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A BOUNDARY METHOD FOR ELASTIC WAVE DIFFRACTION: APPLICATION TO SCATTERING OF *SH* WAVES BY SURFACE IRREGULARITIES

By Franciso J. Sánchez-Sesma, Ismael Herrera, and Javier Avilés

ABSTRACT

A boundary method developed by Herrera is briefly explained in connection with wave scattering. The method is based on the use of complete systems of solutions of the homogeneous equations. A convenient criterion of completeness is the notion of *c*-completeness. The general method grants convergence of the approximating sequence when a least-squares fitting of the boundary conditions is used. As an illustration, the scattering and diffraction of *SH* waves by surface irregularities is treated here. It is shown that plane waves are *c*-complete in a bounded region of arbitrary shape. Scattering is formulated as a problem of connecting solutions in such a region with solutions in an unbounded one where Hankel functions are used. Numerical results for specific cases are reported.

Introduction

Diffraction of elastic waves has interest in seismology and earthquake engineering in several instances. The study of the influence of various kinds of irregularities on the characteristics of ground motion is a subject of great importance. Boundary methods are suitable to deal with such problems because they avoid the introduction of ficticious boundaries and reduce the size of the discretized regions. These facts yield numerical advantages.

There are two main approaches for the formulation of boundary methods: one is based on the use of boundary integral equations (Brebbia, 1978), and the other one, on the use of complete systems of solutions (Herrera, 1981a). The latter approach avoids the introduction of singular integral equations and fundamental solutions which are more difficult to construct than complete systems of solutions. This point is illustrated well in this paper through the use of an extremely simple system of solutions; namely, plane waves.

In some fields of application, procedures which can be identified as particular cases of the approximation by complete systems of solutions have been used. For such studies, the so-called "Rayleigh hypothesis" limits drastically the applicability of the method (Bates, 1975). That such restrictions are due mainly to lack of clarity, can be seen in view of some results due to Millar (1973). In considering diffraction of elastic waves by periodic surfaces and bounded objects, Millar employed a method of series expansion for the scattered wave field in terms of a set of plane waves. The completeness of the set was established which guaranteed that there is "a linear combination of N elements of the set that converges on the boundary to the prescribed values, in the mean-square sense, as $N \to \infty$." Furthermore, it was established that "at the points not on the periodic surfaces, the expansion converges uniformly to the sought solution whether or not the Rayleigh hypothesis is satisfied."

Motivated by this situation, Herrera (1977a, 1979) initiated a systematic research of the subject. The aims of his study have been satisfactorily achieved, to a large extent, and are just being reported (Herrera, 1980a, 1981a). The outcome has been a systematic and rigorous method which expands the versatility of boundary procedures, making them applicable to any problem which is governed by linear partial differential equations.

The method makes extensive use of results that have been derived in the theory of partial differential equations (Lions and Magenes, 1972). In particular, results on the existence and continuity of solutions of elliptic equations, that will be needed in this article are special cases of very general results given in volume I (pp. 188–189) of the books by Lions and Magenes.

The methodology also owes much to contributions by a group of Italian mathematicians (Miranda, 1970) and Kupradze (1967). The systematic development of these procedures, in a manner which is applicable to any linear problem, was made possible, however, by an algebraic theory which has been developed by Herrera (1981b).

There are two additional aspects in which this algebraic theory is relevant; the formulation of variational principles (Herrera, 1977b, 1980b) and the development of biorthogonal functions, which are obtained by separation of variables procedures.

In geophysical research, Herrera's (1964a) orthogonality relation for Rayleigh waves has been known for some time and has been applied by Alsop (1968) and Malischewsky (1976) to problems of elastic wave diffraction. Using a different procedure, similar relations have been obtained for the biharmonic equation (Joseph, 1979). Such developments were lacking, until recently, a general and systematic theoretical framework. It has just been shown (Herrera and Spence, 1981) that the algebraic theory is quite suitable for this purpose.

The introduction of the concept of *c*-completeness allows constructing systems of solutions which are complete, not only with respect to general boundary values, but independently of the specific region considered (Herrera and Sabina, 1978; Herrera, 1980c). In addition, it permits keeping all computations in $\pounds^2$ spaces. A procedure has also been developed for computing boundary information which is complementary to boundary data, e.g., tractions when displacements are prescribed. Applications of the method include problems formulated in discontinuous fields with prescribed jump conditions.

Complete systems of solutions have been used mainly in applications of the method of separation of variables. This has led to the frequent, but false, belief that such systems have to be constructed specifically for a given region. There are available quite general procedures for deriving *c*-complete systems (Herrera and Sabina, 1978). In addition, separation of variables can be used to construct *c*-complete systems which, generally, are biorthogonal, as is the case of Rayleigh waves (Herrera, 1964a). *Ad-hoc* procedures can also be applied; an example is the system of plane waves developed in this article and another one is a general class of *c*-complete systems recently developed for Stokes problems (Herrera and Gourgeon, 1981) and the biharmonic equation.

As an illustration of the method, which is relevant for estimating the influence of local topography and geology, we treat the scattering and diffraction of harmonic *SH* waves by irregularities on the surface of an elastic half-space (Figure 1). This is decomposed into two subregions (Figure 2); a bounded region R and an unbounded region E. Plane waves are used on R and Hankel functions on E. Numerical results are presented for two types of surface irregularities.

Problems similar to the example given here, have been extensively studied using, e.g., perturbations (Herrera, 1964b; Sabina and Willis, 1977), finite differences (Boore, 1972), and boundary methods (Sills, 1978); among the latter, procedures which are similar (Bouchon and Aki, 1977; Bard and Bouchon, 1980) or that can be identified as particular cases of the approximation by complete systems of solutions (Sánchez-Sesma and Esquivel, 1979, 1980; Sánchez-Sesma and Rosenblueth, 1979; England *et al.*, 1980).

FOUNDATIONS OF THE METHOD

The boundary method (Herrera, 1981a) will be presented here with some detail, only in connection with diffraction problems of *SH* waves. The method is, however, general and can be applied to diffraction of other kinds of elastic waves.

The diffraction problem to be considered consists in finding the total field w produced by an incident waves $w^{(i)}$ on a half-space with arbitrary surface irregularities (Figure 1). It is assumed that these waves are periodic with circular frequency $= \omega$. In such a case, the displacement w in the z direction satisfies the reduced wave equation

$$\Delta w + k^2 w = 0 \tag{1}$$

in which $k = \omega/\beta$ = wavenumber, with $\beta = \sqrt{\mu/\rho}$ = shear-wave velocity, μ = shear modulus, and ρ = mass density of the medium.

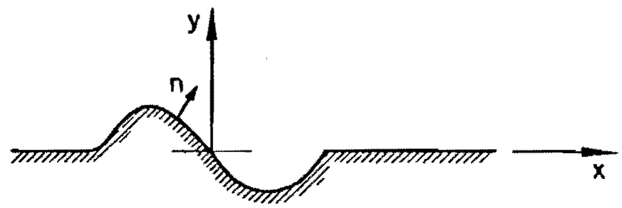

FIG. 1. Surface irregularity.

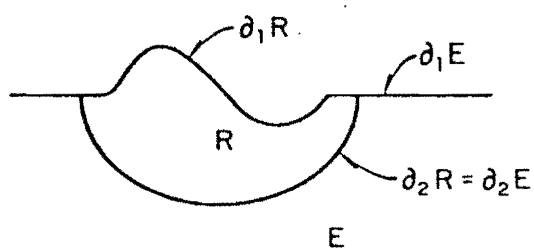

FIG. 2. Interior and exterior regions.

This problem can be formulated as a problem with prescribed jumps, which is a special case of the problem of connecting discussed in detail previously (Herrera, 1977b, 1980b, c). Let $w^{(f)}$ be the free field; i.e., $w^{(f)}$ is the solution when the problem is formulated in a half-space with a plane boundary and no irregularity is present. The total domain can be divided in two regions R and E (Figure 2). For functions w whose domain of definition includes the region E, the notation w_E will be used for its restriction to E; a corresponding convention is adopted for functions whose domain of definition includes R. Define

$$w_E = w_E^{(f)} + u_E. \tag{2}$$

It is convenient to observe at this point that a function $w_R^{(f)}$ will not be defined because the domain of definition of $w^{(f)}$, in general, may not include the region R. On the other hand, we define

$$u_R = w_R. \tag{3}$$

Then, it is easy to see that u_R and u_E, satisfy equation (1) on R and E, respectively.

Also

$$\frac{\partial u_R}{\partial n} = 0; \qquad \text{on } \partial_1 E \tag{4}$$

$$\frac{\partial u_R}{\partial n} = 0; \qquad \text{on } \partial_1 R \tag{5}$$

while the continuity of w implies the following jumps across $\partial_2 R = \partial_2 E$

$$[\hat{u}] = u_E - u_R = -w_E^{(f)}; \qquad \text{on } \partial_2 R \tag{6a}$$

$$\left[\frac{\partial \hat{u}}{\partial n}\right] = \frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n} = -\frac{\partial w_E^{(f)}}{\partial n}; \qquad \text{on } \partial_2 R. \tag{6b}$$

Here, $\partial_1 E$ is the intersection of ∂E with the plane boundary of the half-space (Figure 2), while in (6b), to be definite, the unit normal vector $\underline{n}$, has been taken pointing outward from R. Recall that the pair of functions $\hat{u} = \{u_R, u_E\}$ satisfies in addition equation (1) on E and R separately; this condition together with equations (4) to (6) define the desired problem with prescribed jump discontinuities.

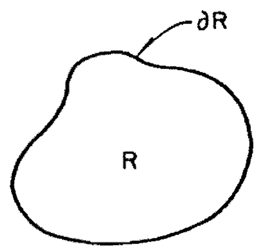

FIG. 3. Region R and its boundary ∂R.

The method to be explained, can be better understood by considering two simpler problems first; these are Dirichlet and Neumann problems for the reduced wave equation (1) in a bounded region R with boundary ∂R (Figure 3).

For definiteness, assume $u \in H^{s+1/2}(R)$, where the standard notation for Sobolev spaces is being used (Lions and Magenes, 1972).

Let us denote by $N^{s+1/2}(R) \subset H^{s+1/2}(R)$ the subspace of functions that satisfy equation (1) in R. If $\mathscr{B} = \{w_1, w_2, \cdots\} \subset N^{s+1/2}(R)$ is a system of such solutions which spans $N^{s+1/2}(R)$, then there is a sequence of approximations

$$u^N = \sum_{n=1}^{N} a_n{}^N w_n \tag{7}$$

such that

$$u^N \to u \quad \text{in} \quad H^{s+1/2}(R). \tag{8}$$

In order for the representation in equation (7) to be useful, it will be required to have a procedure for deriving the coefficients $a_n{}^N$ from boundary data only. This is, indeed, possible.

It is known (Lions and Magenes, 1972) that when $u \in H^{s+1/2}(R)$, then $u \in H^s(\partial R)$

while $\partial u/\partial n \in H^{s-1}(\partial R)$. If the coefficients a_n^N are chosen so that

$$u^N \to u \quad \text{in} \quad H^s(\partial R), \tag{9}$$

then the continuity properties imply (8). [It is assumed that $-k^2$ is not an eigenvalue, either for Dirichlet or for Neumann problems. Otherwise, the argument given here has to be modified (Herrera, 1981a). For simplicity, the corresponding discussion is not included.] Similarly,

$$\frac{\partial u^N}{\partial n} \to \frac{\partial u}{\partial n} \quad \text{in} \quad H^{s-1}(\partial R) \tag{10}$$

also imply (8). Therefore, if the boundary values

$$\{w_1, w_2, \cdots\} \quad \text{span} \quad H^s(\partial R), \tag{11a}$$

the coefficients can be chosen so that (9) is satisfied. On the other hand, if

$$\left\{\frac{\partial w_1}{\partial n}, \frac{\partial w_2}{\partial n}, \cdots\right\} \quad \text{span} \quad H^{s-1}(\partial R) \tag{11b}$$

the coefficients can be chosen so that (10) is satisfied. In the first case, the system $\mathscr{B} \subset N^{s+1/2}(R)$ can be used to solve a Dirichlet problem, while it allows solving a Neumann problem in the second one.

Assume $\mathscr{B} = \{w_1, w_2, \cdots\}$ spans $N^{s+1/2}(R)$, then continuity properties of solutions of elliptic equations grant that both statements (11) hold. Hence, such a system can be used to solve both Dirichlet and Neumann problems. Clearly, (9) holds if u^N is taken as the projection of the boundary values $u \in H^s(\partial R)$ on the subspace spanned by $\{w_1, \cdots, w_N\}$. On the other hand, (10) holds if $\partial u^N/\partial n$ is the projection of the boundary values $\partial u/\partial n \in H^s(\partial R)$ on the subspace spanned by $\{\partial w_1/\partial n, \cdots, \partial w_N/\partial n\}$. Therefore, in both cases, the coefficients can be computed by the standard procedure for projecting on a subspace.

Notice that in the first case, the projections are taken in the sense of the inner product associated with $H^s(\partial R)$, and in the second one, it is associated with $H^{s-1}(\partial R)$. Numerically, it is simpler to use $\mathcal{L}^2(\partial R) = H^0(\partial R)$ inner products, only.

This can be done, if it is granted that

$$\{w_1, w_2, \cdots\} \quad \text{spans} \quad H^0(\partial R) \tag{12a}$$

and simultaneously

$$\left\{\frac{\partial w_1}{\partial n}, \frac{\partial w_2}{\partial n}, \cdots\right\} \quad \text{spans} \quad H^0(\partial R). \tag{12b}$$

This will happen, if and only if,

$$\mathscr{B} = \{w_1, w_2, \cdots\} \subset N^{3/2}(R), \quad \text{spans} \quad N^{3/2}(R). \tag{13}$$

As a matter of fact, conditions (12) are granted whenever $\mathscr{B}$ spans $N^{s+1/2}(R)$ with $s \geq 1$, but the choice (13) is optimal in the sense that it corresponds to the least s that can be taken, granting (12).

There is an alternative manner of imposing condition (13). Let $D_R \subset H^{1/2}(R)$ be the linear subspace (the linear subspace D_R, so defined, is not closed) with the property that for every $u \in D_R$, the boundary values satisfy $u \in H^0(\partial R)$, while $\partial u/\partial n \in H^0(\partial R)$. Define for every $u \in D_R$ and $v \in D_R$, the bilinear functional (Herrera, 1980b, 1981a, b)

$$\langle A_R u, v \rangle = \int_{\partial R} \left\{ v \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n} \right\} n. \tag{14}$$

Let $I_P \subset D_R$ be the linear subspace of D_R with the property that $v \in I_P$, if and only if, there is a solution $w \in D_R$ of equation (1) such that

$$v = w; \quad \text{on} \quad \partial R \tag{15a}$$

and

$$\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n}; \quad \text{or} \quad \partial R. \tag{15b}$$

The subspace I_P can be concisely defined as the quotient space $N^{3/2}(R)/N_{AR}$.

It has been shown (Herrera, 1980c, 1981b) that conditions (12) hold, if and only if, for every $u \in D_R$, one has

$$\langle A_R u, w_\alpha \rangle = 0 \quad \forall \quad \alpha = 1, 2, \cdots \Rightarrow u \in I_P. \tag{16}$$

When (16) is satisfied, the system $\{w_1, w_2, \cdots\} \subset N^{3/2}(R)$ is said to be c-complete. Thus, the system $\mathscr{B}$ is c-complete, if and only if, (12a) and (12b) hold simultaneously. But, since (12) and (13) are equivalent, we can summarize our results as follows; given a system of functions $\mathscr{B} = \{w_1, w_2, \cdots\} \subset N^{3/2}(R)$, satisfying equation (1), the following statements are equivalent

$\mathscr{B}$ is c-complete

$\mathscr{B}$ spans $N^{3/2}(R) \subset H^{3/2}(R)$.

The boundary values, $\{w_1, w_2, \cdots\}$ span $H^0(\partial R)$ and simultaneously $\left\{\frac{\partial w_1}{\partial n}, \frac{\partial w_2}{\partial n}, \cdots\right\}$ span $H^0(\partial R)$.

Notice, finally, that $\mathscr{B}$ spans $N^{1/2}(R)$ whenever $\mathscr{B}$ spans $N^{3/2}(R)$, as it is not difficult to verify.

These results show that a c-complete system can be used to solve both a Neumann and Dirichlet problems. Actually, such a system can be used to solve any boundary value problem associated with what in Herrera's (1980c, 1981b) theory is known as a canonical decomposition.

Going back to the Dirichlet problem, the least-squares or projection condition in $H^0(\partial R)$ leads to the system of equations

$$\sum_{n=1}^{N} M_{nm} a_n^N = c_m \tag{17}$$

where

$$M_{nm} = \int_{\partial R} w_n \bar{w}_m{}^* \, d\underset{\sim}{x} \tag{18a}$$

and

$$c_m = \int_{\partial R} f_{\partial R} \, \bar{w}_m{}^* \, d\underset{\sim}{x} \tag{18b}$$

where the asterisk refers to the complex conjugate. Here $f_{\partial R}$ are the prescribed boundary values for this problem. With this choice, $u^N \to u$ in $H^{1/2}(R)$ (Lions and Magenes, 1972). Similarly, for the Neumann problem, the system of equations (17) also holds, except that equations (18) have to be replaced by

$$M_{nm} = \int_{\partial R} \frac{\partial w_n}{\partial n} \frac{\partial w_m{}^*}{\partial n} \, d\underset{\sim}{x} \tag{19a}$$

and

$$c_m = \int_{\partial R} g_{\partial R} \frac{\partial w_m{}^*}{\partial n} \, d\underset{\sim}{x} \tag{19b}$$

where $g_{\partial R}$ are the prescribed normal derivatives on ∂R. In this case, $u^N \to u$ in $H^{3/2}(R)$ (Lions and Magenes, 1972); therefore, also $u^N \to u$ in $H^{1/2}(R)$.

Computation of complementary boundary values, when they are required, may need a special device. In general, if the normal derivative $\partial u^N/\partial n \to g_{\partial R}$ in $H^0(\partial R)$, then $u^N \to u$ in $H^{3/2}(R)$; hence, on the boundary $u^N \to u$ in $H^1(\partial R)$, which implies $u^N \to u$ in $H^0(\partial R)$. Thus, in the case of the Neumann problem, the unknown boundary values can be derived from the approximating sequence directly and the procedure that is given next is not required; however, such procedure can be used to accelerate the convergence of $u^N \to u$ in $H^0(\partial R)$.

For the Dirichlet problem, the unknown normal derivatives cannot be derived from the approximating sequence u^N, because from $u^N \to f_{\partial R}$ on $H^0(\partial R)$, one can only grant that $\partial u^N/\partial n \to \partial u/\partial n$ in $H^{-1}(\partial R)$ (Lions and Magenes, 1972). If the boundary data is sufficiently smooth; i.e., if $f_{\partial R} \in H^1(\partial R)$, it is known that $\partial u/\partial n \in H^0(\partial R)$ (Lions and Magenes, 1972) and the following approximating sequence can be used

$$\sum_{n=1}^{E} b_n{}^N w_n{}^* \to \frac{\partial u}{\partial n}, \quad \text{in} \quad H^0(\partial R). \tag{20}$$

The coefficients can be obtained from

$$\sum_{n=1}^{N} K_{nm} b_n{}^N = d_m. \tag{21}$$

This procedure is an extension of a relation derived by the Italian mathematicians in the 1940's (Miranda, 1970). Here

$$K_{nm} = \int_{\partial R} w_n{}^* w_m \, d\underset{\sim}{x} \tag{22a}$$

while

$$d_m = \int_{\partial R} \frac{\partial u}{\partial n} w_m \, d\underset{\sim}{x} = \int_{\partial R} u \frac{\partial w_m}{\partial n} \, d\underset{\sim}{x} = \int_{\partial R} f_{\partial R} \frac{\partial w_m}{\partial n} \, d\underset{\sim}{x}. \tag{22b}$$

Notice that d_m is given in terms of boundary data only.

A *C*-Complete System of Plane Waves

The system of functions (given in polar coordinates)

$$\mathscr{B}' = \{J_n(kr)\cos n\theta; \quad J_n(kr)\sin n\theta \mid n = 0, 1, 2, \cdots\} \subset N^{3/2}(R) \tag{23}$$

is *c*-complete (Herrera and Sabina, 1978) in any bounded region R (Figure 3). This fact can be used to prove that the system of plane waves

$$\mathscr{B} = \{\exp[ikr\cos(\theta - \psi_1)], \exp[ikr\cos(\theta - \psi_2)], \exp[ikr\cos(\theta - \psi_3)], \cdots\} \tag{24}$$

where the sequence $\psi_1, \psi_2, \cdots$, is any dense set of real numbers in the interval $[0, 2\pi]$, and is also *c*-complete in any such region.

It is known that (Whittaker and Watson, 1958)

$$J_n(kr)e^{\text{in}\theta} = \frac{(-i)^n}{2\pi} \int_{-\pi}^{\pi} e^{\text{in}\xi + ikr\cos(\theta - \xi)} \, d\xi. \tag{25}$$

This formula exhibits Bessel functions (more precisely, the members of the family $\mathscr{B}'$) as a superposition of plane waves. It is easy to see that

$$\langle A_R u, w \rangle = 0 \quad \forall\, w \in \mathscr{B} \Rightarrow \langle A_R u, w \rangle = 0 \quad \forall\, w \in \mathscr{B}'. \tag{26}$$

Indeed, the implication (26) follows from the fact that the set $\psi_1, \psi_2, \cdots$ is dense in $[0, 2\pi]$ and the continuity of the integrand in equation (25), when use is made of definition (14) of A_R. In view of definition (16) of *c*-completeness, (26) shows that the system of plane waves $\mathscr{B}$ is *c*-complete because $\mathscr{B}'$ is also *c*-complete.

Another result that has been obtained previously (Herrera and Sabina, 1978) is the fact that, in polar coordinates, the system

$$\{H_n{}^{(2)}(kr)\cos n\theta, H_n{}^{(2)}(kr)\sin n\theta \mid n = 0, 1, 2, \cdots\} \tag{27}$$

where $H_n{}^{(2)}$ is the Hankel function of the second kind and order n and is *c*-complete for the space of solutions of equation (1) in the exterior E of a bounded region such as R in Figure 1, which satisfy Sommerfeld radiation conditions.

Another possibility is to take E as a subregion of a half-space, which is exterior to a bounded region like R, in Figure 2 and take the space of solutions of equation (1 in E to be restricted in addition by Sommerfeld radiation conditions, and the boundary condition (4) on $\partial_1 E$ (Figure 2). In this case, the system

$$\{H_n^{(2)}(kr)\cos n\theta \mid n = 0, 1, 2, \cdots\} \tag{28}$$

is c-complete in any such E.

Extension to Diffraction Problems

In this section, we extend the results presented previously to problems with prescribed jumps, in terms of which, the original diffraction problem was formulated in "Foundations of the Methods." Consider the regions R and E illustrated in Figure 2. Let $D_R \subset H^{1/2}(R)$ be the linear space of functions introduced in "Foundations of the Methods," while $N(R) \subset D_R$ will be the linear subspace of D_R whose elements satisfy equation (1) in R; then, $N(R) = N^{3/2}(R)$. The definition of the linear subspace D_E becomes more involved because E is unbounded. Such technical difficulties can be avoided altogether if attention is restricted to boundary values on $\partial_2 E$. Thus, D_E will be taken as the linear space whose elements are pairs $[u_E, \partial u_E/\partial n]$ of functions defined on $\partial_2 E$, such that $u_E \in H^0(\partial_2 E)$ and $\partial u_E/\partial n \in H^0(\partial_2 E)$. In addition, the linear subspace $N(E) \subset D_E$ will be defined by the condition that an element $[u_E, \partial u_E/\partial n] \in N(E)$, if and only if, there is a solution v of the reduced wave equation (1) in E, satisfying the boundary condition (4) on $\partial_1 E$, together with Sommerfeld radiation condition at infinity and such that

$$u_E = v, \quad \text{on } \partial_2 E \tag{29a}$$

$$\frac{\partial u_E}{\partial n} = \frac{\partial v}{\partial n}, \quad \text{on } \partial_2 E. \tag{29b}$$

Conditions (29) are equivalent to require that

$$\int_{\partial_2 R} \left\{ w\frac{\partial u_E}{\partial n} - u_E\frac{\partial w}{\partial n} \right\} d\underset{\sim}{x} = 0 \tag{30}$$

for every element of w of the system (28) because the latter is c-complete. In order to keep the notation simple, elements of D_E will be denoted by u_E; recall that with every $u_E \in D_E$, there is a pair $[u_E, \partial u_E/\partial n]$ where $u_E \in H^0(\partial_2 E)$ and $\partial u_E/\partial n \in H^0(\partial_2 E)$. It will be seen, however, that this ambiguity does not lead to confusion.

We will be interested in pairs of elements $\{u_R, u_E\}$, such that $u_R \in D_R$ while $u_E \in D_E$; the space of such pairs will be denoted by $\hat{D} = D_R \oplus D_E$. The space $\hat{N} \subset D$ will be made by pairs $\{u_R, u_E\}$ such that $u_R \in N(R)$, while $u_E \in N(E)$. In a fashion similar to that in which I_P was introduced in the previous section, the space $\hat{I}_P$ will be defined. An element $\hat{u} = \{u_R, u_E\} \in \hat{D}$ belongs to $\hat{I}_P$, if and only if, there exists $v_R \in N(R)$ such that

$$u_R = v_R, \quad \text{on } \partial R \tag{31a}$$

$$\frac{\partial u_R}{\partial n} = \frac{\partial v_R}{\partial n}, \quad \text{on } \partial R \tag{31b}$$

while $u_E \in N(E)$. The simplicity of this last requirement is due to the fact that we are restricting attention to the boundary values and normal derivatives on $\partial_2 E = \partial_2 R$, of functions defined in the exterior region E.

The problem of diffraction introduced in "Foundations of the Method" can now be formulated as a problem of connecting (Herrera, 1980b, 1981a, b). This consists of finding $\hat{u} = \{u_R, u_E\} \in \hat{N}$ which satisfies (5) and (6). Here, u_R and u_E are related to the total field w by equations (3) and (2), respectively.

In addition to the bilinear form A_R given by equation (14), we introduce A_E by

$$\langle A_E u_E, v_E \rangle = -\int_{\partial_2 R} \left\{ v_E \frac{\partial u_E}{\partial n} - u_E \frac{\partial v_E}{\partial n} \right\} d\underset{\sim}{x}. \tag{32}$$

The minus sign in this definition is motivated by the fact that, in order to be definite, the unit normal vector $\underset{\sim}{n}'$ will be taken pointing outward from R. Define

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle. \tag{33}$$

In this case (Herrera, 1980b),

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle \hat{J}\hat{u}, \hat{v} \rangle - \langle \hat{J}\hat{v}, \hat{u} \rangle + \int_{\partial_1 R} \left\{ v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right\} d\underset{\sim}{x} \tag{34}$$

where the jump operator $\hat{J}$ is given by

$$\langle \hat{J}\hat{u}, \hat{v} \rangle = \int_{\partial_2 R} \left\{ [\hat{u}] \overline{\frac{\partial v}{\partial n}} - \bar{v} \left[\frac{\partial \hat{u}}{\partial n} \right] \right\} d\underset{\sim}{x}. \tag{35}$$

Here, the jumps [ ] are given as in equations (6), while

$$\bar{v} = (v_R + v_E)/2; \qquad \overline{\frac{\partial v}{\partial n}} = \left(\frac{\partial v_R}{\partial n} + \frac{\partial v_E}{\partial n} \right) / 2. \tag{36}$$

A system $\hat{\mathscr{B}} = \{\hat{w}_1, \hat{w}_2, \cdots\} \subset \hat{N}$ is said to be c-complete for the problem of connecting (the diffraction problem) when for every $\hat{u} \in \hat{D}$, one has

$$\langle \hat{A}\hat{u}, \hat{w} \rangle = 0 \quad \forall \quad \hat{w} \in \hat{\mathscr{B}} \Rightarrow \hat{u} \in \hat{I}_P. \tag{37}$$

It has been shown (Herrera, 1980c) that, if the system $\{w_{R1}, w_{R2}, \cdots\} \subset N(R)$ and simultaneously the system $\{w_{E1}, w_{E2}, \cdots\} \subset N(E)$ are c-complete on R and E, respectively, then the system

$$\hat{\mathscr{B}} = \hat{\mathscr{B}}_R \cup \hat{\mathscr{B}}_E \subset \hat{N} \tag{38}$$

is c-complete for the problem of connecting. The notation used in (38) is as follows: given $w_{R\alpha} \in N(R)$ and $w_{E\alpha} \in N(E)$, we define $\hat{w}_{R\alpha} = \{w_{R\alpha}, 0\} \in \hat{N}$ and similarly, $\hat{w}_{E\alpha} = \{0, w_{E\alpha}\}$. Then

$$\hat{\mathscr{B}}_R = \{\hat{w}_{R1}, \hat{w}_{R2}, \cdots\} \subset \hat{N} \tag{39a}$$

$$\hat{\mathscr{B}}_E = \{\hat{w}_{E1}, \hat{w}_{E2}, \cdots\} \subset \hat{N}. \tag{39b}$$

This shows, for example, that the system (28) defined on E, together with the system (24) of plane waves defined on R, are c-complete for the problem of diffraction considered in this paper on region $R \cup E$. By using it, it is possible to approximate as closely as desired, in the least-square sense, any prescribed boundary condition on $\partial_1 R$ and jumps on the function and normal derivatives prescribed on $\partial_2 R$.

This latter statement, however, requires clarification. Notice that with every pair $\hat{u} = \{u_R, u_E\} \in \hat{D}$, there is associated a unique system of three functions: $[\hat{u}] \in H^0(\partial_2 R)$, $[\partial\hat{u}/\partial n] \in H^0(\partial_2 R)$, and $\partial u_R/\partial n \in H^0(\partial_1 R)$. Therefore, the triplet $\{[\hat{u}], [\partial\hat{u}/\partial n], \partial u_R/\partial n\} \in H^0(\partial_2 R) \oplus H^0(\partial_2 R) \oplus H^0(\partial_1 R)$. In addition, there is a second element of this Hilbert space of triplets associated with $\hat{u} \in \hat{D}$, namely, $\{\bar{u}, \overline{\partial u}/\partial n, u_R\} \in H^0(\partial_2 R) \oplus H^0(\partial_2 R) \oplus H^0(\partial_1 R)$. These two systems of triplets constitute what in Herrera's general theory is called a canonical decomposition; however, for the sake of brevity, we will not dwell on that. For our purpose, what matters is that previous results (Herrera, 1980c, 1981b) imply that a system $\{\hat{w}_1, \hat{w}_2, \cdots\} \in \hat{N}$ is c-complete for the problem of connecting, if and only if, each one of the systems of triplets

$$\left\{[\hat{w}_\alpha], \left[\frac{\partial \hat{w}_\alpha}{\partial n}\right], \frac{\partial u_{R\alpha}}{\partial n}\right\}, \qquad \alpha = 1, 2, \cdots \tag{40a}$$

and

$$\left\{\bar{w}_\alpha, \overline{\frac{\partial w_\alpha}{\partial n}}, u_{R\alpha}\right\}, \qquad \alpha = 1, 2, \cdots \tag{40b}$$

span $H^0(\partial_2 R) \oplus H^0(\partial_2 R) \oplus H^0(\partial_1 R)$. To avoid any possible confusion, we remind that given any two elements $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\} \in H^0(\partial_2 R) \oplus H^0(\partial_2 R) \oplus H^0(\partial_1 R)$, the inner product in this Hilbert space, to be denoted by $(\cdot, \cdot)$, is given as

$$(p, q) = \int_{\partial_2 R} p_1 q_1^* \, d\underset{\sim}{x} + \int_{\partial_2 R} p_2 q_2^* \, d\underset{\sim}{x} + \int_{\partial_1 R} p_3 q_3^* \, d\underset{\sim}{x} \tag{41}$$

where the asterisk refers to the complex conjugate. This defines the least squares or projection procedure to be used and the method to be applied for the solution of the diffraction problem.

In a manner similar to what was explained at the end of "Foundation of the Method," we notice that if the complementary boundary values $\{\bar{u}, \partial\bar{u}/\partial n, u\} \in H^0(\partial_2 R) \oplus H^0(\partial_2 R) \oplus H^0(\partial_1 R)$ are required, these can be computed using a generalization (Herrera, 1981b) of the procedure given there. Given any $\hat{u} \in \hat{D}$, define

$$u_1 = \left\{[\hat{u}], \left[\frac{\partial \hat{u}}{\partial n}\right], \frac{\partial u_R}{\partial n}\right\} \tag{42a}$$

$$u_2 = \left\{\overline{\frac{\partial u}{\partial n}}, -\bar{u}, u_R\right\}. \tag{42b}$$

It can be seen that equation (34) can be written as

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = (u_1, v_2{}^*) - (v_1, u_2{}^*). \tag{43}$$

As we want to compute the boundary values having prescribed the jumps on $\partial_2 R$ and the normal derivative on $\partial_1 R$, let us write

$$\sum_{n=1}^{N} C_n{}^N w_{n_1}^* \to u_2 \tag{44a}$$

or

$$\sum_{n=1}^{N} C_n{}^N \left\{ [\hat{w}_n]^*, \left[\frac{\partial \hat{w}_n}{\partial n}\right]^*, \frac{\partial w_{R_n}{}^*}{\partial n} \right\} \to \left\{ \overline{\frac{\partial u}{\partial n}}, -\bar{u}, u_R \right\} \tag{44b}$$

where the coefficients can be obtained using least-squares approximations from the system of equations

$$\sum_{n=1}^{N} K_{nm} C_n{}^N = d_m \quad m = 1, 2, \cdots, N \tag{45}$$

where

$$K_{nm} = (w_{m_1}, w_{n_1}) \tag{46a}$$

and

$$d_m = (w_{m_1}, u_2{}^*). \tag{46b}$$

From equations (37) and (43), it is clear that

$$(w_{m_1}, u_2{}^*) = (u_1, w_{m_2}^*) = d_m. \tag{47}$$

As in equation (22b), d_m is given in terms of boundary data only.

It is interesting to note that the average values of the normal derivate and the function on $\partial_2 R$ are given in terms of the complex conjugate jumps of the approximating sequence and of the normal derivatives, respectively. At the same time, the values of u_R on $\partial_1 R$ can be derived from the complex conjugate of the normal derivatives of the approximating sequence.

A Specific Problem

Assume, for the sake of illustration, that a plane wave of unit amplitude propagates toward the surface of the half-space with an incidence angle ψ, as shown in the Figure 4. This incident wave is given by

$$w^{(i)} = \exp(-ik[x \cos \psi + y \sin \psi]). \tag{48}$$

The time factor $\exp(i\omega t)$ is omitted here and thereafter. In absence of irregularities, the reflected wave is

$$w^{(r)} = \exp(-ik[x \cos \psi - y \sin \psi]), \tag{49}$$

thus the free field $w^{(f)} = w^{(i)} + w^{(r)}$ is given by

$$w^{(f)} = 2\cos(ky\sin\psi)\exp(-ikx\cos\psi). \tag{50}$$

It can be seen that the displacement amplitude of the surface in the free field is two.

Using the notation of "Foundations of the Method," the scattered field u_E in the exterior region E (Figure 2) can be expanded in terms of the c-complete system given by (28) to obtain

$$u_E = \sum_{n=0}^{N} A_n H_n^{(2)}(kr)\cos n\theta \tag{51}$$

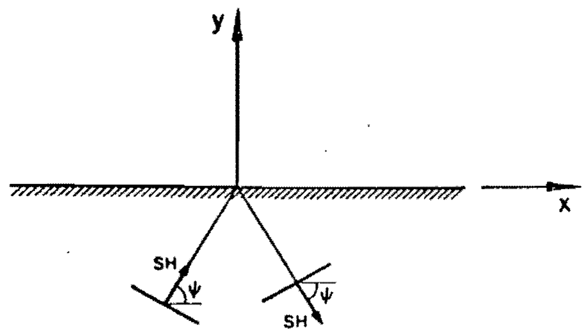

FIG. 4. Incident and reflected *SH* plane waves.

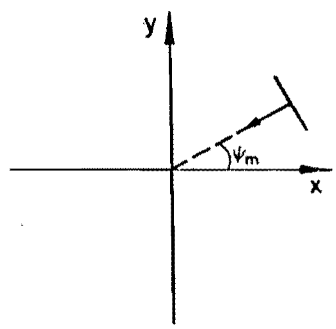

FIG. 5. The *m*-esim *SH* plane wave.

where A_n = complex coefficient, and N = order of multi-pole expansion. On the other hand, the displacements $w_R = u_R$ in the interior region R, which includes the irregularity, can be expanded in terms of the c-complete system [equation (24)] of plane waves to obtain

$$w_R = \sum_{m=1}^{M} B_m \phi_m \tag{52}$$

in which $\phi_m = \exp\{-ikr\cos(\theta - \psi_m)\}$ = plane wave of unitary amplitude and incidence angle ψ_m (Figure 5) = B_m = complex coefficient. It is convenient to choose $\psi_m = \pi(2m-1)/M$, where M = order of plane-wave expansion.

The coefficients of the expansions will be obtained by taking the minimum of the mean-square error which is defined as

$$\epsilon = \int_{\partial_2 R} \left\{ c_1 \,|\, w_E - w_R \,|^2 + c_2 \left| \frac{\partial w_E}{\partial n} - \frac{\partial w_R}{\partial n} \right|^2 \right\} ds + \int_{\partial_1 R} c_3 \left| \frac{\partial w_R}{\partial n} \right|^2 ds \tag{53}$$

in which c_1, c_2, c_3 = normalization factors. As $\epsilon = \epsilon(A_0, A_1, \cdots, A_N, B_1, \cdots, B_M)$, the minimum of ϵ can be found under the conditions

$$\frac{\partial \epsilon}{\partial A_i} = 0, \qquad i = 0, 1, \cdots, N, \quad \text{and} \tag{54a}$$

$$\frac{\partial \epsilon}{\partial B_j} = 0, \qquad j = 1, 2, \cdots, M. \tag{54b}$$

Equations (54) are a system of $N + M + 1$ equations for the $N + M + 1$ unknown complex coefficients and can be arranged in matrix form as

$$[L]\{X\} = \{F\}, \tag{55}$$

in which $[L]$ is an Hermitian positive-definite matrix. Here

$$\{X\}^T = (A_0, A_1, \cdots, A_N, B_1, B_2, \cdots, B_m).$$

Explicit expressions for the coefficients of matrix $[L]$ and vector $\{F\}$ have been obtained by taking the curve $\partial_1 R$ as a semi-circle; they are too long to be presented here. Numerical integration must be used when integrals over $\partial_2 R$ are calculated. However, a direct approach working directly with boundary conditions in a collocation-least-squares procedures has proved to be useful (Sánchez-Sesma and Rosenblueth, 1979; Sánchez-Sesma and Esquivel, 1979, 1980) and leads to a system of equations similar to that of equation (55).

Once the complex coefficients are known, equations (2), (51), and (52) allow us to compute the field at any point of the studied domain. In "Extension to Diffraction Problems," it is explained that the approximating sequences necessarily converge at any interior point and the boundary $\partial_1 E$, but the convergence at the boundary ∂R is not granted. However, the procedure explained in those sections can be used to compute the boundary values.

NUMERICAL EXAMPLES

The method as described above was used to compute displacement amplitudes on the surface of a ridge given by the curve

$$y = h(1 - (x/b)^2)e^{-3(x/b)^2}$$

in $|x| \leqq b$, with $y = 0$ in $|x| > b$. Here h = height of the symmetrical ridge, and $2b$ = width of the base. Numerical results are given for a normalized frequency $\eta = kb/\pi = 2b/\lambda$, where λ = incident wavelength. Three incidence angles = 0°, 45°, 90° and three aspect ratios h/b = 0.25, 0.50, 0.75 were considered. The scattered field was constructed using $N = 10$, and the order of the plane-wave expansion was $M = 20$. Figure 6 shows such results. Agreement with Sills' results (Sills, 1978) is fair. She had obtained the displacement amplitudes at some points on the same surface for a range of frequencies. Here, we fixed the normalized frequency and got results on the interval $|x/b| \leqq 2.5$. The common points have the same amplitude; at least at the scale of the drawings there are not appreciable differences. Results show a very strong dependence on the incidence angle and the aspect ratio. Important reductions and amplifications were found at the edge and at the top of the ridge, respectively.

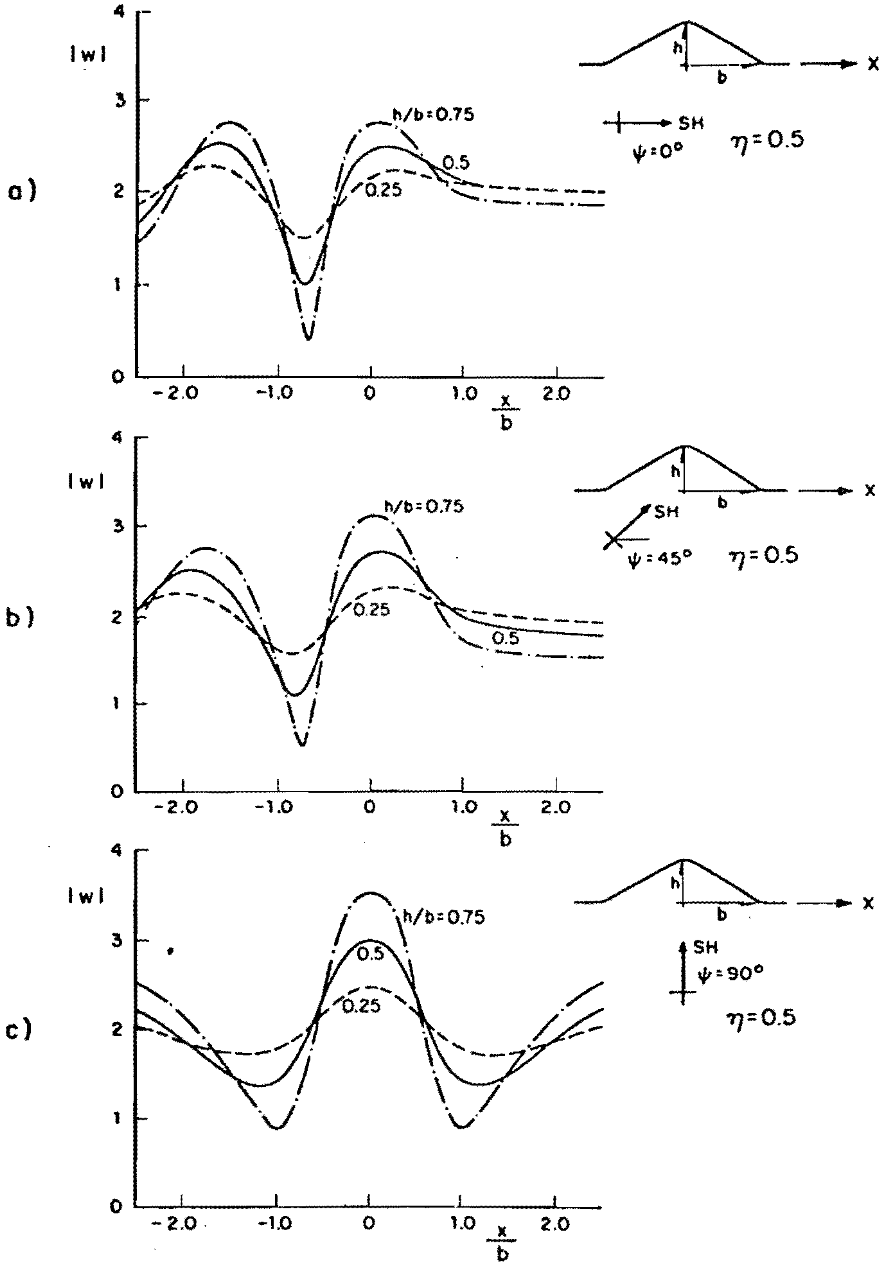

FIG. 6. Displacement amplitudes on the surface of ridges with different aspect ratios and three incidence angles. Normalized frequency $\eta = 0.5$.

Spectacular reductions and amplifications were obtained in a very short horizontal distance on the "incidence side" of the ridge. This fact becomes clear from the graphs a and b in Figure 6.

A mixed topography was also analyzed. It is given, in the interval $|x| \leqq b$, by the curve

$$y = -hc \sin \frac{\pi x}{b} \cos \frac{\pi x}{2b}$$

where $c = 1.29904$, h = height and depth of the mixed topography. Displacement amplitudes are presented in Figure 7 for a normalized frequency $\eta = 0.5$, aspect ratio $h/b = 0.25$, and five incidence angles $\psi = 0°, 45°, 90°, 135°, 180°$. The fields were constructed using $N = 15$ and $M = 30$ for the multi-pole and the plane-wave expansions, respectively. Typically, the relative errors in matching boundary conditions are small, the larger ones are 1.17 and 1.25 per cent of the displacements and the derivatives, respectively. Such errors are defined as

$$\epsilon_w(\%) = \left[\int_{\partial_2 R} | w_E - w_R |^2 \, ds\right] \Big/ \left[\int_{\partial_2 R} | w^{(f)} |^2 \, ds\right] \times 100$$

for displacements, and

$$\epsilon_{\partial w}(\%) = \left[\int_{\partial_2 R} \left|\frac{\partial w_E}{\partial n} - \frac{\partial w_R}{\partial n}\right|^2 ds \Big/ \int_{\partial_2 R} \left|\frac{\partial w^{(f)}}{\partial n}\right|^2 ds\right] \times 100$$

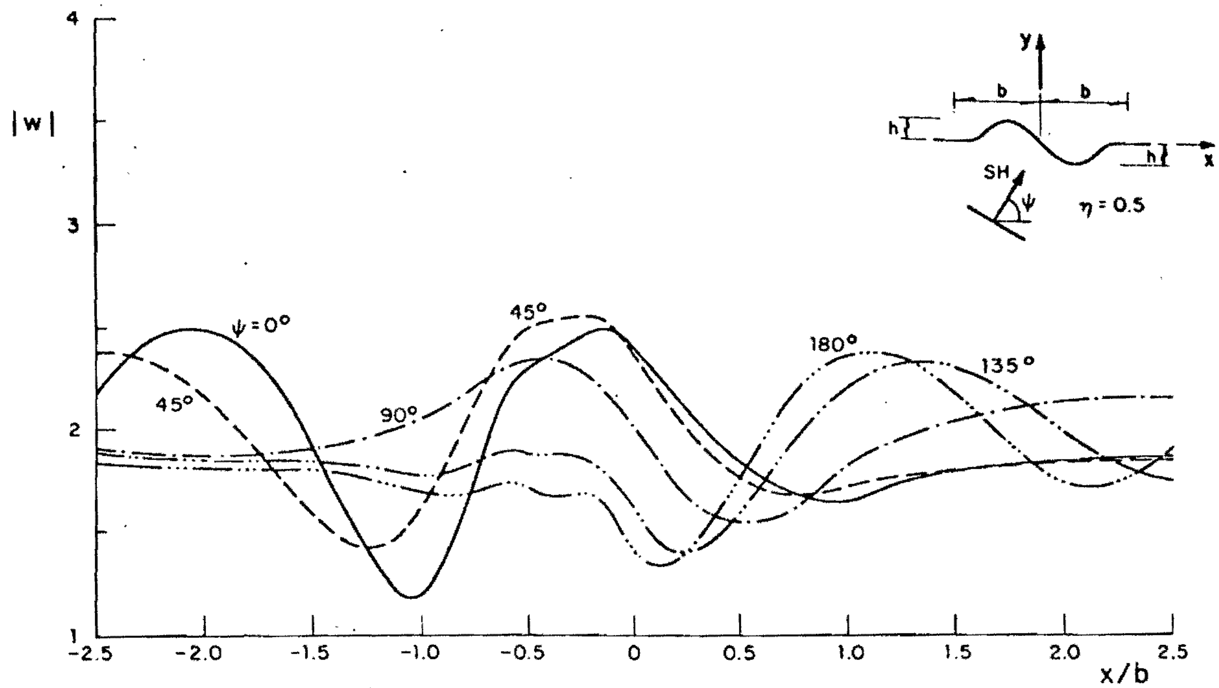

FIG. 7. Displacement amplitudes on the surface of the mixed topography with $h/b = 0.25$ for the five incidence angles and normalized frequency $\eta = 0.5$.

for the derivatives. With larger values of N and M, the relative errors decrease slowly, and the calculated fields practically do not have appreciable changes. In this example, large spatial variations of displacements were found on the incidence side for grazing and inclined incidences. In this case, reductions and amplifications of about 30 per cent were found.

These examples show the significant influence of topographic irregularities in ground motion. Despite their two-dimensional nature and the particularly of the excitation, they throw some light on the basic aspects of the phenomenon. For instance, it could be expected, even with mode conversion, that results for incidence of P or SV waves will follow the general trends observed here.

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Instituto de Ingeniera
Universida Nacional Autonoma
de Mexico
Mexico 20 DF Mexico (F.J.S-S., J.A.)

Instituto de Investigaciones en Mathematic
Aplicadas y en Sistemas
Universidad Nacional Autonoma de Mexico
Mexico 20 DF Mexico (I.H.)

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