WATER RESOURCES RESEARCH, VOL. 18, NO. 3, PAGES 557–562, JUNE 1982

Numerical Treatment of Leaky Aquifers in the Short Time Range

BENITO CHEN AND ISMAEL HERRERA

Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México A. P. 20-726, Mexico City, Mexico

The numerical treatment of leaky aquifers in the short time range is complicated because it requires the use of very refined meshes. This range of time includes not only pumping tests but also many regional studies because the definition of short time range depends on the thickness and diffusivity of the aquitard. The groundwater system of Mexico City is an example in which the short time range extends well beyond the whole life span of the pumping wells. In this paper a procedure which permits the avoidance, to a large extent, of these difficulties is explained; it is based on the integrodifferential equations approach. In addition a method that can be applied to control the error is presented.

1. INTRODUCTION

Mathematically, leaky aquifers are characterized by the assumption of vertical flow in the aquitards, which is well established for most cases of practical interest [*Neuman and Witherspoon,* 1969*a,b*]. There are two main approaches for the numerical modelling of systems of leaky aquifers: one which treats the basic equations in a direct manner without any further development [*Chorley and Frind,* 1978] and the other, which applies a transformation to obtain an equivalent system of integrodifferential equations [*Herrera,* 1976; *Herrera and Rodarte,* 1973; *Herrera and Yates,* 1977; *Hennart et al.,* 1981].

The latter procedure offers considerable computational and analytical advantages [*Herrera et al.,* 1980; *Herrera,* 1976]. In the first approach the aquitard must be discretized, while in the integrodifferential approach the evolution of the aquitards is obtained by means of a series expansion [*Herrera and Yates,* 1977]. The accuracy in the first procedure depends on the number and distribution of the nodes used in the aquitard, while in the second one it depends on the number of terms used in the series expansion; the latter is easier to control.

The treatment of leaky aquifers in the short time range is especially delicate in connection with the facts mentioned above. It is appropriate to say that the short time range is that in which the aquitard can be approximated by a layer of infinite thickness [*Hantush,* 1960; *Herrera and Rodarte,* 1973]. When the latter point of view is adopted, the definition of the short time range depends on the error that one is willing to accept. For example, if the admissible error for the approximation of the aquitard behavior is 10%, then the short time range is $t' < 0.27$, where t' is a dimensionless time (see the notation list at the end of the paper).

The relevance from the practical point of view of the period of operation to which we are referring can be better appreciated by observing that the use of the term 'short time range' may be misleading because the actual physical time can be quite large. In the Mexico City area, for example, clays have exceptionally high specific storage; at Texcoco Lake where artificial reservoirs have been built by inducing land subsidence [*Herrera et al.,* 1974, 1977], there is a layer having a thickness of 32 m, a specific storage of 5.2×10^{-2} m^{-1}, and a permeability of 0.47×10^{-3} m/d. This implies a period of 83.8 yr for the short time range and thus the whole life span of the well field. In this example the specific storage is abnormally high, but the layer is not thick; what determines the span of the short time range is the combination $k'/S_s'b'^2$ which in this case is 8.83×10^{-6}/d. Cases for which this range is above 150 yr are not unusual; the Valley of Guaymas, Mexico, is an example, but many more could be cited.

The numerical modelling of the short time range is difficult. One can understand the nature of such difficulties by looking more carefully into the actual physical situation. When pumping starts in a main aquifer (Figure 1), the effects are first manifested there, and they slowly propagate into the aquitard. The short time range corresponds to that period of time at which the total thickness of the aquitard is much larger than the part that has been affected by pumping. If a uniform mesh were to be used on the whole aquitard, a very refined one would be required; thus too many nodes would be introduced. This could be improved by the use of a nonuniform mesh, but with such procedure it is necessary to have a criterion for distributing the nodes and readjusting the mesh as the time elapses, since the region of the aquitard affected by pumping changes with time.

When the integrodifferential equation approach is used, these problems are reflected in the number of terms that are required in the series expansion of the memory functions. This article is devoted to present a procedure that avoids, to a large extent, these difficulties.

The method is based on the observation that in the short time range the response of the aquitard is approximately the same as if the thickness of the aquitard is infinite, i.e., it does not depend on the actual thickness of the aquitard as long as this is larger than a certain minimum. In the integrodifferential approach this gives rise to a one-parameter family of possible representations for the memory functions, and a procedure is given here to adjust that parameter in such a manner that the number of terms in the series expansion is decreased to a minimum.

In addition an analysis of the error implied by the approximations used for the memory function that is more complete than previous ones [*Herrera and Yates,* 1977] is carried out. This allows supplying a criterion which permits defining in advance the number of terms required to obtain a given accuracy.

Examples of application to actual field situations are given. The results obtained for them exhibit substantial

Paper number 2W0040.
0043-1397/82/002W-0040$05.00

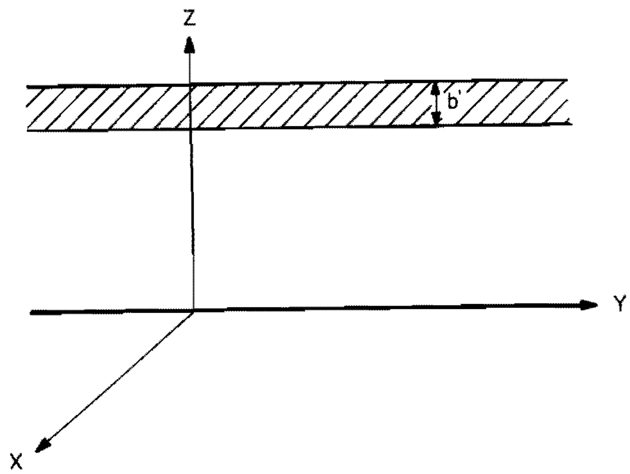

Fig. 1. A single leaky aquifer overlaid by one aquitard.

reductions in the number of terms required for the series expansions. In the worst case this number is 12, but it would have been 128 if the transformation had not been used.

It seems worthwhile to recall that the same basic idea can be used to simplify the numerical treatment of leaky aquifers in the short time range when the aquitard is discretized. The argument presented here can be modified to produce a criterion for distributing the nodes and readjusting the mesh used in the aquitard.

2. Preliminary Notions

Leaky aquifers are characterized by the fact that the permeabilities of the semipervious layers (aquitards) separating the main aquifers (Figure 1) are very small. When the contrast of permeabilities between the main aquifers and the aquitards is one order of magnitude or more, the flow can be taken as vertical in the aquitards [*Neuman and Witherspoon*, 1969*a*,*b*].

To be specific, we restrict attention to the case of one main aquifer limited above by one aquitard and overlying an impermeable bed (Figure 1). The analysis is applicable, however, to a multiaquifer system in the short time range because in this range the interaction between the main aquifers is not perceptible. Also, if the main aquifer overlays another aquitard, the modification of the arguments is straightforward [see, for example, *Herrera*, 1970].

The governing equation when the aquitard is homogeneous is

$$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial s}{\partial y}\right)+K'\frac{\partial s'}{\partial z}(0,t)+Q=S\frac{\partial s}{\partial t} \qquad (1)$$

in the aquifer and

$$\frac{\partial^2 s'}{\partial z^2}=\frac{1}{\alpha'}\frac{\partial s'}{\partial t} \qquad (2)$$

in the semipervious bed. If the system is in hydrostatic equilibrium initially,

$$s(x,y,0)=s'(x,y,z,0)=0 \qquad (3)$$

The boundary condition for s' are

$$s'(x,y,b',t)=0 \qquad (4a)$$

$$s'(x,y,0,t)=s(x,y,t) \qquad (4b)$$

The system of (1)–(4) is equivalent [*Herrera and Rodarte*, 1973] to the integrodifferential equation

$$\frac{\partial}{\partial x}\left(T\frac{\partial s}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial s}{\partial y}\right)-\frac{K'}{b'}$$

$$\cdot\int_0^t\frac{\partial s}{\partial t}(x,y,t-\tau)f(\alpha'\tau/b'^2)\,\partial\tau+Q=S\frac{\partial s}{\partial t} \qquad (5)$$

subjected to

$$s(x,y,0)=0 \qquad (6)$$

When these equations are supplemented by suitable conditions in the boundaries limiting the horizontal extention of the region, one can obtain the drawdown s in the main aquifer without computing the drawdown s' in the aquitard. When s' is required, however, it can be easily derived from the drawdown s [*Herrera and Rodarte*, 1973; *Herrera and Yates*, 1977].

The memory function $f(t')$ has two alternative representations [*Herrera and Rodarte*, 1973]:

$$f(t')=1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2t'}=\frac{1}{(\pi t')^{1/2}}\left(1+2\sum_{n=1}^{\infty}e^{-n^2/t'}\right) \qquad (7)$$

One can define an apparent storage coefficient $S_a(t)$, which changes with time for a leaky aquifer. Consider (5) without horizontal flow; it is

$$S\frac{\partial s}{\partial t}+\frac{K'}{b'}\int_0^t\frac{\partial s}{\partial t}(t-\tau)f(\alpha'\tau/b'^2)\,d\tau=Q \qquad (8)$$

Let the drawdown $s(t)$ be a unit step function; i.e., it is initially zero, and it is then suddenly increased to take the value 1, keeping it fixed thereafter. Then

$$\frac{\partial s}{\partial t}(t)=\delta(t) \qquad (9)$$

where $\delta(t)$ is Dirac's delta function. By substitution of (9) into (8), one gets

$$Q(t)=S\delta(t)+\frac{K'}{b'}f(\alpha't/b'^2) \qquad (10)$$

and the total yield given by the system aquifer-aquitard is

$$\int_0^t Q(\tau)\,d\tau=S+\frac{K'}{b'}\int_0^t f(\alpha'\tau/b'^2)\,d\tau=S+S'F(\alpha't/b'^2) \qquad (11)$$

where F is given by

$$F(t')=\int_0^{t'}f(\tau)\,d\tau \qquad (12)$$

It is natural to define the apparent storage coefficient S_a of the aquifer-aquitard system as the total volume of water per unit area yielded by the system under a unit decline of piezometric head. Hence

$$S_a(t)=\int_0^t Q(\tau)\,d\tau=S+S'F(\alpha't/b'^2) \qquad (13)$$

which is an increasing function of the time elapsed since the application of the unit decline of piezometric head. If there is no aquitard, $S' = 0$ and $S_a(t) = S$, which is independent of time and exhibits the consistency of our definition.

Since $f(t')$ is defined by an infinite series, it is desirable to obtain a good approximation for $F(t')$ using only a small number of terms. We consider two possible approximations. The first is to truncate

$$F(t') = t' + 2 \sum_{n=1}^{\infty} \int_0^{t'} e^{-n^2\pi^2\tau}\, d\tau \tag{14}$$

to N terms:

$$F_N(t') = t' + 2 \sum_{n=1}^{N} \int_0^{t'} e^{-n^2\pi^2\tau}\, d\tau = F(t') - 2 \sum_{n=N+1}^{\infty} \int_0^{t'} e^{-n^2\pi\tau}\, d\tau \tag{15}$$

The second [*Herrera and Yates*, 1977] is to integrate (14) first, sum exactly one of the resulting series, and truncate the other one. This can be obtained observing that

$$\begin{aligned} F(t') &= t' - \frac{2}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2t'}}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2}\right) \\ F(t') &= t' + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-n^2\pi^2t'}}{n^2} \end{aligned} \tag{16}$$

Define

$$F_N(t') = t' + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{e^{-n^2\pi^2t'}}{n^2} \tag{17}$$

Equation (17) can be written alternatively as

$$F_N(t') = F(t') + \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2t'}}{n^2} = F(t') + 2 \sum_{n=N+1}^{\infty} \int_{t'}^{\infty} e^{-n^2\pi^2\tau}\, d\tau \tag{18}$$

The initial value $F_N(0)$, which is different from zero, has been denoted as A_N by *Herrera and Yates* [1977]. This is

$$A_N = F_N(0) = \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \tag{19}$$

Of the two approximations, (15) and (17), the second one yields better results. First, for any N it preserves total yield [see *Herrera and Yates*, 1977]. Second, we are only truncating

$$\sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2t'}}{n^2}$$

which converges very quickly for positive t'. In (15) the series

$$\frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}$$

dominates the error, in contrast to

$$\frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2t'}}{n^2}$$

which is the error in (17). Third, when we integrate the problem numerically, the largest error is in the first time step $\Delta t'$,

$$2 \sum_{n=N+1}^{\infty} \int_{\Delta t'}^{\infty} e^{-n^2\pi^2\tau}\, d\tau$$

and it diminishes rapidly for further time steps. So it is very easy to control the error for any time t'. The same is not true for (15), where the error increases with time.

3. Approximation of the Apparent Storage Coefficient

From (7), it follows that for every $\theta > 0$

$$\begin{aligned} f(\theta t') &= \frac{1}{(\pi\theta t')^{1/2}}\left(1 + 2 \sum_{n=1}^{\infty} e^{-(n^2/\theta t')}\right) \\ &= 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2\theta t'} \end{aligned} \tag{20}$$

Furthermore, define

$$d(t') = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2/t'} \tag{21}$$

Then

$$\frac{f(t')}{f(\theta t')} = \theta^{1/2} \frac{d(t')}{d(\theta t')} \tag{22}$$

Hence

$$f(t') = \sqrt{\theta}\,[1 + \delta(\theta, t')] f(\theta t') \tag{23}$$

where

$$\delta(\theta, t') = \frac{d(t')}{d(\theta t')} - 1 \tag{24}$$

From (23) it follows that

$$F(t') = \theta^{-1/2} F(\theta t') + \Delta(\theta, t') \tag{25}$$

where

$$\Delta(\theta, t') = \sqrt{\theta} \int_0^{t'} \delta(\theta, \tau) f(\theta\tau)\, d\tau \tag{26}$$

A bound for Δ can be obtained when a bound for δ is known. Indeed, if

$$|\delta(\theta, \tau)| \le \varepsilon_1 \quad 0 \le \tau \le t' \tag{27}$$

then

$$|\Delta(\theta, t')| \le \varepsilon_1\, \theta^{-1/2} F(\theta t') \tag{28}$$

In numerical applications of the integrodifferential approach [*Herrera and Yates*, 1977; *Hennart et al.*, 1981] the first of the series expansions given in (7) is used. Equation

(23) can be used to replace this series expansion by one in which t' has been replaced by $\theta t'$. The purpose of this substitution is to accelerate the convergence of the series; clearly, in order to achieve this, we must restrict attention to the case when $\theta > 1$.

Observe that

$$\frac{1}{d(\theta t')} = 1 + \sum_{m=1}^{\infty} [1 - d(\theta t')]^m \quad (29)$$

Hence

$$1 - \frac{d(t')}{d(\theta t')} = d(\theta t') - d(t') + [d(t') - 1][d(\theta t') - 1] - \sum_{m=2}^{\infty} d(t')[1 - d(\theta t')]^m \quad (30)$$

When $d(\theta t') < 2$, (30) can be applied; if, in addition, $\theta > 1$, then

$$0 < -\delta(\theta, t') < d(\theta t') - d(t') + [d(t') - 1][d(\theta t') - 1] \quad (31)$$

Hence

$$|\delta(\theta, t)| < d(\theta t') - 1 + [d(t') - 1][d(\theta t') - 1] = d(t')[d(\theta t') - 1] \quad (32)$$

It is easy to see that the bound given by (32) is monotonically increasing in t'. Suppose a numerical model is being implemented in the range $0 \le t' \le t_{\max}'$. Then

$$|\delta(\theta, t')| < d(t_{\max}')[d(\theta t_{\max}') - 1] \quad (33)$$

Choosing

$$\theta = t_c'/t_{\max}' > 1 \quad (34)$$

where $t_c' > t_{\max}'$ and defining

$$\varepsilon_1 = d(t_{\max}')[d(t_c') - 1] \approx 2 \sum_{n=1}^{\infty} e^{-n^2/t_c'} \quad (35)$$

it is seen that

$$|\delta(\theta, \tau)| \le \varepsilon_1 \quad \text{whenever } 0 < \tau < t_{\max}' \quad (36)$$

Let the approximation F_a be given by

$$F_a(t') = \theta^{-1/2} F_N(\theta t') \quad (37)$$

where F_N is defined by (18). From (25) it follows that

$$F(t') - F_a(t') = \theta^{-1/2}[F(\theta t') - F_N(\theta t')] + \Delta(\theta, t') \quad (38)$$

Notice that $F_N(t')$ is the approximation that has been used in the numerical implementation of the integrodifferential equations approach to leaky aquifers [*Herrera and Yates*, 1977].

Equation (18) clearly implies that

$$F(\theta t') - F_N(\theta t') = -\frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2\theta t'}}{n^2} \quad (39)$$

Therefore

$$F(t') - F_a(t') = -\frac{2\theta^{-1/2}}{\pi^2} \sum_{N+1}^{\infty} \frac{e^{-n^2\pi^2\theta t'}}{n^2} + \Delta(\theta, t') \quad (40)$$

by virtue of (38) and (39). Let ε be the relative error:

$$\varepsilon = \frac{|F(t') - F_a(t')|}{|F(t')|} \quad (41)$$

Using (28) and (40), it is seen that an estimate of ε is given by

$$\varepsilon \le \max\left\{(\pi^3\theta t')^{-1/2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2\theta t'}}{n^2}\right\} + \varepsilon_1 \quad (42)$$

Here the approximation

$$F(t') \approx 2(t'/\pi)^{1/2} \quad (43)$$

implied by (7) was used. If $\Delta t'$ is the time step of the numerical integration, the maximum value achieved by the first term in (42) is attained when the value of t' is $\Delta t'$. This is due to the monotonically decreasing character of the function occurring there. Hence

$$\varepsilon \le (\pi^3\theta\Delta t')^{-1/2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2\theta\Delta t'}}{n^2} + 2\sum_{n=1}^{\infty} e^{-n^2/t_c'} \quad (44)$$

From (13) it is seen that the approximation used corresponds to replacing the storage coefficient S' by

$$S'\left(\frac{t_{\max}'}{t_c'}\right)^{1/2} < S' \quad (45)$$

everywhere, after truncation of the series, in the manner implied by (34) and (37).

The main advantage of this procedure is due to the fact that when $\theta > 1$,

$$\sum_{N+1}^{\infty} \frac{e^{-n^2\pi^2\theta\Delta t'}}{n^2} < \sum_{N+1}^{\infty} \frac{e^{-n^2\pi^2\Delta t'}}{n^2} \quad (46)$$

which shows that the error when (18) is used directly is larger. This permits using fewer terms in the series expansions to achieve a desired accuracy. The procedure can be carried out as follows. Let $\varepsilon > 0$ be the admissible error. If $t_{\max}'$ and $\Delta t'$ are given, then one can choose t_c' so that

$$2\sum_{n=1}^{\infty} e^{-n^2/t_c'} = \varepsilon/2 \quad (47)$$

This equation can be solved for t_c' using a bisection type scheme. For example, if $\varepsilon = 0.1$, then

$$t_c' = 0.27 \quad (48)$$

Once t_c' has been defined, one needs to choose N so that

$$(\pi^3\theta\Delta t')^{-1/2} \sum_{n=N+1}^{\infty} \frac{e^{-n^2\pi^2\theta\Delta t'}}{2} < \varepsilon < (\pi^3\theta\Delta t')^{-1/2} \cdot \sum_{n=N}^{\infty} \frac{e^{-n^2\pi^2\theta\Delta t'}}{n^2} \quad (49)$$

TABLE 1. Results for the Valley of Mexico A

ε	t_c'	θ	N	$N(\theta = 1)$
0.01	0.16690	3.38822	9	17
0.05	0.22821	4.63268	6	13
0.10	0.27108	5.50312	5	11
0.15	0.30455	6.18262	4	10
0.20	0.33379	6.77617	4	9
0.25	0.36064	7.32118	3	9

TABLE 2. Results for the Valley of Mexico B

ε	t_c'	θ	N	$N(\theta = 1)$
0.01	0.16690	3.38822	9	14
0.05	0.22821	4.63268	6	11
0.10	0.27108	5.50312	5	9
0.15	0.30455	6.18262	4	8
0.20	0.33379	6.77617	4	8
0.25	0.36064	7.32118	3	7

4. NUMERICAL EXAMPLES

Two of the leaky aquifers that have been extensively studied in Mexico are the ones under the Valley of Mexico [*Herrera et al.*, 1974, 1977] and Guaymas. The different properties of the aquifers have a wide range of variation.

To exemplify the efficiency of the procedure we have used two sets of values from the Valley of Mexico that we consider to be representative, one set from Guaymas and another from an aquifer with ficticious properties.

For each aquifer the computations were done for both the θ 'optimum' given by (34) and for $\theta = 1$ for a wide range of relative errors. A result with a relative error of 10% is usually very satisfactory. The results are presented below.

Valley of Mexico A (Table 1):

$$S' = 4.8 \qquad T' = 1.816 \times 10^{-5}\ \text{km}^2/\text{yr}$$

$$b' = 4.8 \times 10^{-2}\ \text{km} \qquad t_{\max} = 30\ \text{yr} \qquad \Delta t = 0.5\ \text{yr}$$

or in nondimensional form

$$t_{\max}' = 4.926 \times 10^{-2} \qquad \Delta t' = 8.210 \times 10^{-4}$$

Valley of Mexico B (Table 2):

$$S' = 2.4 \qquad T' = 3.333 \times 10^{-6}\ \text{km}^2/\text{yr}$$

$$b' = 2.4 \times 10^{-2}\ \text{km} \qquad t_{\max} = 30\ \text{yr} \qquad \Delta t = 0.5\ \text{yr}$$

$$t_{\max}' = 7.226 \times 10^{-2} \qquad \Delta t' = 1.200 \times 10^{-3}$$

Guaymas (Table 3):

$$S' = 0.75 \qquad T' = 1.230 \times 10^{-7}\ \text{km}^2/\text{yr}$$

$$b' = 7.5 \times 10^{-2}\ \text{km} \qquad t_{\max} = 50\ \text{yr} \qquad \Delta t = 0.5\ \text{yr}$$

$$t_{\max}' = 1.457 \times 10^{-3} \qquad \Delta t' = 1.457 \times 10^{-5}$$

Aquifer with ficticious properties (Table 4):

$$S' = 4.0 \qquad T' = 3.1536 \times 10^{-6}\ \text{km}^2/\text{yr}$$

$$b' = 1.0 \times 10^{1}\ \text{km} \qquad t_{\max} = 30\ \text{yr} \qquad \Delta t = 0.5\ \text{yr}$$

$$t_{\max}' = 2.365 \times 10^{-3} \qquad \Delta t' = 3.942 \times 10^{-5}$$

NOTATION

$A_N = (2/\pi^2)\ \Sigma_{n=N+1}^{\infty}\ n^{-2}$
b' thickness of aquitard, L.
$d(t')$ function defined by (21).

TABLE 3. Results for Guaymas

ε	t_c'	θ	N	$N(\theta = 1)$
0.01	0.16690	3.38822	12	128
0.05	0.22821	4.63268	8	97
0.10	0.27108	5.50312	6	83
0.15	0.30455	6.18262	5	75
0.20	0.33379	6.77617	5	69
0.25	0.36064	7.32118	4	65

TABLE 4. Results for an Aquifer With Fictitious Properties

ε	t_c'	θ	N	$N(\theta = 1)$
0.01	0.16690	3.38822	9	78
0.05	0.22821	4.63268	6	59
0.10	0.27108	5.50312	5	51
0.15	0.30455	6.18262	4	46
0.20	0.33379	6.77617	4	42
0.25	0.36064	7.32118	3	39

$f(t')$ memory function, equal to $1 + 2\ \Sigma_{n=1}^{\infty}\ e^{-n^2\pi^2 t'}$.
$F(t') = \int_0^{t'} f(\tau)\, d\tau$
$F_a(t')$ approximation of $F(t')$, defined by (37).
$F_N(t')$ approximation of $F(t')$, defined by (38).
K' permeability of aquitard, L/T.
$Q(t')$ pumping rate from aquifer, L/T.
s drawdown in aquifer, L.
s' drawdown in aquitard, L.
S storage coefficient of aquifer.
S' storage coefficient of aquitard.
$S_a(t)$ apparent storage coefficient of system, defined by (13).
t time, T.
t' dimensionless time, equal to $\alpha' t/b'^2$.
t_c' upper bound of short time range.
$t_{\max}'$ maximum value of t'.
T transmissibility of aquifer, L^2/T.
T' transmissibility of aquitard, L^2/T.
x, y, z coordinates, L.
$\alpha' = T'/S'$, L^2/T.
$\delta(t')$ Dirac's delta function.
$\delta(\theta, t')$ function defined by (24).
$\Delta(\theta, t')$ function defined by (26).
ε relative error, defined by (41).
ε_1 error defined by (27).
θ positive parameter of the family of approximations F_a.

Acknowledgments. This paper was presented by invitation at the John Ferris Symposium at the Spring Annual Meeting, AGU, Baltimore, Maryland, May 25–29, 1981.

REFERENCES

Chorley, D. W., and E. O. Frind, An iterative quasi-three-dimensional finite element model for heterogeneous multiaquifer systems, *Water Resour. Res., 14*(5), 943–952, 1978.
Hantush, M. S., Modification of the theory of leaky aquifers, *J. Geophys, Res., 65*(11), 3713–3725, 1960.
Hennart, J. P., R. Yates, and I. Herrera, Extension of the integrodifferential approach to inhomogeneous multiaquifer systems, *Water Resour. Res., 17*(4), 1044–1050, 1981.
Herrera, I., Theory of multiple leaky aquifers, *Water Resour. Res., 6*(1), 185–193, 1970.
Herrera, I., A review of the integrodifferential equations approach to leaky aquifer mechanics, paper presented at the Symposium on Advances in Groundwater Hydrology, Am. Water Resour. Assoc., Minneapolis, Minn., 1976.
Herrera, I., and L. Rodarte, Integrodifferential equations for systems of leaky aquifers and applications, 1, The nature of approximate theories, *Water Resour. Res., 9*(4), 995–1005, 1973.
Herrera, I., and R. Yates, Integrodifferential equations for systems of leaky aquifers, 3, A numerical method of unlimited applicability, *Water Resour. Res., 13*(4), 725–732, 1977.
Herrera, I., J. Alberro, J. L. León, and B. Chen, Análisis de asentamientos para la construcción de los lagos del plan Texcoco, *Rep. 340,* Inst. de Ing., Univ. Nac. Auton. de Mex., Mexico City, 1974.
Herrera, I., J. Alberro, R. Graue, and J. J. Hanel, Development of

artificial reservoirs by inducing land subsidence, *IAHS-AISH Publ. 121*, 39–45, 1977.

Herrera, I., J. P. Hennart, and R. Yates, A critical discussion of numerical models for multiaquifer systems, *Advan. Water Resour., 3*(4), 159–163, 1980.

Neuman, S. P., and P. A. Witherspoon, Theory of flow in a confined two-aquifer system, *Water Resour. Res., 5*(4), 803–816, 1969*a*.

Neuman, S. P., and P. A. Witherspoon, Transient flow of ground water to wells in multiple aquifer systems, *Geotech. Eng. Rep. 69-1*, Univ. of Calif., Berkeley, Jan. 1969*b*.

(Received June 24, 1981;
revised December 15, 1981;
accepted December 22, 1981.)