and assume $I_P \subset D$ *is completely regular for* $A \quad D \to D^*$. *Then, if*

$$(N_P \cap N_{A1}) + (N_P \cap N_{A2}) \subset N_A \tag{12.11}$$

the pair $\{N_P^1, N_P^2\}$ *constitutes a canonical decomposition of* N_P, *with respect to* $A_0 : N_P \to N_P^*$, *as given by (12.5). When this is the case,*

$$N_{A0} = I_P \cap N_A \tag{12.12}$$

Proof. The proof is given in the appendix.

We illustrate the material contained in this section by considering a very simple example. Take, as in Sec. XI, $D = C^\infty(R)$ and let R be the unit square $0 < x < 1, 0 < y < 1$ (Fig. 3). Define

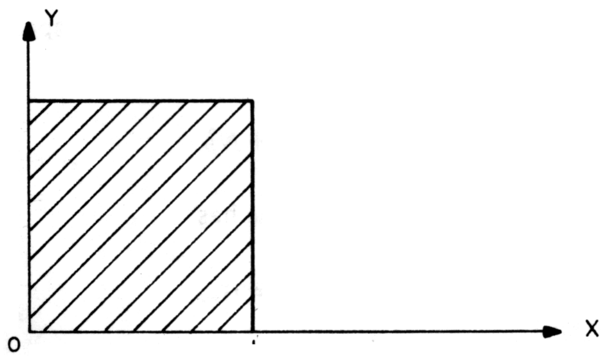

Figure 3

$$\langle Pu, v\rangle = \int_R v\nabla^2 u dx + \int_0^1 u \frac{\partial v}{\partial x}\bigg|_{x=1} dy - \int_0^1 u\frac{\partial v}{\partial x}\bigg|_{x=0} dy \tag{12.13}$$

Then

$$\langle Au, v\rangle = \int_0^1 \left\{ v \frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right\}_{y=0}^{y=1} dx \tag{12.14}$$

Define

$$\langle A_1 u, v\rangle = \int_0^1 \left\{ v\frac{\partial u}{\partial y} \quad u\frac{\partial v}{\partial y}\right\}_{y=1} dx \tag{12.15a}$$

$$\langle A_2 u, v\rangle = -\int_0^1 \left\{ v\frac{\partial u}{\partial y} \quad u\frac{\partial v}{\partial y}\right\}_{y=0} dx \;. \tag{12.15b}$$

Then equations (12.2) are satisfied, because

$$\{u \in D \mid u = \frac{\partial u}{\partial y} = 0 \;, \quad \text{at } y = 1 \tag{12.16a}$$

$$= \{u \in D \mid u = \frac{\partial u}{\partial y} = 0 \;, \quad \text{at } y = 0\} \tag{12.16b}$$

Notice that the space of solutions $N_P \subset D$, is made, in this case, of the functions which are harmonic in the unit square and vanish on the sides of the square; i.e.,

$$N_P = \{u \in D \quad \nabla^2 u = 0 \;, \quad \text{on } R, \; u = 0 \text{ at } x = 0, 1\} \tag{12.17}$$

The space of solutions can be decomposed into two subspaces

$$N_P^1 = \{u \in N_P \mid u = 0 \;, \quad \text{at } y = 1\} \;, \tag{12.18a}$$

$$N_P^1 = \{u \in N_P \mid u = 0 \;, \quad \text{at } y = 0\} \;. \tag{12.18b}$$

Then, it is straightforward to verify (i) to (iii) as well as (12.11). Hence, $\{N_P^1, N_P^2\}$ is a canonical decomposition of the space of solutions N_P.

The assumption (12.11) is similar to the condition that an overdetermined problem has only the trivial solution. It can also be derived, in some applications, by means of analytic continuation arguments. For the specific example given here, it follows from the fact that the only function which vanishes at $x = 0, 1$, is harmonic in the square and vanishes together with its normal derivative, either at the top or at the bottom of the square, is the zero function (i.e. the function which is identically zero in the square).

In applications, $A_0 : N_P \to N_P^*$, has many alternative expressions. For example, if one defines the bilinear functional $A(\lambda)$, by

$$\langle A(\lambda)u, v\rangle = \int_0^1 \{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\}_{y=\lambda} dx \; ; \quad 0 \leqslant \lambda \leqslant 1 \; . \tag{12.19}$$

Then, in the example here considered, for every $u \in N_P$ and $v \in N_P$, one has

$$\langle A_0 u, v\rangle = \langle A(\lambda)u, v\rangle \; , \quad \forall \lambda \in [0, 1 \tag{12.20}$$

A corollary of theorem 12.1, that will be used when discussing biorthogonal functions, is that for every $u \in N_P$, one has that N_P^1 and N_P^2 are completely regualr for A_0; i.e.,

$$u \in N_P^1 \Leftrightarrow \langle A_0 u, v\rangle = 0 \quad \forall v \in N_P^1 \tag{12.21a}$$

and

$$u \in N_P^2 \Leftrightarrow \langle A_0 u, v\rangle = 0 \quad \forall v \in N_P^2 \tag{12.21b}$$

In the specific example given in this section, relation (12.21a) implies that a harmonic function u that vanishes at the sides $x = 0, 1$ of the square, vanishes at the top, if and only if, the integral (12.19) vanishes for every harmonic function v, that satisfies the same conditions. Clearly, harmonic functions that vanish at the bottom of the square have a similar property due to (12.21b).

XIII. FOURIER BIORTHOGONAL SYSTEMS

Let $\{N_P^1, N_P^2\}$ be a canonical decomposition of the space of solutions N_P with respect to A_0.

Definition 13.1. *Let* $\mathcal{B}_1 = \{w_1, w_2, ...\} \subset N_P^1$ *and* $\mathcal{B}_2 = \{w_1^*, w_2^*, ...\} \subset N_P^2$ *be c-complete for* N_P^1 *and* N_P^2*, respectively, then* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are biorthogonal with respect to each other when*

$$\langle A_0 w_n, w_m^*\rangle| = 0 \; , \quad \textit{whenever } n \neq m \; . \tag{13.1}$$

They are biothonormal when

$$\langle A_0 w_n, w_m^*\rangle = \delta_{nm} \tag{13.2}$$

Lemma 13.1. *Assume the pair* $\mathcal{B}_1 = \{w_1, w_2, ...\} \subset N_P^1$ *and* $\mathcal{B}_2 = \{w_1^*, w_2^*, ...\} \subset N_P^2$, *is a c-complete pair of biorthogonal systems for* $A_0 : N_P \to N_P^*$ *such that*

$$A_0 w_n \neq 0 \quad \forall n = 1, 2 \ , \tag{13.3}$$

Then, it can be normalized (i.e., by multiplication, by a scalar of every one of its elements, one can derive a pair which is biorthonormal).

Proof. Clearly, the assertion of the lemma is true if $\langle A_0, w_n, w_n^* \rangle \neq 0$ for every $n = 1, 2, ...$. Assume

$$\langle A_0 w_n, w_n^* \rangle = 0 \tag{13.4}$$

for some n. Then

$$\langle A_0 w_n, w_m^* \rangle = 0 \quad \forall m = 1, 2, \tag{13.5}$$

This implies $w_n \in N_P^2$; i.e. $w_n \in N_P^1 \cap N_P^2 = N_{A0}$. This contradicts (13.3).

Notice that when biorthogonal systems $\mathcal{B}_1 \subset N_P^1$ and $\mathcal{B}_2 \subset N_P^2$, which are c-complete, are given, with every $u \in N_P = N_P^1 + N_P^2$, one can associate unique sequences $[a_1, a_2, ...]$, $[b_1, b_2, ...]$ by means of

$$a_\alpha = \langle A_0 u, w_\alpha^* \rangle \ ; \quad b_\alpha = \langle A_0 w_\alpha, u \rangle \ , \quad \alpha = 1, 2, ... \ . \tag{13.6}$$

Let $\mathcal{E} \subset N_P/N_{A0}$ be

$$\mathcal{E} = \{u \in N_P/N_{A0} \mid \sum_{\alpha=1}^{\infty} |a_\alpha|^2 < \infty, \sum_{\beta=1}^{\infty} |b_\alpha|^2 < \infty\} \ . \tag{13.7}$$

Then, on $\mathcal{E}$, one can define the inner product

$$(u, v) = \sum_{\alpha=1}^{\infty} a_\alpha \bar{a}'_\alpha + \sum_{\beta=1}^{\infty} b_\alpha \bar{b}'_\alpha \ , \tag{13.8}$$

where a'_α, b'_α are associated with v by means of equations corresponding to (13.6); in addition, the bars in (13.8), denote the complex conjugates. Let H be the closure of $\mathcal{E}$ in this inner product.

Of special interest is the case when $H \subset N_P/N_{A0}$. In this case one can show that the system $B_1 \cup B_2$ is orthonormal for the Hilbert-space H, with inner product given by (13.8). This inner product and the corresponding metric, will be said to be induced by the biorthogonal system B_1, B_2. Notice that

$$u = \sum_{\alpha=1}^{\infty} a_\alpha w_\alpha + \sum_{\beta=1}^{\infty} b_\beta w_\beta^* , \tag{13.9}$$

$$(u, v) = \langle A_0 u, v^* \rangle \tag{13.10}$$

$$v^* = -\sum_{\alpha=1}^{\infty} \bar{b}'_\alpha w_\alpha + \sum_{\alpha=1}^{\infty} \bar{a}'_\beta w_\beta^* . \tag{13.11}$$

Convergence in (13.9) is with respect to the induced metric, or any equivalent metric.

For applications, it is of course extremely important to establish criteria under which the induced metric is equivalent to a metric which is relevant for the problem considered. Some aspects of this question were discussed in section X.

As recalled in Sec. XII, equation (12.19), one usually has many alternative expressions for the operator $A_0 : N_P \to N_P^*$. Let $A(\lambda)$ be a family of bilinear functionals, susch that

$$\langle A_0 u, v \rangle = \langle A(\lambda) u, v \rangle \tag{13.12}$$

for every $u \in N_P$ and $v \in N_P$. Consider, as before, a canonical decomposition $\{N_P^1, N_P^2\}$ of N_P. Let $w_n \in N_P^1$ and $w_n^* \in N_P^2$, $n = 1, 2, ...$, be two families of solutions such that

$$\langle A(\lambda) w_n, w_m^* \rangle = f_{nm}(\lambda) \langle A(\lambda_0) w_n, w_m^* \rangle \tag{13.13}$$

in some range $a < \lambda < b$. Here λ_0 is a fixed value belonging to this range and $f_{nm}(\lambda)$ is for every $n, m = 1, 2, ...$ a function of λ. Then

"Either $f_{nm}(\lambda)$ is a constant or

$$\langle A_0 w_n, w_m^* \rangle = 0" \tag{13.14}$$

This is a general form of the alternative previously formulated by the author [51].

As an example, let N_P be the linear space of functions which are harmonic everywhere in the plane, except, possibly, at the origin. Let $A_0 : N_P \to N_P^*$ be

$$\langle A_0 u, v \rangle = \int_C \left\{ v \frac{\partial u}{\partial r} - u \frac{\partial v}{\partial r} \right\} dx \ , \tag{13.15}$$

where C is any circle with center at the origin and $\partial/\partial r$ stands for the directional derivative in the radial direction. By the procedure explained in Sec. XII, it can be shown that a canonical decomposition of N_P is the pair $\{N_P^1, N_P^2\}$, where N_P^1 is the set of functions which are harmonic in the whole plane, including the origin, while N_P^2 is made of the function $u \in N_P$, such that $u - b_0 \log r$ is square integrable in any region of the plane that excludes a neighborhood of the origin. Here

$$b_0 = \frac{1}{2\pi} \int_{C(\lambda)} \frac{\partial u}{\partial r} dx$$

It can be shown that the only element of N_{A_0} is the zero function.

A family of bilinear functionals $A(\lambda)$, with property (13.12), is

$$\langle A(\lambda) u, v \rangle = \int_{C(\lambda)} \left\{ v \frac{\partial v}{\partial r} - u \frac{\partial v}{\partial r} \right\} dx \tag{13.16}$$

where $C(\lambda)$ is a circle of radius λ and center at the origin. If $w_n \in N_P^1$ and $w_n^* \in N_P^2$, $n = 1, 2, \ldots$ are families of solutions of product form; i.e., if

$$w_n = f_n(r) p_n(\theta) \ ; \quad w_n^* = g_n(r) q_n(\theta) \ ; \tag{13.17}$$

then, using (13.16) it is seen that

$$\langle A(\lambda) w_n, w_m^* \rangle = [g_m(\lambda) f_n'(\lambda) - f_n(\lambda) g_m'(\lambda)] \lambda \langle A(1) w_n, w_m^* \rangle \ . \tag{13.18}$$

Application of the alternative (13.14), yields

$$\int_{C1} \left\{ w_m^* \frac{\partial w_n}{\partial r} \quad w_n \frac{\partial w_m^*}{\partial r} \right\} dx = 0 \tag{13.19}$$

unless

$$\{g_n(\lambda) f_n'(\lambda) - f_n(\lambda) g_n'(\lambda)\}\lambda = \text{const.} \quad 0 < \lambda < \infty \tag{13.20}$$

Solutions of product form are

$$\{w_1, w_2, \ldots\} = \{1, \ r\cos\theta, \quad \sin\theta, \ldots\} \subset N_P^1 \tag{13.21a}$$

and

$$\{w_1^*, w_2^*, \ldots\} = \{\log r, \ \ r^{-1}\cos\theta, \ \ r^{-1}\sin\theta\} \subset N_P^2 \tag{13.21b}$$

With these definitions, equations (13.19) and (13.20) imply that

$$\langle A_0 w_n, w_m^* \rangle = 0, \text{ if } n \neq m \text{ and } \begin{cases} n+1 \neq m & \text{when } n \text{ is even} \\ n-1 \neq n & \text{when } n \text{ is odd.} \end{cases} \tag{13.22}$$

This would give groups of two functions which are orthogonal to all the others. However, due to the manner in which they have been chosen, equation (13.22) holds whenever $n \neq m$.

XIV. BIORTHOGONAL FUNCTIONS FOR STRIPS AND WAVE GUIDES

The procedure explained in Sec. XII, for deriving biorthogonal systems, is applicable to arbitrary formally symmetric systems of equations. When the equation is not formally symmetric the procedure of Sec. III, theorem 3.2, can be used to transform it into a formally symmetric one.

There are many problems of mathematical physics which can be formulated using auxiliary potential functions. An example is linear elasticity which can be formulated in terms of potentials from which the displacement fields are derived. When a potential is used for elastostatic problems in a strip, the potential is biharmonic.

This leads to two alternative procedures when dealing with systems which admit a potential function representation; one is to obtain directly displacement fields of product form to which the alternative (13.14) applies directly, and the other one is to obtain product form potentials which satisfy the biharmonic or any other corresponding equations, to which the alternative (13.14) is applied.

The first approach was first used by Herrera [51] to obtain orthogonality relations for Rayleigh waves. It yields product form displacement fields which are biorthogonal with respect to the bilinear form A_0, which involve the displacement and the associated tractions and is given by equation (14.10).

The second approach, on the other hand, yields product form protentials which satisfy biorthogonal relations with respect to a bilinear form which does not involve the displacement fields directly. For example, when the potentials satisfy the biharmonic equation, the bilinear form is [25, 26]:

$$\langle A_0\phi, \psi\rangle = \int_{R_x} \left\{ \psi \frac{\partial \nabla^2 \phi}{\partial n} - \nabla^2 \phi \frac{\partial \psi}{\partial n} + \nabla^2 \psi \frac{\partial \phi}{\partial n} \quad \phi \frac{\partial \nabla^2 \psi}{\partial n} \right\}$$

The fact that in the bilinear form (14.1), the quantities involved in the boundary value problems relevant in elasticity do not occur, is a shortcoming of this approach that may lead to complications [46].

Application of the first approach, on the other hand, does not preclude the use of potentials; indeed, the use of them may be very valuable to construct the displacement fields of product form which satisfy the equations of elasticity, as will be seen in some of the examples given here.

For the development of biorthogonal systems of functions, it is convenient to consider a cylindrical region $R = R_x \oplus (-\infty, \infty)$, (Fig. 4), where R_x is a region of the n-dimensional Euclidean space $\mathcal{R}^n$ (here, only $n = 1, 2$, will be considered). The points of such region will be denoted by (x, y), where $\mathrm{x} \in R_x$, while $-\infty < y < \infty$. It will be assumed that the elastic tensor C_{ijpq} satisfying the usual symmetry conditions [54]

$$C_{ijpq} = C_{pqij} = C_{jipq}$$

is defined in R. When considering biorthogonal systems of functions it is convenient to let C_{ijpq} be a function of $\mathrm{x} \in R_x$ but to be independent

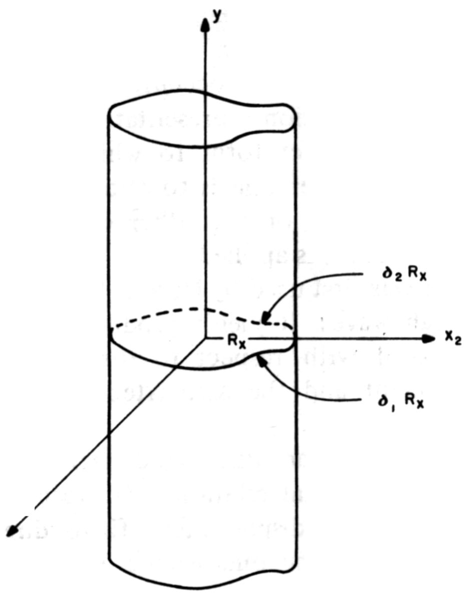

Figure 4

of y. For simplicity, let C_{ijpq} be C^∞ in R_x. In addition, C_{ijpq} will be strongly elliptic; i.e.,

$$C_{ijpq}\,\xi_i\eta_j\xi_p\eta_q > 0 \quad \text{whenever} \quad \xi_i\xi_i \neq 0 \;; \quad \eta_i\eta_i \neq 0 \tag{14.3}$$

The reduced equations of elastodynamics are

$$\frac{\partial \tau_{ij}}{\partial x_j}(\mathbf{u}) + \rho\omega^2 u_i = 0 \ . \tag{14.4}$$

Suitable boundary conditions on the lateral boundary of the cylinder are

$$\mathbf{u} = 0 \;; \quad \mathbf{x} \in \partial_1 R_x \ . \tag{14.5a}$$

$$\mathbf{T}(\mathbf{u}) = 0 \;; \quad \mathbf{x} \in \partial_2 R_x \ . \tag{14.5b}$$

Here, as it is usual

$$\tau_{ij}(u) = C_{ijpq}\frac{\partial u_p}{\partial x_q} \tag{14.6a}$$

and

$$\mathbf{T}(\mathbf{u}) = \tau_{ij}(\mathbf{u})n_j \tag{14.6b}$$

It is also assumed that the boundary ∂R_x of R_x, is decomposed into two parts $\partial_1 R_x$ and $\partial_2 R_x$. This induces a decomposition of the lateral boundary of the cylinder into $\partial_1 R = \partial_1 R_x \oplus (-\infty, \infty)$ and $\partial_2 R = \partial_2 R_x \oplus (-\infty, +\infty)$. The linear space N_P of functions to be considered will satisfy (14.4) and (14.5) in a distributional sense [55] in every subregion Ω of the cylinder $R_x\, x(-\infty, \infty)$; for every such subregion the displacement field is assumed to be such that $\mathbf{u} \in H^{3/2}(\Omega)$. Observe that equation (14.4) becomes the equation of elastostatics when $\omega = 0$.

Clearly

$$\int_\Omega v_i \frac{\partial \tau_{ij}}{\partial x_j}(\mathbf{u})d\mathbf{x} + \int_{\partial_1 \Omega} u_i T_i(\mathbf{v})d\mathbf{x} - \int_{\partial_2 \Omega} v_i T_i(\mathbf{u})d\mathbf{x} = 0 \tag{14.7}$$

for every $\mathbf{u} \in N_P$ and $\mathbf{v} \in N_P$. Here

$$\partial_1 \Omega = \partial\Omega \cap \partial_1 R \;; \quad \partial_2 \Omega = \partial\Omega \cap \partial_2 R \tag{14.8}$$

For any real number λ, $R_x(\lambda) = R_x \oplus \lambda$ denotes the cross-section of the cylinder at $y = \lambda$. Then, equation (14.7) implies that the bilinear functional

$$\langle A(\lambda)\mathbf{u}, v\rangle = \int_{R_x(\lambda)} \{v_i T_i(\mathbf{u}) - u_i T_i(\mathbf{v})\} d\mathbf{x} \tag{14.9}$$

defined for every $\mathbf{u} \in N_P$ an $\mathbf{v} \in N_P$ is independent of λ, when $-\infty < \lambda < \infty$. In order to have $A(\lambda)$ uniquely defined, it is assumed that the unit normal vector used in the computation of the tractions T, points upwards in the direction of increasing y.

Hence, one can define $A_0 : N_P \to N_P^*$ by

$$\langle A_0 \mathbf{u}, \mathbf{v}\rangle = \langle A(\lambda)\mathbf{u}, \mathbf{v}\rangle \,. \tag{14.10}$$

It can be shown that the only elements belonging to tne null subspace N_{A_0} is the identically zero displacement field.

We decompose the cylinder R, into two subregions R_+ and R_-; a point $(x, y) \in R_+$ when $y > 0$. R_- is defined correspondingly. The linear space N_P is decomposed into two subspaces $N_P^1 \subset N_P$ and $N_P^2 \subset N_P$, defined as follows: An element $u \in N_P$ belongs to N_P^1, if and only if, u is bounded in R_+ while $\tau_{ij}(u)$ is square integrable in R_+; N_P^2 is defined replacing R_+ by R_-. Under suitable assumptions of regularity for the region R_x, it can be shown that with these definitions assumptions (12.6) to (12.8) are satisfied. Thus, the pair $\{N_P^1, N_P^2\}$ constitutes a canonical decomposition of N_P with respect to $A_0 : N_P \to N_P^*$, as given by (14.10).

Let $w_n \in N_P^1$ and $w_n^* \in N_P^2$, be given by

$$w_n(x, y) = e^{-k_n y}\phi_n(x) \;;\quad w_n^*(x, y) = e^{k_n y}\phi_n^*(x) \;, \tag{14.11}$$

where $\text{Re}(k_n) > 0$. Application of the alternative (13.14), using equations (14.9) and (14.10), yields

$$\int_{R_{x(\lambda)}} \{w_m^* \; T(w_n) - w_n \cdot T(w_m^*)\} dx = 0 \;;\; \text{if} \;\; k_n \neq k_m \tag{14.12}$$

These are Herrera's [51] orthogonality relations. The generality of equation (14.12) must not be overlooked; it holds for any elastic fields of the form (14.11), in a cylinder or wave guide, for general inhomogeneous and anisotropic materials. We recall also that the range of the indexes in the elastic tensor C_{ijpq} may also be varied; we will be mainly interested in cases for which they can take the values 1 to 3 or, alternatively, only 1 and 2. Of course, the relevance of relations (14.12) depends on the existence of product form solutions; this has been discussed for a few special cases but a systematic discussion of the subject is lacking.

For applications to plane strain, one must take

$$C_{ijpq} = \lambda\delta_{pq}\delta_{ij} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \tag{14.13}$$

and when considering generalized plane stress λ must be replaced by $2\lambda\mu/(\lambda + 2\mu)$. In such applications, the range of Latin indexes is 1 and 2. Then, relation (14.12) is available and all what is required is to construct displacement fields of product form which satisfy (14.4) and (14.5). Notice that when R is a strip, R_x is a segment which will be taken to be $(-1,1)$. The boundary ∂R_x is made of two points, -1 and 1. It will be assumed that $\partial_2 R_x = \partial R$; i.e., only the case when the lateral boundary of the strip is stress-free will be considered.

For static problems, biharmonic potentials of product form have been given by Joseph [47]. Let

$$U_n(x, y) = e^{-k_n y} \psi_n(x) \qquad (14.14a)$$

and

$$U_n^*(x, y) = e^{k_n y} \psi_n(x) \qquad (14.14b)$$

be such potentials. When they are taken as the Airy functions and the boundary condition

$$\tau_{xx} = \tau_{xy} = 0 \; ; \quad \text{at } x = 0, 1 \qquad (14.15)$$

is imposed, one gets

$$k_n \sin k_n \cos k_n x - k_n x \cos k_n \sin k_n x \qquad (14.16a)$$

for even eigenfunctions. The odd eigenfunctions are

$$k_n \cos k_n \sin k_n x \quad k_n x \sin k_n \cos k_n x \; . \qquad (14.16b)$$

The eigenvalues corresponding to even and odd eigenfucntions, satisfy

$$2k_n + \sin 2k_n = 0 \; . \quad 2k_n - \sin 2k_n = 0 \qquad (14.17)$$

respectively.

The displacement fields associated with the potentials (14.14), can be derived from [46]:

$$2\mu \frac{\partial w_{n1}}{\partial x} = (1 - \nu) \frac{\partial^2 U_n}{\partial y^2} \quad \nu \frac{\partial^2 U_n}{\partial x^2} \qquad (14.18a)$$

$$2\mu \frac{\partial w_{n1}}{\partial y} = \frac{\partial^2 U_n}{\partial x \partial y} - (1 - \nu) Q_n \qquad (14.18b)$$

$$2\mu w_{nr} = \frac{\partial^2 U_n}{\partial r \partial y}, \quad 2\mu w_{ny} = 2(1 \quad \nu)\nabla^2 U_n - \frac{\partial^2 U_n}{\partial y^2} \tag{14.24}$$

Thus

$$2\mu w_{nr} = \lambda_n \psi_n' e^{-\lambda_n y} \; ; \quad 2\mu w_{ny} = [2(1-\nu)(\psi_n'' + \frac{\psi_n'}{r})$$

$$+ (\; - 2\nu)\lambda_n^2 \psi_n]e^{-\lambda_n y} \tag{14.25}$$

Therefore, the orthogonality relations derived from (14.12) are

$$\int_0^1 \{w_{mr}^* \tau_{ry}(\mathrm{w}_n) + w_{my}^* \tau_{yy}(\mathrm{w}_n) \; - w_{nr}\tau_{ry}(\mathrm{w}_m^*)$$

$$- w_{ny}\tau_{yy}(\mathrm{w}_m^*)\} r dr = 0 \; ; \quad k_n \neq k_m \tag{14.26}$$

where

$$\tau_{ry}(\mathrm{w}_n) = [(1-\nu)(\psi_n'' + \psi_n''/r - \psi_n'/r^2) \quad \nu\psi_n^2 \psi_n']e^{-\lambda_n y} \tag{14.27a}$$

$$\tau_{yy}(\mathrm{w}_n) = -\lambda_n [(2-\nu)(\psi_n'' + \psi_n'/r) + (1-\nu)\lambda_n^2 \psi_n]e^{-\lambda_n y} \tag{14.27b}$$

The expressions for the displacements $w_n^*(x, y)$ and the tractions associated with them can be obtained by changing λ_n by $-\lambda_n$ everywhere in equations (14.25) and (14.27).

XV. SOLUTION OF BOUNDARY VALUE PROBLEMS USING BIORTHOGONAL FUNCTIONS

Boundary value problems can be formulated as problems with linear restrictions of section V. Let $D = N_P$ be, as before, the linear space of solutions of a homogeneous equation and $A_0 : N_P \rightarrow N_P^*$ the corresponding bilinear form, which is assumed to be antisymmetric. Take an operator $B : N_P \rightarrow N_P^*$ that decomposes A_0. When $N_P^1 \subset N_P$ is a linear subspace and the linear functional $Bu \in N_P^*$ (generally, defined by some boundary values) is given, the problem with linear restrictions consists in finding $u \in N_P^1$. By virtue of theorem 5.1, when this problem with linear restrictions satisfies existence $\{N_P^1, N_B\}$ is a canonical decomposition of $D = N_P$ with respect to A_0.

Assume there are availabe c-complete biorthogonal systems $\mathcal{B}_1 = \{w_1, w_2, ...\} \subset N_P^1$ and $\mathcal{B}_B = \{w_1^*, w_2^*, ...\} \subset N_B$, then

$$u = \sum_{\alpha=1}^{\infty} a_\alpha w_\alpha + v_0^1 \tag{15.1a}$$

while

$$U - u = \sum_{\alpha=1}^{\infty} b_\alpha w_\alpha^* + v_0^2 \ . \tag{15.1b}$$

$$a_\alpha = \langle AU, w_\alpha^* \rangle \ ; \quad b_\alpha = \langle Aw_\alpha, U \rangle \tag{15.1c}$$

while v_0^1 and v_0^2 belong to the null subspace of A_0.

Frenquently, a c-complete biorthogonal system is available for a canonical decomposition $\{N_P^1, N_P^2\}$, but one is interested in a problem with linear restrictions in which, as before, a different canonical decomposition $\{N_P^1, N_B\}$, is involved. In such cases the following construction is useful.

Therem 15.1. *Assume* $\{w_1, w_2, ...\} \subset N_P^1$ *and* $\{w_1^*, w_2^*, ...\} \subset N_P^2$ *are biorthonormal. Define* $w_n' \in N_B$, $n = 1, 2, ...$ *by*

$$w_n^* - w_n' \in N_P^1 \ ; \quad w_n' \in N_B \tag{15.2}$$

Then $\{w_1', w_2', ...\} \subset N_B$ *is c-complete and biorthonormal with* $\{w_1, w_2, ...\} \subset N_P^1$.

Proof. Under the assumptions, the construction of $\{w_1', w_2', ...\} \subset N_B$ is always possible, because $\{N_P^1, N_B\}$ is a canonical decomposition of N_P. Now, due to (15.2) it is possible to write

$$w_n^* = w_n' + \epsilon_n \ ; \tag{15.3}$$

where $\epsilon_n = w_n^* - w_n' \in N_P^1$. Hence

$$\langle Aw_n, w'_m \rangle = \langle Aw_n, w'_m \rangle + \langle Aw_n, \epsilon_m \rangle$$

$$= \langle Aw_n, w^*_m \rangle = \delta_{n\,m} \qquad (15.4)$$

Any $u \in N_P$, can be written as $u = u_1 + u_B$ with $u_1 \in N_P^1$ and $u_B \in N_B$. Therefore

$$\langle Au, w'_n \rangle = \langle Au_1, w'_n \rangle = \langle Au_1, w^*_n \rangle \qquad (15.5)$$

This shows that

$$\langle Au, w'_n \rangle = 0 \quad \forall n = 1, 2, \quad \Rightarrow \langle Au_1, w^*_n \rangle = 0$$

$$\forall n = 1, 2, \ldots \Rightarrow u_1 \in I_1 \cap I_2 = N_{A0} \qquad (15.6)$$

Hence

$$\langle Au, w'_n \rangle = 0 \quad \forall n = \ \ , 2, .. \Rightarrow u = u_1 + u_B \in N_B \qquad (15.7)$$

Examples of the applications of these results to boundary value problems in elasticity were given in [27].

APPENDIX

CANONICAL DECOMPOSITION OF I_P

Let $B : D \to D^*$ decompose A and $N_B = I$. In addition, assume $A_1 : D \to D^*$ and $A_2 : D \to D^*$ be antisymmetric operators that satisfy (12.2). Then by lemma 2.2.

$$N_A = N_{A1} \cap N_{A2} \, . \qquad (A.1)$$

Given any $u \in D$ write $u = u_1^0 + u_2^0$ where $u_1^0 \in N_{A2}$ and $u_2^0 \in N_{A1}$ Then

$$\langle Au, v\rangle = \langle A_1 u_1^0, v_1^0\rangle + \langle A_2 u_2^0, v_2^0\rangle \tag{A.2}$$

Notice that

$$\langle A_1 u, v\rangle = \langle Au_1^0, v_1^0\rangle \ ; \quad \langle A_2 u, v\rangle = \langle Au_2^0, v_1^0\rangle \tag{A.3}$$

Clearly, we can define operators $B_1 : D \to D^*$ and $B_2 : D \to D^*$ such that B_1 decomposes A_1, while B_2 decomposes A_2. They are given by

$$\langle B_1 u, v\rangle = \langle Bu_1^0, v_1^0\rangle \ ; \quad \langle B_2 u, v\rangle = \langle Bu_2^0, v_2^0\rangle \ . \tag{A.4}$$

Define

$$D_1 = N_{B2} \supset I \supset N_A \ ; \quad D_2 = N_{B1} \supset I \supset N_A \ . \tag{A.5}$$

Then

$$D = D_1 + D_2 \ ; \quad I = D_1 \cap D_2 \ . \tag{A.6}$$

When $u = u_1 + u_2$, with $u_1 \in D_1$, $u_2 \in D_2$ and similarly for v, one has

$$\langle A_1 u_2, v_2\rangle = \langle A_2 v_1, u_1\rangle = 0 \tag{A.7a}$$

Because

$$\langle Au, v\rangle = \langle A_1 u_1, v_1\rangle = \langle A_1 u_1, v_2\rangle + \langle A_1 u_2, v_2\rangle$$

$$+ \langle A_2 u_2, v_2\rangle + \langle A_2 u_1, v_2\rangle + \langle A_2 u_2, v_1\rangle \ . \tag{A.7b}$$

Hence

$$\langle Au_1, v_1\rangle = \langle A_1 u_1, v_1\rangle \tag{A.8a}$$

$$\langle Au_2, v_2 \rangle = \langle A_2 u_2, v_2 \rangle \ . \tag{A.8b}$$

Write

$$I_{P1} = I_P \cap D_1 \ ; \quad I_{P2} = I_P \cap D_2 \ . \tag{A.9}$$

Define for $\alpha = 1, 2$, the operator $\check{A}_\alpha : I_P \to I_P^*$ by

$$\langle \check{A}_\alpha u, v \rangle = \langle A_\alpha u, v \rangle \ , \quad \forall \, u \in I_P \ \& \ v \in I_P \tag{A.10}$$

Lemma A.1. *When $\{I, I_P\}$ is a canonical decomposition of D, with respect to $A : D \to D^*$, one has*

$$I_P = I_{P1} + I_{P2} \tag{A.11}$$

Proof. Notice that $I_{P1} \subset D_1$ while $I_{P2} \subset D_2$. In general any $u \in I_P$ can be written as

$$u = u_1 + u_2, \quad u_1 \in I_{P1} \ \& \ u_2 \in I_{P2} \ , \tag{A.12}$$

because $u = u_1' + u_2'$, with $u_1' \in D_1$ and $u_2' \in D_2$, by virtue of (A.6). Write $u_1' = u_1 + w$ with $u_1 \in I_P$ and $w \in I$; this is possible because $\{I, I_P\}$ is a canonical decomposition of D. The fact that $u_1' \in D_1$ while $w \in I \subset D$ (Eq. A.6), implies that $u_1 \in D_1$; hence, $u_1 \in I_P \cap D_1 = I_{P1}$. Writing $u_2 = u_2' - w$, it is easy to see that $u = u_1 + u_2$ and $u_2 \in I_{P2}$.

Theorem A.1. *Assume $I_P \cap (N_{A1} \oplus N_{A2}) \subset N_A$. Then when $\{I, I_P\}$ is a canonical decomposition of D with respect to A, $\{I_{P1}, I_{P2}\}$ is a canonical decomposition of I_P, with respect to $\check{A}_\alpha : I_P \to I_P^*$ ($\alpha = 1$ or 2).*

Proof. In view of lemma A.1, it is only necessary to prove that I_{P1} and I_{P2} are regular subspaces. Denote by $\check{N}_{A\alpha}$ the null subspace of $\check{A}_\alpha$. Take $\alpha = 1$ and assume $u \in \check{N}_{A1} \subset I_P$, then

$$\langle A_1 u, v \rangle = \langle A u_1^0, v_1^0 \rangle = \langle A u_1^0, v \rangle \ , \quad \forall \, v \in I_P \tag{A.13}$$

This shows that $u_1^0 \in I_P$; hence $u_1^0 \in I_P \cap N_{A1} \subset N_A$. A similar argument for $\alpha = 2$, shows that $N_A \subset I_{P\alpha}$ ($\alpha = 1, 2$). To show that I_{P1} and I_{P2} are commutative subspaces, notice that $u \in I_P$ and $v \in I_P$ implies $\langle Au, v \rangle = 0$. Hence, the desired result follows from (A.7a) and (A.8).

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RESUMEN

Se presenta una teoría abstracta de problemas de frontera desarrollada recientemente por el autor. Ella exhibe la estructura algebraica asociada a problemas lineales. Se da una caracterización de sistemas completos de soluciones para regiones de forma arbitraria. También se sistematiza la utilización de sistemas de funciones biortogonales que contribuyen a ampliar la teoría de series de Fouries generalizadas. Se desarrollan principios variacionales generales para problemas de condiciones de frontera, con saltos prescritos y sujetos a condiciones de tipo de continuación. Se exhiben aplicaciones a mecánica de fluidos y de sólidos; entre ellas, teoría de placas, flujos de Stokes, problemas de difracción elástica, etcétera.