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SOLUTION OF FREE BOUNDARY PROBLEMS USING C-COMPLETE SYSTEMS.

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ABSTRACT.

There are two main approaches to the formulation of boundary methods, these are boundary integral equations and approximations by complete systems of solutions. The latter has been the subject of extensive studies by one of the authors oriented to clarifying the foundations of the method and increasing its versatility. The present paper is devoted to explain the application of this procedure to free boundary problems such as Signorini's and contact problem [1-6].

1. INTRODUCTION

There are two main approaches for the formulation of boundary methods; one is based on boundary integral equations and the other one, on the use of complete systems of solutions. One of the authors has given previously extensive descriptions of the latter method [1-11]. Its theoretical foundations and development embrace the following aspects: a) approximating procedures and conditions for their convergence; b) formulation of variational principles; and c) development of complete systems of solutions. It has been shown that a suitable criterion for completeness is c-completeness. A method of considerable generality, for generating such systems is described in [6] and [10]. A general version of separation of variables procedures yields biorthogonal systems which are c-complete [9]. Convenient features of c-complete systems are their simplicity, as in the case of plane waves [11], and the fact that the same system can be applied to large classes of regions and boundary conditions.

It has been shown [3,10] that under general conditions a system which is c-complete for a region has this property for any region which contains the first one. The possibility of using this property to treat problems subjected to floating boundary conditions such as seepage flow was suggested previously [12]. In the present paper we initiate the systematic development of this subject. First, a very simple version of a contact problem is presented as an example and then theoretical results that can be applied to a general class of contact problems are developed. These theoretical developments are based on the theory of variational inequalities [13-16].

2. AN EXAMPLE

Let Ω (Fig. 1) be a bounded and connected set in R^n with a Lipschitz continuous boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \phi$. It will be assumed that meas$(\Gamma_1) \neq 0$.

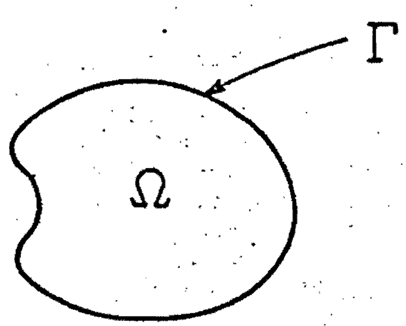

FIGURE 1

Write

$$V = H^1(\Omega) \hookrightarrow H = L^2(\Omega) \tag{1}$$

where the symbol $\hookrightarrow$ means densely contained. Given the functions

$$\begin{aligned} F &\in L^2(\Omega), \\ g &\in H^{1/2}(\Gamma), \\ \hat{u} &\in H^{3/2}(\Gamma), \\ h_o &\in H^{3/2}(\Gamma), \end{aligned} \tag{2}$$

one can formulate the distributional boundary value problem which consists in finding $u \in V$, such that

$$\left.\begin{array}{r}-\Delta u = F \quad \text{in } \Omega, \\ u = \hat{u} \quad \text{on } \Gamma_1, \\ \left.\begin{array}{r} u \geq h_o \\ \partial u/\partial\nu \geq g \\ (\partial u/\partial\nu - g)(u-h) = 0 \end{array}\right\} \text{on } \Gamma_2 \end{array}\right\} \tag{3}$$

Consider the continuous bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad u,v \in H^1(\Omega), \tag{4}$$

As it is usual $\gamma_o : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H^1(\Omega) \to H^{-1/2}(\Gamma)$ are the trace operators. Given $\hat{u} \in H^{3/2}(\Gamma_1)$ and $h_o \in H^{3/2}(\Gamma_2)$, define

$$K = \{v \in H^1(\Omega) : \gamma_o v = \hat{u} \quad \text{on } \Gamma_1, \; \gamma_o v \geq h_o \; \text{on } \Gamma_2\} \tag{5}$$

and $f \in V'$ by

$$\langle f, v \rangle = \int_\Omega F v dx + \int_{\Gamma_2} g \gamma_o v ds \quad , \; v \in H^1(\Omega), \tag{6}$$

Clearly, K is a non-empty, closed and convex subset of V.

Using standard techniques [13,14] it follows that problem (3) is characterized by the variational problem, find $u \in K$ such that

$$\int_\Omega \nabla u \cdot (\nabla v - \nabla u) d\underset{\sim}{x} \geq \int_\Omega F(v-u) d\underset{\sim}{x} + \int_{\Gamma_2} g(\gamma_o v - \gamma_o u) d\underset{\sim}{x}, \; \forall \; v \in K \tag{7}$$

In order to transform the variational problem (7) into a boundary variational problem, let $w \in H^2(\Omega)$ be such that

$$-\Delta w = F \quad , \quad \text{in } \Omega \tag{8}$$

and define the functions

$$\left.\begin{array}{l} g_o = g - \gamma_1 w \in H^{1/2}(\Gamma) \\ \hat{u}_o = \hat{u} - \gamma_o w \in H^{3/2}(\Gamma) \\ h_o = h - \gamma_o w \in H^{3/2}(\Gamma) \end{array}\right\} \tag{9}$$

Consider the closed convex set

$$K_o = \{v \in V_o | \gamma_o v = \hat{u}_o \; \text{on } \Gamma_1, \; \gamma_o v \geq h_o \; \text{on } \Gamma_2\} \tag{10}$$

where

$$V_o = \{v \in H^1(\Omega) | \Delta v = 0 \quad \text{in } \Omega) \tag{11}$$

Then under the transformation

$$u_o = u - w \tag{12}$$

where $w \in H^2(\Omega)$ is a fixed element satisfying (8), problem (7) and therefore (3) is equivalent to the boundary variational problem, find $u_o \in K_o$ such that

$$\int_{\Gamma_2} (v-u_o)\partial u_o/\partial n d\underset{\sim}{x} \geq \int_{\Gamma_1} (v-u_o) g_o d\underset{\sim}{x}, \quad \forall\ v \in K_o \qquad (13)$$

The method we propose for solving the boundary variational problem (13), is based on the use of a basis $\{\varphi_1, \varphi_2, \ldots\}$ of V_o. The construction of such basis has been extensively studied by Herrera [6,10] recently for a large class of systems of partial differential equations. For Laplace's equation, for example, it has been shown that a basis of V_o, when Ω is bounded and simply connected, is the system of harmonic polynomials [6]

$$\{Re\ z^n,\ Im\ z^n \ ;\ n = 0,1,2,\ldots\} \qquad (14)$$

Given such basis define

$$V_{om} = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\},\ m \geq 1 \qquad (15)$$

Let

$\{p_1, p_2, \ldots\}$ be a basis of $H^o(\Gamma_1)$ (16)

$\{q_1, q_2, \ldots\}$ be a basis of $H^o(\Gamma_2)$ (17)

It will be assumed that $q_j \geq 0$, $j = 1,2,\ldots$. Elements v_m of the convex subset K_{om}, will be required to satisfy

$$\int_{\Gamma_1} p_j (v_m - \hat{u}_o) d\underset{\sim}{x} = 0 \ ; \ \int_{\Gamma_2} q_j (\gamma_o v_m - h_o) d\underset{\sim}{x} \geq 0 \ ; \ j=1,\ldots,m. \qquad (18)$$

More precisely

$$K_{om} = \{v_m \in V_{om} | v_m \text{ satisfies } (18)\} \qquad (19)$$

If K_o is replaced by K_{om} in the definition of the boundary variational problem (13), one obtains a family of variational problems. Let $u_{om} \in K_{om}$ be the solution of such problem, then it can be shown that $u_{om} \to u_o$. The example given here is a particular case of the general theory explained next.

3. NOTATION

$(V, \|\cdot\|)$ is a real Hilbert space with topological dual $(V', \|\cdot\|_*)$; $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $V' \times V$. $(H, (\cdot,\cdot), |\cdot|)$ is a real Hilbert space identified with its dual, in which V is densely and continously embedded: $V \quad H \quad V'$. K is a nonempty, closed and convex subset of V.

$a: V \times V \to \mathcal{R}$ is a continuous bilinear form (not necessarily symmetric) which satisfies the condition c: (K-ellipticity) there is a constant $\alpha > 0$ such that

$$a(u-v, u-v) \geq \alpha \|u-v\|^2, \quad \forall \; u, v \in K.$$

$\mathcal{A} \in \mathcal{L}(V, V')$ is the corresponding continuous linear operator:

$$a(u,v) = \langle \mathcal{A}u, v \rangle, \quad u, v \in V. \tag{20}$$

$\gamma \in \mathcal{L}(V, B)$ is a linear continuous surjection with kernel V_o dense in H, B being a Hilbert space isomorphic to the quotient space V/V_o, and the quotient map $\hat{\gamma}: V/V_o \to B$ norm-preserving.

$A \in \mathcal{L}(V, V_o')$ is the linear continuous composition $\rho\mathcal{A}$, where $\rho: V' \to V_o'$ is the restriction to V_o of functionals on V, called the formal operator determined by $a(\cdot,\cdot)$, V and V_o:

$$a(u,v) = \langle Au, v \rangle, \quad u \in V, \; v \in V_o. \tag{21}$$

Hence (cf. [13]), the following abstract Green's formula holds:

$$\langle \mathcal{A}u, v \rangle - (Au, v) = [\partial u, \gamma v], \quad u \in D_o, \; v \in V, \tag{22}$$

where $D_o = \{u \in V : Au \in H\}$, $\partial \in L(D_o, B')$ is the abstract Green's operator and $[\cdot,\cdot]$ is the duality pairing on $B \times B'$.

$j: K \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous functional. In addition $f \in V'$.

4. ABSTRACT BOUNDARY VALUE PROBLEM

With the above notation in force, we now consider the variational problem (P) corresponding to a kind of abstract boundary value problems. Toward this end, let K be characterized only by boundary constraints:

$$K + V_o \subset K, \tag{23}$$

Clearly, condition (23) is equivalent to

$$K + V_o = K \tag{24}$$

Let $a_1: V \times V \to \mathcal{R}$ and $a_2: B \times B \to \mathcal{R}$ be continuous bilinear forms such that

$$a(u,v) = a_1(u,v)+a_2(\gamma u,\gamma v), \quad u,v \in V, \qquad (25)$$

satisfies condition (c). Similarly, let $h: B \to (-\infty,+\infty]$ be a functional such that

$$j(v) = h(\gamma v), \quad v \in K \qquad (26)$$

is proper, convex and lower semi-continuous. Let $F \in H$ and $g \in B'$, and define

$$f(v) = (F,v)+[g,\gamma v], \quad v \in V, \qquad (27)$$

which belongs to V'. Then the variational problem (P) takes the following form:

$$\left.\begin{aligned} &\text{Find } u \in K \text{ such that} \\ &a_1(u,v-u)+a_2(\gamma u,\gamma v-\gamma u)+h(\gamma v)-h(\gamma u) \\ &\qquad \geq (F,v-u)+[g,\gamma v-\gamma u], \quad \forall\ v \in K \end{aligned}\right\} \qquad (28)$$

The following theorem determines the abstract boundary value problem to which (28) is equivalent.

<u>Theorem 1</u>. *Let $A_1 \in \mathcal{L}(V,V')$ and $A_2 \in \mathcal{L}(B,B')$ be the operators corresponding to $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$, respectively. Let $A \in \mathcal{L}(V,V_o')$ be the formal operator determined by $a_1(\cdot,\cdot)$ (or, equivalently, $a(\cdot,\cdot)$), V and V_o, and let $\partial_1 \in L(D_o,B')$ be the abstract Green's operator defined by (22):*

$$\langle A_1 u,v\rangle - (Au,v) = [\partial_1 u,\gamma v], \quad u \in D_o, \ v \in V. \qquad (29)$$

Then the problem (28) is equivalent to the problem

$$\left.\begin{aligned} &\textit{Find } u \in K \textit{ such that} \\ &\qquad Au = F \quad \textit{in} \quad H, \\ &[\partial_1 u+A_2(\gamma u),\gamma v-\gamma u]+h(\gamma v)-h(\gamma u) \\ &\qquad \geq [g,\gamma v-\gamma u], \quad \forall \quad v \in K \end{aligned}\right\} \qquad (30)$$

<u>Proof</u>. Let $u \in K$ be the solution of problem (28). Then, in accordance with (23), we can set $v = u - v_o \in K$, $v_o \in V_o$, and obtain $Au = F$ in H, since V_o is dense in H, and $u \in D_o$. Now, upon using (29), the boundary inequality of (30) follows from (28).

Conversely, let $u \in K$ be a solution of problem (30). Then $u \in D_o$ and because of (29), u is a solution of (28).

5. BOUNDARY VARIATIONAL PROBLEM

In order to transform the abstract boundary value problem (30) into a boundary variational problem, we introduce equation "Au=F in H" in K as an additional constraint. Hence, reconsider problem (30) on the closed convex set

$$\tilde{K} = \{v \in K : Au = F \text{ in } H\}. \tag{31}$$

Then problem (30) transforms into the problem

$$\left.\begin{array}{l} \text{Find } u \in K \text{ such that} \\ [\partial_1 u + A_2(\gamma u), \gamma v - \gamma u] + h(\gamma v) - h(\gamma u) \\ \qquad \geq [g, \gamma v - \gamma u], \quad \forall\, v \in \tilde{K} \end{array}\right\} \tag{32}$$

Theorem 2. *The abstract boundary value problem (11) [or, equivalently, the variational problem (28)] is equivalent to the boundary variational problem (32).*

Proof. It is clear that solutions of (30) are also solutions of (32). The converse follows by observing that given $v \in K$, there exist a $\tilde{v} \in \tilde{K}$ such that $\tilde{v} - v \in V_0$ and, consequently, (32) holds for such a v.

6. HOMOGENEIZATION OF THE DOMAIN CONSTRAINT

For convenience in our study on internal approximations of problem (32), we make homogeneous the domain constraint of $\tilde{K}$ and incorporate it into the space V. Toward this end, let

$$w \in D_0 : Aw = F \text{ in } H \tag{33}$$

be a known function, and consider the change of dependent variable

$$u_0 = u - w. \tag{34}$$

Then, according to the definitions

$$\left.\begin{array}{l} g_0 = g - \partial_1 w - A_2(\gamma w) \in B' \\ h_w(v) = h(v + \gamma w), \quad \forall\, v \in B \end{array}\right\} \tag{35}$$

and

$$\left.\begin{array}{l} V_A = \{v \in V : Av = 0 \text{ in } H\} \\ K_A = \{v \in V_A : v + w \in K\} \end{array}\right\} \tag{36}$$

the following result is easily established:

Theorem 3. *Via the relation (34), the problem (32) is equivalent to the problem*

Find $u_o \in K_A$ such that

$$\left.\begin{array}{r}[\partial_1 u_o + A_2(\gamma_o), \gamma v - \gamma u_o] + h_w(v) - h_w(u_o) \\ \geq [g_o, \gamma v - \gamma u_o], \ \forall\ v \in K_A\end{array}\right\} \quad (37)$$

7. INTERNAL APPROXIMATIONS

Let $\{V_m, K_m\}_{m\geq 1}$ be an internal approximation of $\{V_A, K_A\}$ in the sense of [15]:

(38)

i) $\{V_m\}_{m\geq 1}$ is a family of finite dimensional subspace of V_A with parameter $m=\dim V_m$;

ii) $\forall\ v \in V_A,\ \exists\ v_m \in V_m$: $v_m \to v$ in V as $m\to\infty$;

iii) For each $m\geq 1$, K_m is a nonempty, closed and convex subset of V_m;

iv) $\forall\ v \in K_A,\ \exists\ v_m \in K_m$: $v_m \to v$ in V as $m\to\infty$;

v) If $v_m \in K_m$ and $v_m \to v$ (weakly) in V as $m\to\infty$, then $v \in K_A$.

For each $m\geq 1$, we associate to problem (37) the discrete problem:

Find $u_m \in K_m$ such that

$$\left.\begin{array}{r}[\partial_1 u_m + A_2 \gamma u_m, \gamma v - \gamma u_m] + h_w(v) - h_w(u_m) \\ \geq [g_o, \gamma v - \gamma u_m], \ \forall\ v \in K_m\end{array}\right\} \quad (39)$$

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