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CONTRIBUTION TO FREE BOUNDARY PROBLEMS USING BOUNDARY ELEMENTS: TREFFTZ APPROACH

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1. Introduction

In recent years, by a boundary method is usually understood a numerical procedure in which a subregion or the entire region is left out of the numerical treatment, by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problem leading to considerable economy in the numerical work and constitute a very convenient manner of treating adequately unbounded regions by numerical means. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced [1, 2].

There are two main approaches for the formulation of boundary methods; one is based on boundary integral equations and the other one on the use of complete systems of solutions. In numerical applications the first of these methods has received most of the attention [3]. This is in spite of the fact that the use of complete systems of solutions presents important numerical advantages; e.g., it avoids the introduction of singular integral equations and it does not require the construction of a fundamental solution. The latter is especially relevant in connection with complicated problems, for which it may be extremely laborious to build up a fundamental solution. This is illustrated by the fact that there are methods for synthesizing fundamental solutions starting from plane waves, which can be shown to be a complete system [4].

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[5]. The idea of his method [6] consists in looking for approximate solutions from among the appropriate class of functions that satisfy exactly the differential equation, but do not necessarily satisfy the prescribed boundary conditions. Although Trefftz's original formulation was linked to a variational principle, this is not required. Indeed, complete systems of solutions can be used to treat differential equations which are linear but otherwise arbitrary [7, 8].

The method has been used in many fields. For example, applications to Laplace's equation are given by Mikhlin [9], to the biharmonic equation by Rektorys [10] and to elasticity by Kupradze [11]. Also many scattered contributions to the method can be found in the literature. Special mention is made here of work by Amerio, Fichera, Kupradze, Picone and Vekua [12–16]. Colton [17, 18] constructed families of solutions which are complete for parabolic equations.

However, general discussions applicable to arbitrary linear differential equations were lacking until recently. Motivated by this situation Herrera started a systematic research of the subject. The aim of the study has been two-fold; firstly to clarify the theoretical foundations required for using complete systems of solutions in a reliable manner, and secondly, to expand the versatility of such methods, making them applicable to any problem which is governed by partial differential equations which are linear.

The aims of the research have been satisfactorily achieved, to a large extent. A first survey article has already appeared [7], but two more complete ones soon will be published [8, 19]. The systematic development includes:

(a) approximating procedures and conditions for their convergence [4, 8, 20–22];

(b) formulation of variational principles [23–28]; and

(c) development of complete systems of solutions [29–33].

In addition, the algebraic framework [34] in which the theory has been constructed has been used for developing biorthogonal systems of solutions [35].

Thus far, the theory has been applied to linear problems only. This article is devoted to extend it to free boundary problems; these are nonlinear even when the governing differential equations are linear. Typical examples are contact problems in elasticity [36, 37] and seepage problems in flow through porous media [38, 39]. They have been treated using finite element approximations. Boundary integral equations have also been applied [40–43]. Trefftz method has been applied [22] to seepage problems, but the theoretical analysis is wanting.

Liggett and Liu [40] have concentrated on the discussion of the boundary integral equation approach as applied to flow in porous media while Kikuchi [41–43] has given extensive discussions of that approach for contact problems under the general heading of reciprocal variational inequalities.

Advantages of Trefftz method (complete systems of solutions) are the simplicity of the formulation and the possibility of using the same system of solutions independently of the detailed shape of the region considered and of the specific boundary conditions. Frequently, this is possible [8, 30, 33, 34] when the system is c-complete. In the case of seepage problems, for which the region is not known in advance, this fact is specially relevant [22].

In the present paper, a very simple version of a contact problem is introduced as an example and then general theoretical results that can be applied to a wide class of problems are developed. These results are derived from general properties of variational inequalities [44–49].

2. An example

Let Ω be a bounded and connected set in $\mathbb{R}^n$ with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \cap \Gamma_2 = \emptyset$ and Γ is a C^2-manifold. It will be assumed that $\text{meas}(\Gamma_1) > 0$.

Write

$$V = H^1(\Omega) \hookrightarrow H = L^2(\Omega) \tag{1}$$

where the symbol $\hookrightarrow$ means densely and continuously embedded. Given the functions

$$F \in L^2(\Omega), \qquad g \in H^{1/2}(\Gamma), \qquad \hat{u} \in H^{3/2}(\Gamma), \qquad d \in H^{3/2}(\Gamma)$$

one can formulate the distributional boundary value problem which consists in finding $u \in V$ such that

$$-\Delta u = F \quad \text{in } \Omega \qquad u = \hat{u} \quad \text{on } \Gamma_1$$
$$u \geq d, \qquad \partial u/\partial\nu \geq g, \qquad (\partial u/\partial\nu - g)(u - d) = 0 \quad \text{on } \Gamma_2$$

Consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad u, v \in H^1(\Omega)$$

which is continuous in $H^1(\Omega)$.

As usual, $\gamma_0 : H^m(\Omega) \to H^{m-1/2}(\Gamma)$ and $\gamma_1 : H^m(\Omega) \to H^{m-3/2}(\Gamma)$, $m \geq 1$, will be the trace operators. Define

$$K = \{v \in H^1(\Omega) : \gamma_0 v = \hat{u} \text{ on } \Gamma_1, \gamma_0 v \geq d \text{ on } \Gamma_2\} \tag{5}$$

and $f \in V'$ by

$$\langle f, v\rangle = \int_\Omega Fv \, dx + \int_{\Gamma_2} g\gamma_0 v \, ds, \quad v \in H^1(\Omega) \tag{6}$$

Clearly, K is a non-empty, closed and convex subset of V.

Using standard techniques [44, 46] it is seen that problem (3) is characterized by the variational problem:

Find $u \in K$ such that

$$\int_\Omega \nabla u \cdot (\nabla v - \nabla u) dx \geq \int_\Omega F(v - u) dx + \int_{\Gamma_2} g(\gamma_0 v - \gamma_0 u) ds \quad \forall v \in K \tag{7}$$

In order to transform the variational problem (7) into a boundary variational problem, let $w \in H^2(\Omega)$ be such that

$$-\Delta w = F \quad \text{in } \Omega \tag{8}$$

and define the functions

$$g_0 = g - \gamma_1 w \in H^{1/2}(\Gamma), \qquad \hat{u}_0 = \hat{u} - \gamma_0 w \in H^{3/2}(\Gamma), \qquad d_0 = d - \gamma_0 w \in H^{3/2}(\Gamma) \tag{9}$$

There are available many procedures, both analytical and numerical, for constructing such a function w. Indeed, (8) is a linear equation which is not subject to any boundary conditions in contrast with problem (3) which is a nonlinear boundary value problem. For example, using a fundamental solution for Laplace's equation, w can be obtained by quadrature.

Consider the closed and convex set

$$K_A = \{v \in V_A \colon \gamma_0 v = \hat{u}_0 \text{ on } \Gamma_1,\ \gamma_0 v \geqslant d_0 \text{ on } \Gamma_2\}$$

where

$$V_A = \{v \in H^1(\Omega) \colon -\Delta v = 0 \text{ in } \Omega\}$$

Then, under the transformation

$$u_0 = u - w$$

where $w \in H^2(\Omega)$ is the fixed element satisfying (8), problem (7) and therefore equivalent to the boundary variational problem:

Find $u_0 \in K_A$ such that

$$\int_\Gamma \gamma_1 u_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}s \geqslant \int_{\Gamma_2} g_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}s, \quad \forall v \in K_A \tag{13}$$

Notice that $u_0 \in K_A$ only grants that $\gamma_1 u_0 \in H^{-1/2}(\Gamma)$. However, because of the regularity of the region and the data [48], $u_0 \in K_A \cap H^2(\Omega)$ and $\gamma_1 u_0 \in H^{1/2}(\Gamma)$.

The method we propose for solving the boundary variational problem (13) is based on the use of a basis $\{\varphi_1, \varphi_2, \ldots\}$ of V_A. Its application is simplified if the equality condition in definition (10) of K_A is relaxed. This can be done if the variational inequality (13) is slightly modified. Thus, let the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_1} \gamma_0 u \gamma_0 v \,\mathrm{d}\boldsymbol{x}$$

while

$$K_A = \{v \in V_A \colon \gamma_0 v \geqslant d_0 \text{ on } \Gamma_2\}$$

Then system (3) is equivalent to the boundary variational problem

Find $u_0 \in K_A$ such that

$$\gamma_0 u_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}x + \int_{\Gamma_2} \gamma_1 u_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}x \geq$$
$$\geq \int_{\Gamma_1} \hat{u}_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}x + \int_{\Gamma_2} g_0(\gamma_0 v - \gamma_0 u_0)\mathrm{d}x\,, \quad \forall v \in K_A \tag{13'}$$

A systematic study about the construction of complete systems applicable to boundary methods has been carried out by one of the authors [33]. For Laplace's equation, for example, it is known that a basis of V_A, when Ω is bounded and simply connected, is the system of harmonic polynomials [29]

$$\{\mathrm{Re}\, z^n, \mathrm{Im}\, z^n, n = 0, 1, 2, \tag{14}$$

Given such a basis define

$$V_m = \mathrm{span}\{\varphi_1, \varphi_2, \quad , \varphi_m\}\,, \quad m \geq \tag{15}$$

Let $\{p_1, p_2, \ldots\}$ be a basis of $H^0(\Gamma_2)$. It will be assumed that $p_j \geq 0$, $j =\ ., 2$ Elements v_m of the convex subset K_m will be required to satisfy

$$\int_{\Gamma_2} p_j(\gamma_0 v_m - d_0)\mathrm{d}s \geq 0\,, \quad = 1, \quad , m \tag{16}$$

More precisely,

$$K_m = \{v_m \in V_m : v_m \text{ satisfies (16)}\} \tag{17}$$

Even more, it will be assumed that the system $\{p_1, p_2, \quad \subset H^0(\Gamma_2)$ is such that for every $r \in H^0(\Gamma_2)$ one has

$$\int_{\Gamma_2} r p_j \mathrm{d}s \geq 0\ \forall j = 1, 2, \quad \Rightarrow \quad r \geq 0 \text{ a.e. on } \Gamma_2 \tag{18}$$

Here a.e. means almost everywhere. For example, when $\Gamma_2 = [0, 1]$, a system $\{p_1, p_2, \quad .\}$ that satisfies this condition is

$$p_j(x) = \begin{cases} 1, & 2^{-n}(j - 2^n) \leq x \leq 2^{-n}(j + \quad - 2^n), \\ 0, & \text{otherwise}, \end{cases} \quad 2^n \leq j < 2^{n+1}, n = 0, 1, 2,$$

If K_A is replaced by K_m in the definition of the boundary variational problem (13′), one obtains a family of variational problems. Let $u_m \in K_m$ be the solution of such problem, then, as will be demonstrated in Section 8, $u_m \to u_0$ in $H^1(\Omega)$.

The example given here is a particular case of the general theory explained in the next sections.

Observe, for later use, that

$$V_m \cap K_A \subset K_m \tag{19}$$

3. Notation

$(V, \|\cdot\|)$ is a real Hilbert space with topological dual $(V', \|\cdot\|_*)$; $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $V' \times V$. $(H, (\cdot,\cdot), |\cdot|)$ is a real Hilbert space identified with its dual, in which V is densely and continuously embedded: $V \hookrightarrow H \hookrightarrow V'$. K is a non-empty, closed and convex subset of V.

$a : V \times V \to \mathbb{R}$ is a continuous bilinear form (not necessarily symmetric) which satisfies the condition (C): (K-ellipticity)

there is a constant $\alpha > 0$ such that

$$a(u - v, u - v) \geq \alpha \|u - v\|^2 \quad \forall u, v \in K$$

$\mathscr{A} \in \mathscr{L}(V, V')$ is the corresponding continuous linear operator,

$$a(u, v) = \langle \mathscr{A}u, v\rangle, \quad u, v \in V \tag{20}$$

$\gamma \in \mathscr{L}(V, B)$ is a linear continuous surjection with kernel V_0 dense in H, B being a Hilbert space isomorphic to the quotient space V/V_0, and the quotient map $\hat{\gamma} : V/V_0 \to B$ norm-preserving.

$A \in \mathscr{L}(V, V_0')$ is the linear continuous composition $\rho\mathscr{A}$, where $\rho : V' \to V_0'$ is the restriction to V_0 of functionals on V, called the formal operator determined by $a(\cdot,\cdot)$, V and V_0,

$$a(u, v) = \langle Au, v\rangle, \quad u \in V, v \in V_0 \tag{21}$$

Hence (cf. [45]), the following abstract Green's formula holds

$$\langle \mathscr{A}u, v\rangle - (Au, v) = [\partial u, \gamma v], \quad u \in D_0, v \in V \tag{22}$$

where $D_0 = \{u \in V : Au \in H\}$, $\partial \in L(D_0, B')$ is the abstract Green's operator and $[\cdot,\cdot]$ is the duality pairing on $B' \times B$.

$j : K \to (-\infty, +\infty]$ is a proper, convex and lower semi-continuous functional. In addition $f \in V'$.

4. Abstract boundary value problem

With the notation of Section 3 in force, we now consider the variational problem

Find $u \in K$ such that

$$a(u, v - u) + j(v) - j(u) \geq f(v - u), \quad \forall v \in K$$

which possesses a unique solution (cf. [47]), corresponding to a kind of abstract boundary value problems. Let K be characterized only by boundary constraints,

$$K + V_0 \subset K. \tag{23}$$

Clearly, condition (23) is equivalent to

$$K + V_0 = K$$

Let $a_1 : V \times V \to \mathbb{R}$ and $a_2 : B \times B \to \mathbb{R}$ be continuous bilinear forms such that

$$a(u, v) = a_1(u, v) + a_2(\gamma u, \gamma v), \quad u, v \in V$$

satisfies condition (C). Similarly, let $h : B \to (-\infty, +\infty]$ be a functional such that

$$j(v) = h(\gamma v) \qquad v \in K \tag{26}$$

proper, convex and lower semi-continuous. Let $F \in H$ and $g \in B'$, and define

$$f(v) = (F, v) + [g, \gamma v] \qquad v \in V \tag{27}$$

which belongs to V'. Then the variational problem (P) takes the following form

Find $u \in K$ such that

$$a_1(u, v - u) + a_2(\gamma u, \gamma v - \gamma u) + h(\gamma v) - h(\gamma u) \geq$$
$$\geq (F, v - u) + [g, \gamma v - \gamma u], \quad \forall v \in K.$$

The following theorem determines the abstract boundary value problem to which (28) is equivalent.

THEOREM 4.1. Let $\mathcal{A}_1 \in \mathcal{L}(V, V')$ *and* $\mathcal{A}_2 \in \mathcal{L}(B, B')$ *be the operators corresponding to* $a_1(\cdot,\cdot)$ *and* $a_2(\cdot,\cdot)$*, respectively. Let* $A \in \mathcal{L}(V, V_0')$ *be the formal operator determined by* $a_1(\cdot,\cdot)$ *(or, equivalently,* $a(\cdot,\cdot)$*),* V *and* V_0*, and let* $\partial_1 \in L(D_0, B')$ *be the abstract Green's operator defined by* (22),

$$\langle \mathcal{A}_1 u, v\rangle - (Au, v) = [\partial_1 u, \gamma v], \quad u \in D_0, v \in V$$

Then problem (28) *is equivalent to the problem*

Find $u \in K$ *such that*

$$Au = F,$$

$$[\partial_1 u + \mathcal{A}_2(\gamma u), \gamma v - \gamma u] + h(\gamma v) - h(\gamma u) \geq [g, \gamma v - \gamma u] \qquad \forall v \in K$$

PROOF. Let $u \in K$ be the solution of problem (28). Then, in accordance with (23), we can set $v = u \pm v_0 \in K$, $v_0 \in V_0$, and obtain $Au = F$ in H, since V_0 is dense in H, and $u \in D_0$. Now, upon using (29), the boundary inequality of (30) follows from (28).

Conversely, let $u \in K$ be a solution of problem (30). Then $u \in D_0$ and because of (29), u is a solution of (28).

5. Boundary variational problem

In order to transform the abstract boundary value problem (30) into a boundary variational problem, we introduce equation $Au = F$ as an additional constraint in K. Hence, reconsider problem (30) on the closed and convex set

$$\tilde{K} = \{v \in K: Av = F\}$$

Then problem (30) transforms into the problem:

$$\text{Find } u \in \tilde{K} \text{ such that}$$
$$[\partial_1 u + \mathscr{A}_2(\gamma u), \gamma v - \gamma u] + h(\gamma v) - h(\gamma u) \geqslant [g, \gamma v - \gamma u]\,, \quad \forall v \in \tilde{K}$$

THEOREM 5.1. The abstract boundary value problem (30) (*or, equivalently, the variational problem* (28)) *is equivalent to the boundary variational problem* (32).

PROOF. It is clear that solutions of (30) are also solutions of (32). The converse follows by observing that given $v \in K$, there exists a $\tilde{v} \in \tilde{K}$ such that $\tilde{v} - v \in V_0$ and, consequently, (30) holds for such a v.

6. Homogenization of the domain constraint

For convenience in our study on internal approximations of problem (32), we make homogeneous the domain constraint of $\tilde{K}$ and incorporate it into the space V. Toward this end, let

$$w \in D_0: Aw = F$$

be a known function, and consider the change of dependent variable

$$u_0 = u - w$$

Then, according to the definitions

$$g_0 = g - \partial_1 w - \mathscr{A}_2(\gamma w) \in B'\,, \qquad h_w(v) = h(v + \gamma w)\,, \quad \forall v \in B\,,$$

and

$$V_A = \{v \in V: Av = 0\}\,, \qquad K_A = \{v \in V_A: v + w \in K\}$$

the following result is easily established.

THEOREM 6.1. Via relation (34), *problem* (32) *is equivalent to the problem*

$$\textit{Find } u_0 \in K_A \textit{ such that}$$
$$[\partial_1 u_0 + \mathscr{A}_2(\gamma u_0), \gamma v - \gamma u_0] + h_w(\gamma v) - h_w(\gamma u_0) \geqslant [g_0, \gamma v - \gamma u_0] \quad \forall v \in K_A$$

7. Internal approximations

In what follows, all the topological notions to be used, are with respect to the V_A-topology. Let K_A be a non-empty, closed and convex subset of V_A, and $\{V_m, K_m\}_{m\geq 1}$ be an internal approximation of $\{V_A, K_A\}$ in the sense of [46]:

(i) $\{V_m\}_{m\geq 1}$ is a family of finite-dimensional subspaces of V_A with parameter $m = \dim V_m$

(ii) For each $m \geq 1$, K_m is a non-empty, closed and convex subset of V_m;

(iii) $\forall v \in K_A, \exists v_m \in K_m : v_m \to v$ in V_A as $m \to \infty$

(iv) If $v_m \in K_m$ and $v_m \rightharpoonup v$ (weakly) in V_A as $m \to \infty$ then $v \in K_A$. 38)

A useful sufficient condition for property (iii) is given by

THEOREM 7.1. Let conditions (i) *and* (ii) *of* (38) *hold. Assume the interior (in* V_A*)* $\mathring{K}_A$ *of* K_A *is not empty and*

$$V_m \cap K_A \subset K_m, \quad \forall m \geq \tag{39}$$

Then the assumption

(iii′) $\forall v \in V_A, \exists \{v_m \in V_m : m \quad , 2, \quad v_m \to v$ *in* V_A *as* $m \to \infty$

implies assumption (iii) *of* (38).

PROOF. When $\mathring{K}_A$ is non-empty, $\mathring{K}_A \subset K_A$ is dense in K_A. In order to prove this assertion, let $v \in K_A$; it is easy to see that

$$\mathring{K}_A = \{w \in V_A : \exists z \in \mathring{K}_A : w = (1 \quad \lambda)v + \lambda z, 0 < \lambda < 1\}$$

Then v belongs to the closure of $\mathring{K}_A$ because if $z \in \mathring{K}_A$ then $(1-\lambda)v + \lambda z \to v$ as $\lambda \to 0$. Therefore, $\mathring{K}_A$ is dense in K_A since this holds for every $v \in K_A$.

Once this has been shown, it will be enough to prove (iii) of (38) for the case when $v \in \mathring{K}_A$. Let $v \in \mathring{K}_A$ and choose $\{v_m \in V_m : m = 1, 2, \ldots\}$ as in (iii′). Then there exists $M > 0$ such that $v_m \in \mathring{K}_A \subset K_A$ whenever $m > M$ since $\mathring{K}_A$ is an open set which contains v. This shows

$$v_m \in V_m \cap K_A \subset K_m$$

because $v_m \in V_m$, and the restricted form of (iii′) follows.

For each $m \geq 1$, we associate with problem (37) the discrete problem

Find $u_m \in K_m$ such that

$$[\partial_1 u_m + \mathcal{A}_2(\gamma u_m), \gamma v - \gamma u_m] + h_w(\gamma v) - h_w(\gamma u_m) \geqslant [g_0, \gamma v - \gamma u_m], \quad \forall v \in K_m \tag{40}$$

It is clear that existence and uniqueness for problem (40) are granted if the bilinear continuous form $a : V_A \times V_A \to \mathbb{R}$, which is K_A-elliptic, and the lower semi-continuous proper convex functional $j_w = h_w \circ \gamma : K_A \to (-\infty, +\infty]$ satisfy the uniform conditions:

(C1) (Uniform K_m-ellipticity) there is $\tilde{\alpha} > 0$ such that

$$a(u - v, u - v) \geqslant \tilde{\alpha}\|u - v\|^2, \quad \forall u, v \in K_m, \forall m \geqslant 1$$

(C2) $j_w : K_m \to (-\infty, +\infty]$ is convex, and uniformly proper and lower semi-continuous in m; i.e., it is a lower semi-continuous proper convex functional on each K_m, there exist $l \in V'_A$ and $\mu \in \mathbb{R}$ such that

$$j_w(v) \geqslant l(v) + \mu, \quad \forall v \in K_m, \quad \forall m \geqslant 1,$$

and for every $v_m \in K_m$, converging weakly to $v \in K_A$,

$$\liminf_{m\to\infty} j_w(v_m) \geqslant j_w(v)$$

However, in general, these conditions do not guarantee weak convergence of the approximation process. They do imply weak convergence (cf. [49]), if in addition

(C3) $\quad a(u - v, u - v) \geqslant 0, \quad \forall u \in K_m, v \in K_A, \forall m \geqslant$

(C4) $j_w : K_m \to (-\infty, +\infty]$ is uniformly continuous in m; i.e., for every $v_m \in K_m$, converging (strongly) to $v \in K_A$,

$$\lim_{m\to\infty} j_w(v_m) = j_w(v)$$

and strong convergence, if furthermore

(C5) there is $\hat{\alpha} > 0$ such that

$$a(u - v, u - v) \geqslant \hat{\alpha}\|u - v\|^2, \quad \forall u \in K_m, v \in K_A, \forall m \geqslant 1$$

Since in some applications conditions (C2) and (C4) are not satisfied, it is then necessary to approximate the functional $j_w : K_A \to (-\infty, +\infty]$ by a family $\{j_m\}_{m\geqslant 1}$, for example, in the sense of [49]:

(C2′) $j_m : K_m \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous for each $m \geqslant 1$, there exist $l \in V'_A$ and $\mu \in \mathbb{R}$ such that

$$j_m(v) \geqslant l(v) + \mu \quad \forall v \in K_m, \forall m \geqslant 1$$

and for every $v_m \in K_m$, converging weakly to $v \in K_A$,

$$\liminf_{m\to\infty} j_m(v_m) \geq j_w(v)$$

(C4′) For every $v_m \in K_m$, converging (strongly) to $v \in K_A$

$$\lim_{m\to\infty} j_m(v_m) = j_w(v)$$

Then we approximate problem (37) by the family of discrete problems:

Find $u_m \in K_m$ such that

$$[\partial_1 u_m + \mathscr{A}_2(\gamma u_m), \gamma v - \gamma u_m] + j_m(v) - j_m(u_m) \geq [g_0, \gamma v - \gamma u_m], \quad \forall v \in K_m$$

THEOREM 7.2. Let conditions (C1) *and* (C2′) *be satisfied. Then, for each* $m \geq 1$ *problem* (41) *has a unique solution* $u_m \in K_m$. *Furthermore, the sequence of discrete solutions* $\{u_m\}_{m\geq 1}$ *is weakly convergent in* V_A *to the solution* $u_0 \in K_A$ *of problem* (37) *if, in addition, conditions* (C3) *and* (C4′) *hold. The convergence is strong if condition* (C5) *is also satisfied.*

PROOF. These convergence results are established via the usual arguments (cf. [49]), and we omit the details.

8. Convergence result of the example

In this final section we demonstrate that the convergence conditions of Theorem 7.2 are satisfied by the example of Section 2. In this case, since $j_w \equiv 0$, conditions (C2′) and (C4′) are trivially satisfied. Also conditions of Theorem 7.1 are fulfilled by K_A as given by (10′) by virtue of (19) and the fact that the interior $\mathring{K}_A$ of K_A is non-empty. As a matter of fact, the main reason for changing definition (10) by (10′) is that the interior of K_A, as given by (10), is empty. On the contrary, K_A as given by (10′) has a non-empty interior. Taking all this into account, only conditions (iv) of (38), (C1) and (C5) remain to be proved.

PROPOSITION 8.1. The family $\{V_m, K_m\}_{m\geq 1}$ *defined by* (15) *and* (17) *is an internal approximation of* $\{V_A, K_A\}$ *of* (11) *and* (10′).

PROOF. To prove (iv) of (38), let $v_m \in K_m$ be a weakly convergent sequence to v in V_A. Then, since $\gamma_0 \in \mathscr{L}(V_A, L^2(\Gamma))$,

$$\gamma_0 v_m \rightharpoonup \gamma_0 v \quad \text{(weakly) in } L^2(\Gamma).$$

Therefore,

$$\int_{\Gamma_2} p_j(\gamma_0 v - d_0)\mathrm{d}s \geq 0 \qquad \forall j = 1, 2,$$

by virtue of (16) and (17). This shows that $v \in K_A$ by virtue of (18).

PROPOSITION 8.2. The bilinear form a $H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ *of* (4′), *is* V_A*-elliptic; i.e.*

$$\exists\beta>0\colon a(v,v)\geq\beta\|v\|^2\quad\forall v\in V_A$$

Here $\|v\|$ *stands for the norm in* $H^1(\Omega)$.

PROOF. This is established using standard arguments (e.g. see [50, the proof of Theorem 1.2.1]. Consider the semi-norm

$$[v]^2=a(v,v)=\int_\Omega\nabla v\cdot\nabla v\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_1}(\gamma_0 v)^2\mathrm{d}\boldsymbol{x}\,,\quad v\in V_A$$

in V_A. Even more, $[v]$ is a norm because

$$[v]^2=0\Rightarrow\nabla v=0\text{ and }\gamma_0 v=0\text{ on }\Gamma_1\Rightarrow v=\text{const}=0$$

Here, the fact that $\text{meas}(\Gamma_1)\neq0$, was used. Taking into account the continuity of $\gamma_0\colon H^1(\Omega)\to H^{1/2}(\Gamma)$, it can be shown that there exists $\delta>0$ such that

$$\delta\|v\|^2\geq[v]^2\tag{46}$$

Assume condition (43) is not satisfied, then there exists a sequence $\{v_1,v_2,\quad.\}\subset V_A$ such that

$$\|v_n\|^2=\qquad n=\ ,2,$$

while

$$[v_n]\xrightarrow[n\to\infty]{}0$$

Condition (47) implies that there exists a subsequence $\{v_1',v_2',\ldots\}\subset\{v_1,v_2,\ldots\}$ which is Cauchy in the $L^2(\Omega)$ norm, because the imbedding of $H^1(\Omega)$ in $H^0(\Omega)$ is compact. Therefore, the same is true in the norm

$$\|v_n'\|^2=\int_\Omega\nabla v_n'\cdot\nabla v_n'\mathrm{d}\boldsymbol{x}+\|v_n'\|^2_{L^2(\Omega)}\tag{49}$$

because $\int_\Omega\nabla v_n'\cdot\nabla v_n'\mathrm{d}\boldsymbol{x}\to0$ by virtue of (44) and (48).

Thus, there exists $v\in H^1(\Omega)$ such that $v_n'\to v$ in $H^1(\Omega)$. However, (46) implies that

$$[v_n'-v]\xrightarrow[n\to\infty]{}0\,,$$

e. $v=0$ by virtue of (48). This contradicts (47) and the proof of Proposition 8.2 is complete

Once this proposition has been established, conditions (C1) and (C5) are clear because

$$V_A\supset K_1\supset K_2\supset\qquad\text{and}\qquad V_A\supset K_A$$

9. Conclusions

The numerical treatment of free boundary problems can be simplified when a complete system of solutions of the homogeneous partial differential equations is available. In particular, if the approach is based on variational inequalities, they can be transformed into boundary inequalities, i.e., variational inequalities involving boundary values only. Generally, in order to obtain formulations suitable for numerical treatment and easy to analyze theoretically, it is necessary to modify the functionals which are usually applied in the finite element handling of such problems. The results presented in this paper supply the theoretical basis for the case when the region of definition of the problem is fixed, e.g. Signorini and friction problems in elasticity.

Notice that although the example discussed in the present paper refers to a scalar case, the general results are indeed applicable to elasticity. As usual, the vector case is treated in terms of product Hilbert spaces [51]. Inequalities analogous to (13) and (13′) hold in which the inner product of the tractions by the displacements vectors occur [51].

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