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BOUNDARY METHODS: DEVELOPMENT OF COMPLETE SYSTEMS OF SOLUTIONS

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1.- INTRODUCTION.

There are two main approaches for the formulation of boundary methods; one is based on boundary integral equations and the other one, on the use of complete systems of solutions. The author has given previously extensive descriptions of the latter method [1-3]. Its theoretical foundations and development embrace the following aspects: a) Approximating procedures and conditions for their convergence; b) Formulation of variational principles; and c) Development of complete systems of solutions. The first two of these subjects have been studied and discussed in previous publications [1-8]. However, a systematic discussion of the last one is wanting.

There are a number of scattered publications related with this matter [6, 9-11]. It has been shown that a suitable criterion for completeness is c-completeness [1-3, 8]. A method of considerable generality, for generating such systems is described in [6] and [9]. A general version of separation of variables procedures yields biorthogonal systems which are c-complete [12-13]. Convenient features of c-complete systems are their simplicity, as in the case of plane waves [14], and the fact that the same system can be applied to large classes of regions and boundary conditions. In this paper we present a brief but systematic exposition of the methods available for the construction of c-complete systems.

2.- C-COMPLETE SYSTEMS AND HILBERT-SPACES.

In general a linear subspace $N_P \subset D$ of a linear space D is considered. Here, N_P stands for the null subspace of an operator $P:D \to D^*$. For example, if Laplace equation in a bounded region R (Fig. 1), is considered, a convenient definition of P is

$$\langle Pu,v\rangle = \int_R v\Delta u \, d\underset{\sim}{x} \tag{2.1}$$

In this case the linear space D may be taken as $H^s(R)$, $s>3/2$. The elements of the null subspace are the harmonic functions on R. The antisymmetric operator $A=P-P^*$ plays an important role. In the example, above, it is

$$\langle Au,v\rangle = \int_{\partial R} \left\{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right\} d\underset{\sim}{x} \tag{2.2}$$

For formally symmetric operators, the null subspace N_A of $A=P-P^*$, is used to introduce a classification of boundary value [1-3, 7]. When P is given by (2.1), for example,

$$N_A = \{u \in D \mid u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial R\} \tag{2.3}$$

and the relevant boundary values can be taken to be u and $\partial u/\partial n$ on ∂R. In general such classification yields the space of boundary values $\hat{D}$, defined by [1-2]:

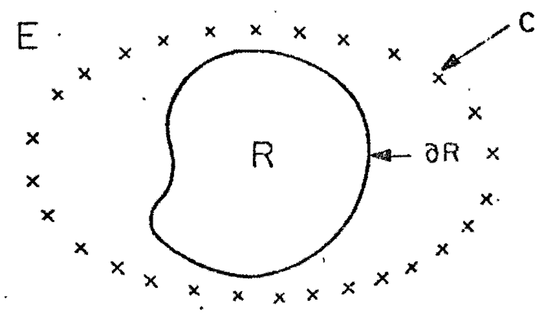

Figure 1

$$\hat{\mathcal{D}} = D/N_A \tag{2.4}$$

Using the concept of canonical decomposition, it is always possible to take $\hat{\mathcal{D}}$ as a product space. More precisely, the relevant boundary values can be expressed in terms of two groups of functions. For Laplace equation elements $\hat{u} \in \hat{\mathcal{D}}$ are characterized by a pair of functions $\{u, \partial u/\partial n\}$ which are defined only on the boundary. Note the difference between such elements and those of D, which are defined in the whole region R.

There are, however, many alternative ways of decomposing the boundary values into two groups; each associated with a canonical decomposition. If a_1, a_2, b_1 and b_2 are constants such that

$$a_1 b_2 - b_1 a_2 = 1 \tag{2.5}$$

another possibility in the example here discussed, is to take $\{a_1 u + a_2 \frac{\partial u}{\partial n}, b_1 u + b_2 \frac{\partial u}{\partial n}\}$. In general, when $\hat{u} \in \hat{\mathcal{D}}$ we write

$$\hat{u} = \{u_1, u_2\} \tag{2.6}$$

The elements of the space $I_P = N_P + N_A$, on the other hand, are characterized by the fact they attain the same boundary values as some solution of the homogeneous equation (a harmonic function, in our example). The range of boundary values reached by solutions of the homogeneous equation, is the quotient space

$$\hat{N}_P = N_P/N_A \tag{2.7}$$

Under quite general conditions [1-3], I_P is a completely regular subspace. By this we mean

$$\langle Au, v\rangle = 0 \quad \forall \quad v \in I_P \Leftrightarrow u \in I_P \tag{2.8}$$

The same is true of $\hat{N}_P$. A subset $\mathcal{B} \subset I_P$ is said to be c-complete for I_P, when for every $u \in D$, one has

$$\langle Au,w\rangle = 0 \quad \forall \quad w\in\mathcal{B} \Rightarrow u\in I_P \tag{2.9}$$

A c-complete subset is said to be a connectivity basis, when in addition it is linearly independent. Completely regular subspaces always possess c-complete systems [2]. One of the main advantages of c-completeness is that this criterion is purely algebraic; this allows greater generality and flexibility for some of the results of the theory.

As an illustration, we give the following example. For Laplace equation it has been shown [6] that the system of harmonic polynomials, whose expression in polar coordinates is

$$\mathcal{B} = \{1,\ r^n\cos n\theta,\ r^n\sin n\theta \mid n=1,2,\dots\}\subset N_P \tag{2.10}$$

is c-complete in any bounded region of the plane, which is simply connected (Fig. 1).

Under very general conditions [2-3] there is a Hilbert-space H such that the product space $\hat{\mathcal{D}}$ of boundary values satisfies

$$\hat{\mathcal{D}} \subset H \oplus H \tag{2.11}$$

Even more

$$\langle A\hat{u},\hat{v}\rangle = (u_2,\ \overline{v_1}) - (v_2,\ \overline{u_1}) \tag{2.12}$$

Here (,) is the inner product in H and the bar above stands for the complex conjugate (antilinear mapping). If the coefficients in the Hilbert-space are real, the bar must be deleted.

Let N_{P1} and N_{P2} be the range spanned by the first and second component, respectively, of elements $\hat{u} = \{u_1,\ u_2\}\in\hat{N}_P$. Given a subset $\mathcal{B}\subset N_P\subset I_P$, there is associated to it a $\hat{\mathcal{B}}\subset\hat{N}_P$ whose elements $\hat{u} = \{u_1,\ u_2\}$ are the traces of elements of $\mathcal{B}$. It has been shown that the following statements are equivalent

i).- $\mathcal{B}\subset N_P$ is c-complete for I_P; (2.13)

ii).- span $\mathcal{B}_1 = \overline{N}_{P1}$ while span $\mathcal{B}_2 = \overline{N}_{P2}$ (2.14)

A proof, as well as, more precise and ellaborated forms of this result are given in References [2-3].

3.- CHANGE OF BOUNDARY CONDITIONS.

Let $H = H^\circ(\partial R) = \mathcal{L}^2(\partial R)$ where ∂R is the boundary of the simply connected region R, illustrated in Figure 1. The inner product of two functions $u\in H^\circ(\partial R)$, $v\in H^\circ(\partial R)$ is given by

$$(u,v) = \int_R u\overline{v}\,dx \tag{3.1}$$

In view of (2.2), it is clear that equations (2.11) and (2.12) are satisfied if $u_1 = u$ and $u_2 = \partial u/\partial n$ on ∂R.

As it is well-known [15]

$$\overline{N}_{P1} = H^\circ(\partial R) \quad ; \quad \overline{N}_{P2} = \{1\}^\perp \tag{3.2}$$

where $\{1\}^\perp$ stands for the orthogonal complement of the constant function 1 on ∂R. Corresponding to the c-complete system (2.10), we have the boundary values on ∂R:

$$\mathcal{B}_1 = \{1,\ r^n\cos n\theta,\ r^n\sin n\theta\} \tag{3.3a}$$

and

$$\mathcal{B}_2 = \{0,\ \partial r^n \cos n\theta/\partial n,\ \partial r^n \sin n\theta/\partial n\} \qquad (3.3b)$$

According to equations (2.14), $\mathcal{B}_1$ spans $H^\circ(\partial R)$ while $\mathcal{B}_2$ spans $\{1\}^\perp$. The direct verification of this statement is straight-forward if the region R is taken as a circle with center at the origin.

From the above discussion it is clear that the c-complete system $\mathcal{B}$ as given by (2.10), can be used to approximate any Dirichlet and Neuman boundary data. Even more, the same system $\mathcal{B}$ can be used to solve any linear boundary condition. Let such condition be given by the specification of the linear combination $a_1 u + a_2 \partial u/\partial n$. Choose b_1 and b_2 so that (2.5) is fulfilled. Define

$$u_1 = b_1 u + b_2 \partial u/\partial n \quad ; \quad u_2 = a_1 u + a_2 \partial u/\partial n \qquad (3.4)$$

Then, it is easy to verify that relations (2.11) and (2.12) are satisfied when A is given by (2.2). In this case

$$\overline{N}_{P1} = H^\circ(\partial R) \quad , \quad \overline{N}_{P2} = H^\circ(\partial R) \qquad (3.5)$$

unless b_2 or a_2 are zero. For the sake of brevity this latter case is left out of the discussion. Equations (2.14) show that

$$\text{span } \mathcal{B}_1 = \text{span } \mathcal{B}_2 = H^\circ(\partial R) \qquad (3.6)$$

This shows that the system $\mathcal{B}$ given by (2.10) can be used to solve a problem where either $a_1 u + a_2 \partial u/\partial n$ or alternatively $b_1 u + b_2 \partial u/\partial n$ is specified, if the boundary values of the functions belonging to $\mathcal{B}$ are organized in the manner implied by equations (3.4).

4.- CHANGE OF REGION.

In a previous publication [2], the author expressed the belief that a system that is c-complete in a region R also has this property in any subregion of R. (Prof. S. Antman had the suspicion that such result had actually been shown. Unfortunately a partial survey of the literature seems to indicate that such results, if available, are not sufficiently general for applications of boundary methods. Thus, the matter had to be tackled ab-initio). Recently, this has been shown and also, that conversely, if a system is c-complete in a region, then it is also c-complete in any region that contains it. The precise conditions under which this is true, as well as more general questions about modifications of a region that do not alter the c-completeness property are given in [3].

An example is the system of functions (2.10) which has been shown to be c-complete for Laplace equation, in any bounded and simply connected region [6]. For an exterior region such as E in Figure 1, on the other hand, N_p is the linear space of harmonic functions such that $u - d\,\mathrm{Ln}r$ is square-integrable on E. Here

$$d = \frac{1}{2\pi} \int_C \frac{\partial u}{\partial n}\, d\underset{\sim}{x} \qquad (4.1)$$

where C is any closed curve which encloses the only hole of E. It has been shown [6], that the system

$$\mathcal{B} = \{\mathrm{Ln}r,\ r^{-n}\cos n\theta,\ r^{-n}\sin n\theta \,|\, n=1,2,\ldots\} \qquad (4.2)$$

is c-complete for N_p. In this case an interesting phenomenon occurs. In general $\overline{N}_{p1} = H^\circ(\partial E)$ and $\overline{N}_{p2} = H^\circ(\partial E)$. However [16], there are anomalous regions for which $\overline{N}_{p1} = \{1\}^\perp$. An example is a unit circle; evaluating

(4.2) at r=1, it is seen that

$$\mathcal{B}_1 = \{0, \cos n\theta, \sin\theta\} \quad ; \quad \mathcal{B}_2 = \{1, -n \cos n\theta, -n \sin n\theta\} \quad (4.3)$$

Clearly, equations (2.14) are satisfied. Thus, the same c-complete system (4.2) can be used, even if the region is anomalous.

5.- THE SOURCE METHOD.

By this it is usually meant a procedure in which a system of sources is used to represent any field [17]. Consider a simply connected bounded region R (Fig. 1), whose exterior is E. Let

$$G(\underset{\sim}{x},\underset{\sim}{y}) = \frac{1}{2\pi} \mathrm{Ln}|\underset{\sim}{x}-\underset{\sim}{y}| \qquad (5.1)$$

For every $\underset{\sim}{y} \in E$, define

$$w_{\underset{\sim}{y}}(\underset{\sim}{x}) = G(\underset{\sim}{x},\underset{\sim}{y}) \quad , \quad \underset{\sim}{x} \in R \qquad (5.2)$$

and

$$\mathcal{B}_E = \{w_{\underset{\sim}{y}}(\underset{\sim}{x}) \,|\, \underset{\sim}{y} \in E\}.$$

Then $w_{\underset{\sim}{y}} \in N_P$ and it is well-known that

$$W(\underset{\sim}{y}) = \langle Au, w_{\underset{\sim}{y}} \rangle = \int_{\partial R} \{G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n}\} d\underset{\sim}{x} = 0 \quad \forall \quad \underset{\sim}{y} \in E \Rightarrow u \in I_P \qquad (5.3)$$

This shows that $\mathcal{B}_E$ is c-complete. Given a curve C, enclosing R, take a dense subset $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \ldots\} \subset C$ of C. Define

$$w_\alpha(\underset{\sim}{x}) = w_{\underset{\sim}{y}\alpha}(\underset{\sim}{x}) \qquad (5.4)$$

and

$$\mathcal{B}_C = \{w_{\underset{\sim}{y}}(\underset{\sim}{x}) \,|\, \underset{\sim}{y} \in C\} \quad ; \qquad (5.5)$$

For any harmonic function in R, one has

$$\langle Au, 1 \rangle = \int_{\partial R} \frac{\partial u}{\partial n} d\underset{\sim}{x} = 0 \qquad (5.6)$$

Using results derived by Christiansen [16] it can be seen that in the presence of restriction (5.6), if W(y) vanishes identically on C, then it vanishes identically on E. This shows that the system $\{1\} \cup \mathcal{B}_C$ is c-complete. Therefore, the system

$$\mathcal{B} = \{1, w_1, w_2, \ldots\} \qquad (5.7)$$

is c-complete, because the condition

$$\langle Au, w \rangle = 0 \quad \forall \quad w \in \mathcal{B}$$

is tantamount to require condition (5.6) and that the function W(y) vanishes in a dense subset of C. This implies that the premise in (5.3) is satisfied.

A procedure similar to the one explained here, based on Green's third identity [18], can be used to establish the foundations of the source method in many other problems (Stokes, biharmonic equation, etc.),

6.- SEPARATION OF VARIABLES.

A procedure which is more general than the standard separation of variables is presented in this Section. Herrera's orthogonality relations for Rayleigh waves [19], which have been used in the geophysical literature [20-21] to treat diffraction problems, are an example of its application.

A pair of completely regular subspaces $\{N_p^1, N_p^2\}$ of N_p, is said to be a canonical decomposition when they span N_p. Under the general assumptions of the theory [2, 13], the results can be summarized as follows:

"There are two families of separable solutions $\mathcal{B} = \{\phi_1, \phi_2, \ldots\} \subset N_p^1$ and $\mathcal{B}^* = \{\phi_1^*, \phi_2^*, \ldots\} \subset N_p^2$ which are orthonormal for the anti-symmetric bilinear functional A, and c-complete for the subspaces of a canonical decomposition $\{N_p^1, N_p^2\}$ of the space N_p".

An example is the biharmonic equation in a horizontal strip $(-1 \leq y \leq 1)$, subjected to the boundary condition $u = \partial u/\partial y = 0$ at $y = \pm 1$. In this case the subspaces of the canonical decomposition are characterized by $u \to 0$ as $x \to +\infty$ when $u \in N_p^1$, while $u \to 0$ as $x \to -\infty$ when $u \in N_p^2$.

Taking Re $\lambda_n > 0$, separable solutions satisfy [12]

$$\phi_n(x,y) = f_n(y) e^{-\lambda_n x} \quad ; \quad \phi_n^*(x,y) = f_n^*(y) e^{\lambda_n x} \tag{6.1}$$

where $\sin^2 2\lambda_n - 4\lambda_n^2 = 0$. The corresponding families of separables solutions are biorthogonal with respect to

$$\langle Au, v\rangle = \int_C \{v \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} + \Delta v \cdot \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n}\} d\underset{\sim}{x} \tag{6.2}$$

where C is any curve that goes across the strip (Fig. 2). It is easy to see that this definition of A is independent of the specific choice of C.

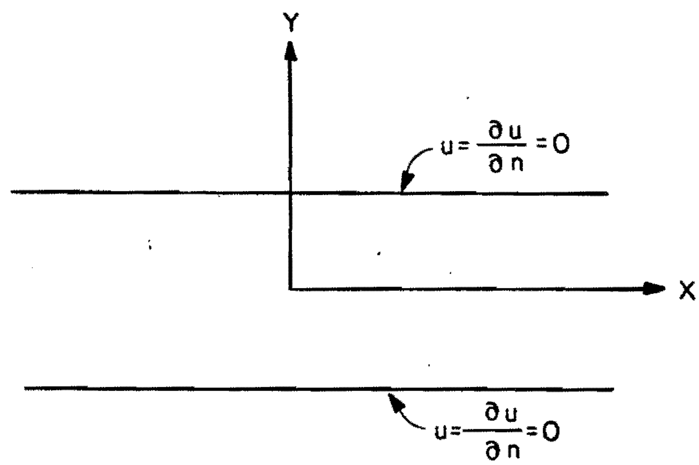

Figure 2

Similarly, the systems (2.10) and (4.4) are biorthogonal, when A is given by (2.2) and R is any bounded and simply connected region that contains the origin in its interior.

7.- AD-HOC PROCEDURES.

It is frequently possible to construct c-complete systems for complicated differential equations starting from simpler ones. We illustrate this by building up such systems for the biharmonic equation and Stokes problem, starting from c-complete systems for Laplace equation.

Consider again the biharmonic equation, but this time in a region R. Thus

$$\langle Au,v\rangle = \int_{\partial R} \{v \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n}\} d\underset{\sim}{x} \tag{7.1}$$

Let $\{\psi_1, \psi_2, \ldots\}$ be harmonic functions and let $\{w_1, w_2, \ldots\}$ be such that

$$\Delta w_j = \psi_j \qquad j = 1,2,\ldots \tag{7.2}$$

By an extension of the arguments used in Ref. [10], it can be shown that the system

$$\mathcal{B} = \{\psi_1, \psi_2, \ldots\} \cup \{w_1, w_2, \ldots\} \tag{7.3}$$

is c-complete in R, for the biharmonic equation whenever the system $\{\psi_1, \psi_2, \ldots\}$ is c-complete for Laplace equation.

An easy manner of satisfying (7.2) is choosing

$$w_j = x\phi_j \quad ; \quad \text{where } \frac{\partial \phi_j}{\partial x} = \psi_j \tag{7.4}$$

and ϕ_j harmonic. When this procedure is applied to the system (2.10) one gets the system

$$\{1, r^n \cos n\theta, r^n \sin n\theta\} \cup \{r^2 \cos^2\theta, r^{n+2} \cos(n+1)\theta \cos\theta, r^{n+2} \sin(n+1)\theta \cos\theta\} \tag{7.5}$$

as a c-complete system for the biharmonic equation in any bounded and simply connected region.

The same result holds in an exterior domain. Applying a similar procedure to system (4.3), one gets the system

$$\{\log r, r^{-n} \cos n\theta, r^{-n} \sin \theta\} \cup \{\phi, r \log r \cos\theta, r \log r \sin\theta, r^{-n+1} \cos n\theta \cos\theta, r^{-n+1} \sin n\theta \cos\theta\} \tag{7.6}$$

where $\phi = r^2[\mathrm{Log}\, r - 1]$.

As a last example let us consider Stokes problem in a region R (Fig. 1). In this case the differential equations are

$$\nu \Delta \underset{\sim}{u} - \nabla p = 0 \tag{7.7a}$$

$$\nabla \cdot \underset{\sim}{u} = 0 \tag{7.7b}$$

In connection with system (7.7), one is led to consider the space of functions $\hat{D}$ whose elements are pairs $\hat{u} = [\underset{\sim}{u}, p]$ where $\underset{\sim}{u}$ is a vector while p is a scalar. A suitable formally symmetric operator $\hat{P}: \hat{D} \to \hat{D}^*$ is

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = \int_R \{\nu \underset{\sim}{v} \cdot \Delta \underset{\sim}{u} - \underset{\sim}{v} \cdot \nabla p + q \nabla \cdot \underset{\sim}{u}\} d\underset{\sim}{x} \tag{7.8}$$

Here $\hat{v} = [\underset{\sim}{v}, q]$. This yields

$$\langle \hat{A}\hat{u}, \hat{v}\rangle = \int_{\partial R} \{\underset{\sim}{v} \cdot (\nu \frac{\partial \underset{\sim}{u}}{n} - p\underset{\sim}{n}) - \underset{\sim}{u} \cdot (\nu \frac{\partial \underset{\sim}{v}}{\partial n} - q\underset{\sim}{n})\} d\underset{\sim}{x} \tag{7.9}$$

It can be shown [11] that equations (7.7) are satisfied, if and only if, there exist functions ϕ and $\underset{\sim}{H}$ in R such that

$$\Delta \underset{\sim}{H} = 0 \tag{7.10a}$$

$$\Delta\phi + \nabla \cdot \underset{\sim}{H} = 0 \tag{7.10b}$$

with the property

$$\underset{\sim}{u} = \frac{1}{\nu}(\nabla\phi + \underset{\sim}{H}) \quad ; \quad p = \nabla\phi \tag{7.11}$$

Conversely, equations (7.10) are satisfied, if and only if, there are functions p and u fulfilling (7.7) and such that

$$\Delta\phi = p \tag{7.12a}$$

$$\underset{\sim}{H} = \nu\underset{\sim}{u} - \nabla\phi \tag{7.12b}$$

These two representations which will be used in connection with Stokes problem, leads to consider two linear spaces $\hat{D}_u$ and $\hat{D}_H$ of functions. Elements $\hat{u} = \hat{D}_u$ will be pairs $\hat{u} = [u, p]$ where u is a vector while p is a scalar. Similarly, elements $\hat{H} \in \hat{D}_H$ will be pairs $\hat{H} = [\underset{\sim}{H}, \phi]$.

Let N be the dimension of the space (2 or 3). Assume

$$B_L = \{w_1, w_2, \ldots\} \tag{7.13a}$$

and

$$B_o = \{\phi_{10}, \phi_{20}, \ldots\}$$

are c-complete systems for Laplace equation (the possibility $B_L = B_o$ is not excluded). Take $\phi_{i\alpha}$ (α=1, ..., N), so that

$$\Delta\phi_{i\alpha} + \frac{\partial w_i}{\partial x_\alpha} = 0 \quad , \quad i=1, 2, \ldots \tag{7.14}$$

Define

$$\underset{\sim}{w}_{i1} = (w_i, 0) \quad ; \quad \underset{\sim}{w}_{i2} = (0, w_i) \tag{7.15}$$

Here, for simplicity, N=2 was assumed; the modification required for the case N=3 is straight-forward. For every i=1, 2, ..., let be

$$\hat{H}_{i\alpha} = [\phi_{i\alpha}, \underset{\sim}{w}_{i\alpha}] \quad ; \quad \alpha = 0,1,\ldots,N \tag{7.16}$$

where $w_{i0} = 0$. Define

$$\hat{B}_\alpha = \{H_{1\alpha}, H_{2\alpha}, \ldots\} \quad \alpha = 0,1,\ldots,N \tag{7.17}$$

and

$$\hat{B}_H = \bigcup_{\alpha=0}^{N} \hat{B}_\alpha \tag{7.18}$$

With every $\hat{H} \in \hat{B}_H$, associate an element $\hat{u} = \tau(\hat{H}) \in \hat{D}_u$ where τ is the transformation (7.11). Let

$$\hat{B} = \{\hat{u} \in \hat{D}_u \mid \hat{u} = \tau(\hat{H}), \hat{H} \in \hat{B}_H\} \tag{7.19}$$

It has been shown [11] that the system $\hat{B}$ is c-complete for Stokes problem.

A specific application of this result, is a c-complete system for Stokes problems in a bounded region R, which is derived from the c-complete system B for Laplace equation, given in equation (2.10). This is constructed taking

$$B_0 = B_L = B \tag{7.20}$$

In addition

$$\phi_{i\alpha} = -2 \frac{\partial w_i}{\partial x_\alpha} \tag{7.21}$$

The same applies to an exterior region (Fig. 1) if $\mathcal{B}$, in the above construction, is replaced by the system given in equation (4.3)

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